International Journal of Statistical Sciences *ISSN 1683–5603*
Vol. 8, 2008, pp 23-31

Detecting Images Using Laplace Random Field

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[Received October 10, 2006; Revised June 22, 2008; Accepted September 9, 2008]

Abstract

Mammogram is a good technique for early detection of the breast cancer. This technique generates digital images, where the radiodense is used as a marker for a breast cancer. This kind of images do not follow the Gaussian random field. In this paper, the Laplace random field is introduced as a model for such images. We give a theorem which characterizes the Laplace random field. The expected value of the Euler characteristic of its excursion set is calculated. We also derive an approximation to the mean size of one connected component of its excursion set above a high threshold.

Keywords and Phrases: Euler characteristic; Excursion Set; Laplace Random Field; Multivariate Laplace Distribution.

AMS Classification: 60G60.

1 Introduction

New technologies provide us with informative images about astrophysics, living human brain and breast cancer, etc. These images are modelled by a smooth and stationary random field, $Z(\mathbf{t}),\ \mathbf{t} \in \mathcal{C}$, where $\mathcal{C}$ is some search region. If $Z(\mathbf{t})$ is the value of the image at $\mathbf{t} \in \mathcal{C}$, and $\mu(\mathbf{t})$ is the mean value of the image at $\mathbf{t}$, then the null hypothesis $\mathrm{H}_0 : \mu(\mathbf{t}) = 0$ for all $\mathbf{t} \in \mathcal{C}$ against $\mathrm{H}_1 : \mu(\mathbf{t}) = 0$ for some $\mathbf{t} \in \mathcal{C}$, will be rejected if the statistic $\sup_{\mathbf{t}\in\mathcal{C}} Z(\mathbf{t})$ is large. Then the probability $P\{\sup_{\mathbf{t}\in\mathcal{C}} Z(\mathbf{t}) \geq u\}$, for large u, represents the P-value of this test statistic. It is not possible in general to find its exact value. One of the most famous methods to approximate this probability is to consider the *excursion set* of the random field $Z(\mathbf{t})$ above a threshold u. Adler (1981) defines the excursion set of the random field $Z(\mathbf{t})$ above u in $\mathcal{C}$ as the set

$$A_u = A_u(Z, \mathcal{C}) = \{\mathbf{t} \in \mathcal{C} : Z(\mathbf{t}) \geq u\}. \tag{1}$$

If $Z(\mathbf{t})$ is a homogeneous and smooth Gaussian random field, then with probability tending to one as $u \to \infty$, the excursion set is a finite union of convex sets such that each convex set contains a local maximum for $Z(\mathbf{t})$ (see Figure 1). Then $P\{\sup_{\mathbf{t}\in\mathcal{C}} Z(\mathbf{t}) \geq u\}$ is approximated by the expected number of local maxima of $Z(\mathbf{t})$ in $\mathcal{C}$ above u.

Mammograms are digital images generated by an X-ray special machine to do a breast cancer early detection. The value of the image at each pixel is the gray-level. Some mammogram images are given in Heine, Deans, and Clarke (1999). The histograms of these images show that the Gaussian random field is not a suitable model for these images. In this paper, we will introduce the Laplace random field as a model for this kind of images. Under the null hypothesis, i.e., given a woman has a negative mammogram result, we study the geometry of its excursion set above a high threshold. In Sec. 2, we introduce the Laplace random field and we give a characterization theorem for this field. In Sec. 3, we find the expected Euler characteristic of its excursion set. In Sec. 4, we find an approximation to the expected number of local maxima of the Laplace Random field. In Sec. 5, we give an example to check the validity of the approximations. In Sec. 6, we give an approximation to the mean size of one component of the excursion set. In Sec. 7, we propose an algorithm to simulate a Laplace random field. A summary is given in Sec. 8.

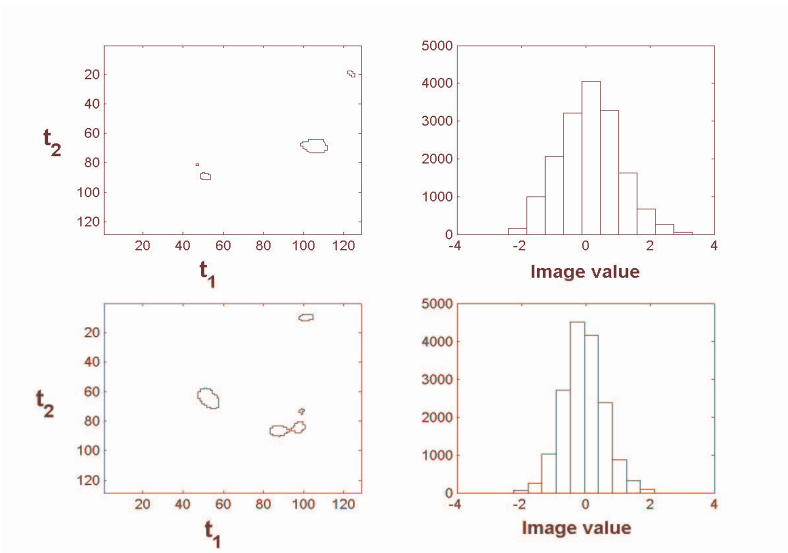

Figure 1: Excursion set (top left) and histogram (top right) of a simulated Gaussian random field above the threshold $u = 2.5$. Excursion set (bottom left) and histogram (bottom right) of a simulated Laplace random field above the threshold $u = 1.5$.

The following notations will be used. For equality in distribution we use $=^d$. The transpose, determinant of a matrix A and diagonal blocking of two matrices A and B will be denoted by A^T, $\det(A)$ and diag(A,B), respectively. $R \sim \exp(1)$ will mean an exponential random variable with mean 1, and $\mathbf{R}_Z(\mathbf{t})$ will denote the covariance function of a homogeneous random field $Z(\mathbf{t})$. We use $W \sim N_D(\mu, \Sigma)$ for a D-dimensional multivariate normal random vector with mean vector μ and covariance matrix Σ. The modified Bessel function of the third kind of order ν, denoted by $K_\nu(x)$, is given by

$$K_\nu(x) = \frac{1}{2}\left(\frac{x}{2}\right)^\nu \int_0^\infty s^{-\nu-1}\exp(-s-\frac{x^2}{4s})ds.$$

For $\nu = \frac{1}{2}$, $K_{\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2x}}\exp(-x)$.

2 A Laplace Random Field

In this paper, we will assume that all random variables and vectors have densities.

Definition 2.1. A random vector W $(D\times 1)$ is said to be a D-dimensional multivariate Laplace with parameters μ $(D \times 1)$ and Σ $(D \times D)$ if its joint density function is given by

$$f_W(w) = 2(2\pi)^{-\frac{D}{2}}\det(\Sigma)^{-\frac{1}{2}}\left(\frac{1}{2}(w-\mu)^T\Sigma^{-1}(w-\mu)\right)^{\frac{\nu}{2}} K_\nu\left(\sqrt{2(w-\mu)^T\Sigma^{-1}(w-\mu)}\right),$$

$\mathrm{w} \in \mathbb{R}^D$, $\nu = \frac{1}{2}(2-D)$.

It can be shown that $E\{W\} = \mu$ and $\mathrm{cov}(W) = \Sigma + \mu\mu^T$. For more information about the properties of the multivariate Laplace distribution, see Kotz et al. (2001). We will use the notation $W \sim LA_n(\mu, \Sigma)$ for a multivariate Laplace distribution with parameters μ and Σ. Every Multivariate Laplace random vector W with parameters μ and Σ admits the following representation $W =^d \mu + \sqrt{R}V$, where R is independent of V, $R \sim \exp(1)$ and $V \sim N_D(\mathbf{0}, \Sigma)$.

Definition 2.2. A real valued random field $Y(\mathbf{t})$, $\mathbf{t} \in \mathcal{C}$, is said to be Laplace random field if for every k, every set of points $\mathbf{t}_1, \ldots, \mathbf{t}_k \in \mathcal{C}$, the vector $(Y(\mathbf{t}_1), \ldots, Y(\mathbf{t}_k))$ has a multivariate Laplace distribution, i.e., all finite dimensional distributions of $Y(\mathbf{t})$ are multivariate Laplace.

The following theorem characterizes the Laplace random field.

Theorem 1. *A homogeneous Laplace random field, $Y(\boldsymbol{t})$, with zero mean and covariance function $\boldsymbol{R}_Y(\boldsymbol{t})$, admits the stochastic representation*

$$Y(\boldsymbol{t}) = \sqrt{R}X(\boldsymbol{t}), \tag{2}$$

where $X(\boldsymbol{t})$ is a homogeneous Gaussian random field with zero mean, unit variance, covariance function $\boldsymbol{R}_X(\boldsymbol{t}) = \boldsymbol{R}_Y(\boldsymbol{t})$, and R is an exponential random variable with mean 1, *independent of $X(\boldsymbol{t})$.*

We omit the proof of this theorem.

For a random field $Z(\mathbf{t})$, let $\dot{\mathbf{Z}}(\mathbf{t}) = (\dot{Z}_1(\mathbf{t}), \ldots, \dot{Z}_D(\mathbf{t}))$, $\ddot{\mathbf{Z}}(\mathbf{t})$ be the gradient and the Hessian matrix of $Z(\mathbf{t})$ with respect to $\mathbf{t}$, respectively. Let $\ddot{\mathbf{Z}}_{D-1} = \ddot{\mathbf{Z}}_{D-1}(\mathbf{t})$ be $(D-1)\times(D-1)$ matrix of second order partial derivatives of $Z(\mathbf{t})$ with respect to $\mathbf{t}$. For a random field $Z(\mathbf{t})$, the *moduli of continuity* $\omega_Z(h)$, $\omega_{\dot{Z}_j}(h)$, $\omega_{\ddot{Z}_{ij}}(h)$ of Z, $\dot{Z}$ and $\ddot{Z}$ are defined by

$$\begin{aligned}
\omega^Z(h) &= \sup_{\|\mathbf{t}-\mathbf{s}\|<h} |Z(\mathbf{t}) - Z(\mathbf{s})|, \\
\omega^Z_j(h) &= \sup_{\|\mathbf{t}-\mathbf{s}\|<h} |\dot{Z}_j(\mathbf{t}) - \dot{Z}_j(\mathbf{s})|, \quad \text{for } j = 1,\ 2, \ldots,\ D, \\
\omega^Z_{ij}(h) &= \sup_{\|\mathbf{t}-\mathbf{s}\|<h} |\ddot{Z}_{ij}(\mathbf{t}) - \ddot{Z}_{ij}(\mathbf{s})|, \quad \text{for } i,j = 1,\ 2, \ldots,\ D.
\end{aligned}$$

We will assume that the Gaussian random field used in equation (2) is ergodic and satisfies the condition

$$\max_{i,j} E\{|\ddot{X}_{ij}(\mathbf{t}) - \ddot{X}_{ij}(\mathbf{0})|\} \leq C\|\mathbf{t}\|,$$

for some constant $C > 0$ and all $\mathbf{t}$ in some neighborhood of $\mathbf{0}$.

3 Expected Euler Characteristic

Consider a homogeneous random field $Z(\mathbf{t})$, $\mathbf{t} \in \mathcal{C}$, a compact subset of $\mathbb{R}^D$ with a twice differentiable boundary $\partial\mathcal{C}$. Adler (1981) defines the differential topology characteristic of $A_u(Z, \mathcal{C})$ as

$$\chi(A_u(Z,\mathcal{C})) = (-1)^{D-1} \sum_{j=0}^{D-1} (-1)^j \chi_j(A_u(Z,\mathcal{C})),$$

where $\chi_j(A_u(Z,\mathcal{C}))$ is the number of points $\mathbf{t} \in \mathcal{C}$ such that: (i) $Z(\mathbf{t}) = u$, (ii) $\dot{Z}_1(\mathbf{t}) = 0, \ldots, \dot{Z}_{D-1}(\mathbf{t}) = 0$, (iii) $\dot{Z}_D(\mathbf{t}) > 0$ and (iv) the index of $\ddot{\mathbf{Z}}_{D-1}(\mathbf{t})$ is j. According to Adler (1981), provided that $A_u(Z,\mathcal{C}) \bigcap \partial\mathcal{C} = \emptyset$, it can be shown that $\chi(A_u(Z,\mathcal{C}))$ is equivalent to the Euler-Poincare characteristic of $A_u(Z,\mathcal{C})$ which counts the number of connected components minus the number of holes. As the level z gets large, the excursion set has simpler topology, and we are left with isolated connected components. According to Hasofer (1978), the following approximation is accurate

$$P\{\sup_{\mathbf{t}\in\mathcal{C}} Z(\mathbf{t}) \geq u\} \approx E\{\chi(A_u(Z,\mathcal{C}))\},$$

as $u \to \infty$. The mean value of $\chi(A_u(Z,\mathcal{C})$ is given by Theorem 5.2.1 of Adler (1981) which will be reported here.

Theorem 2. *Assume that $\ddot{Y}_{ij}$'s have finite variances, $f^Y(.)$, the joint density of $(Y,\ \dot{\boldsymbol{Y}},\ \ddot{\boldsymbol{Y}})$, is continuous in its arguments, and the conditional density of $(Y,\dot{Y}_1,\ldots,\dot{Y}_{D-1})$, given $\dot{Y}_D$ and $\ddot{Y}_{ij}$, $1 \le i \le D$, $1 \le j \le D-1$, is bounded above. Let $\mathcal{C}$ be a convex subset of $\mathbb{R}^D$, and*

$$P\{\max_{i,j}\{\omega_i^Y(h),\ \omega_{i,j}^Y(h)\} > \epsilon\} \quad = \quad o(h^D), \quad \text{as } h \downarrow 0,$$

for every $\epsilon > 0$. Then

$$E\{\chi(A_u(Y,\mathcal{C}))\} = \lambda(\mathcal{C})(-1)^{D-1}\int_0^\infty\int_{\mathbb{R}^{\frac{D(D-1)}{2}}} \dot{y}_D^+ \det(\ddot{\boldsymbol{y}}_{D-1}) f^Y(u,0,\ldots,\dot{y}_D,\ddot{\boldsymbol{y}}_{D-1}) d\ddot{\boldsymbol{y}}_{D-1} d\dot{y}_D \tag{3}$$

When $Y(\mathbf{t})$ is Gaussian, Adler (1981) gives the following expression for $E\{\chi(A_u(Y,\mathcal{C}))\}$

$$E\{\chi(A_u(Y,\mathcal{C}))\} \quad = \quad \lambda(\mathcal{C})\frac{\exp(-\frac{u^2}{2})\det(\Lambda)^{\frac{1}{2}}}{(2\pi)^{(D+1)/2}}H_{D-1}(u), \tag{4}$$

where

$$H_n(x) = n!\sum_{j=0}^{[n/2]}\frac{(-1)^j x^{n-2j}}{j!(n-2j)!2^j} \tag{5}$$

is the n-th Hermite polynomial and $\Lambda = \mathrm{Var}(\dot{\mathbf{Y}}(\mathbf{0}))$. If $Y(\mathbf{t})$ is a Laplace random field, then, using total probability law and Fubni's theorem for interchanging the integration order, we can write $E\{\chi(A_u(Y,\mathcal{C}))\}$ as follows,

$$E\{\chi(A_u(Y,\mathcal{C}))\} =$$

$$\lambda(\mathcal{C})(-1)^{D-1}\times\int_0^\infty\int_0^\infty\int \dot{y}_D^+ \det(\ddot{\mathbf{y}}) f^Y(u,0,\ldots,\dot{y}_D,\ddot{\mathbf{y}}_{D-1}|R=r)\exp(-r)d\ddot{\mathbf{y}}_{D-1}d\dot{y}_D dr, \tag{6}$$

where the inner integration is taken over $\mathbb{R}^{\frac{D(D-1)}{2}}$. Since $(Y,\dot{\mathbf{Y}},\ddot{\mathbf{Y}}))|R \sim N_m(\mathbf{0},\ R\Sigma)$, $m = 1+D+D(D-1)/2$, and Σ is the covariance matrix of $(X,\dot{\mathbf{X}},\ddot{\mathbf{X}}))$, then

$$\begin{aligned} f^Y(u,0,\ldots,\dot{y}_D,\ddot{\mathbf{y}}_{D-1}|R=r) \quad &= \quad (2\pi)^{-\frac{m+1}{2}}r^{-\frac{m}{2}}\det(\Sigma)^{-\frac{1}{2}}\times \\ &\exp\left(-\frac{1}{2r}(u,0,\ldots,\dot{y}_D,\ddot{\mathbf{y}}_{D-1})^T\Sigma^{-1}(u,0,\ldots,\dot{y}_D,\ddot{\mathbf{y}}_{D-1})\right). \end{aligned}$$

Change the variables to $v_D = \dot{y}_D/\sqrt{R}$, $v_{ij} = \ddot{y}_{ij}/\sqrt{R}$, for $1 \le i < j \le D-1$, $i = 1,\ldots,D-1$, we can write (6) as

$$E\{\chi(A_u(Y,\mathcal{C}))\} \quad = \quad E\{E\{\chi(A_{u/\sqrt{R}}(X,\mathcal{C}))\}\}, \tag{7}$$

where the outer expectation is taken over R. Using (4) and (5), we can find a closed form expression for the right hand-side of (7) as

$$\begin{aligned}
E\{\chi(A_u(Y,\mathcal{C}))\} &= \int_0^\infty E\{\chi(A_{u/\sqrt{r}}(X,\mathcal{C}))\}\exp(-r)dr, \\
&= \frac{\lambda(\mathcal{C})\det(\Lambda)^{\frac{1}{2}}}{(2\pi)^{\frac{D+1}{2}}}\int_0^\infty \exp(-r-\frac{u^2}{2r})H_{D-1}(\frac{u}{\sqrt{r}})dr, \\
&= \frac{\lambda(\mathcal{C})\det(\Lambda)^{\frac{1}{2}}}{(2\pi)^{\frac{D+1}{2}}}\int_0^\infty \exp(-r-\frac{u^2}{2r})H_{D-1}(\frac{u}{\sqrt{r}})dr, \\
&= \frac{\lambda(\mathcal{C})\det(\Lambda)^{\frac{1}{2}}\Gamma(D)}{(2\pi)^{\frac{D+1}{2}}}\times \\
&\qquad \sum_{j=0}^{[\frac{D-1}{2}]}\frac{(-1)^j u^{D-1-2j}}{j!(D-1-2j)!2^j}\int_0^\infty r^{-\frac{D-1}{2}+j}\exp(-r-\frac{u^2}{2r})dr, \\
&= \frac{2^{\frac{D+1}{4}}\lambda(\mathcal{C})\det(\Lambda)^{\frac{1}{2}}\Gamma(D)}{(2\pi)^{\frac{D+1}{2}}}\sum_{j=0}^{[\frac{D-1}{2}]}\frac{(-1)^j u^{\frac{D}{2}-j+\frac{1}{2}}}{j!(D-1-2j)!2^{\frac{3j}{2}}}K_{\frac{D-1}{2}-j-1}(\sqrt{2}u),
\end{aligned} \tag{8}$$

4 Expected Number of Local Maxima

For a random field $Y(\mathbf{t})$, $\mathbf{t} \in \mathcal{C}$, let $\mathcal{M}(A_u(Y,\mathcal{C}))$ denote the number of local maxima of $Y(\mathbf{t}) \in \mathcal{C}$ above the level u. The following theorem, reported in Adler (1981), gives the expectation of $\mathcal{M}(A_u(Y,\mathcal{C}))$

$$E\{\mathcal{M}(A_u(Y,\mathcal{C}))\} = \lambda(\mathcal{C})\int_u^\infty\int_{\mathcal{D}}|\det(\ddot{\mathbf{y}})|f^Y(y,\mathbf{0},\ddot{\mathbf{y}})d\ddot{\mathbf{y}}dy, \tag{9}$$

where $\mathcal{D} \subset \mathbb{R}^{D(D+1)/2}$ for which $\ddot{\mathbf{y}}$ is negative definite. By conditioning on R, using Fubuni's theorem and then changing the variables, it is easy to show that for the Laplace random field $Y(\mathbf{t})$,

$$E\{\mathcal{M}(A_u(Y,\mathcal{C}))\} = E\{E\{\mathcal{M}(A_{u/\sqrt{R}}(X,\mathcal{C}))\}\},$$

where the outer expectation is taken over R. For smooth Gaussian random field, $X(\mathbf{t})$, Adler (1981) gives the following asymptotic formula as $u \to \infty$:

$$E\{\mathcal{M}(A_u(X,\mathcal{C}))\} = \frac{\lambda(\mathcal{C})\det(\Lambda)^{\frac{1}{2}}u^{D-1}}{(2\pi)^{\frac{D+1}{2}}}\exp(-\frac{u^2}{2})\left(1+\mathrm{O}(\frac{1}{u})\right) \tag{10}$$

In general, it is difficult to simplify the inner multiple integral in (9). We still can approximate $E\{\mathcal{M}(A_u(Y,\mathcal{C}))\}$, when $Y(\mathbf{t})$ is the Laplace random field, by approximating the value of $P\{\sup_{\mathbf{t}\in\mathcal{C}} Y(\mathbf{t}) \geq u\}$. By conditioning on R and splitting the integral we get

$$P\{\sup_{\mathbf{t}\in\mathcal{C}} Y(\mathbf{t}) \geq u\} = \int_0^{u^k} P\{\sup_{\mathbf{t}\in\mathcal{C}} X(\mathbf{t}) \geq \frac{u}{\sqrt{r}}\}\exp(-r)dr + \int_{u^k}^{\infty} P\{\sup_{\mathbf{t}\in\mathcal{C}} X(\mathbf{t}) \geq \frac{u}{\sqrt{r}}\}\exp(-r)dr,$$

where $k \geq 2$. The first integral in the last equation is the dominating one since the second term is dominated by the $\exp(-u^k)$, which is very small as $u \to \infty$. This yields the following approximation

$$P\{\sup_{\mathbf{t}\in\mathcal{C}} Y(\mathbf{t}) \geq u\} \quad \approx \quad \int_0^{u^k} P\{\sup_{\mathbf{t}\in\mathcal{C}} X(\mathbf{t}) \geq \frac{u}{\sqrt{r}}\}\exp(-r)dr.$$

Since u^k is large we can replace the integrand in the last equation by the right hand side of (10) to get

$$P\{\sup_{\mathbf{t}\in\mathcal{C}} Y(\mathbf{t}) \geq u\} \quad \approx \quad \frac{\lambda(\mathcal{C})\det(\Lambda)^{\frac{1}{2}}u^{D-1}}{(2\pi)^{\frac{D+1}{2}}}\int_0^{u^k} r^{-\frac{D-1}{2}}\exp(-r-\frac{u^2}{2r})dr$$

If u^k is large, then

$$\int_0^{u^k} r^{-\frac{D-1}{2}}\exp(-r-\frac{u^2}{2r})dr \approx 2(\frac{2}{\sqrt{2}u})^{\frac{D-3}{2}}K_{\frac{D-3}{2}}(\sqrt{2}u).$$

So

$$P\{\sup_{\mathbf{t}\in\mathcal{C}} Y(\mathbf{t}) \geq u\} \approx \frac{\lambda(\mathcal{C})\det(\Lambda)^{\frac{1}{2}}2^{\frac{D+1}{4}}u^{\frac{D+1}{2}}}{(2\pi)^{\frac{D+1}{2}}}K_{\frac{D-3}{2}}(\sqrt{2}u).$$

Since $E\{\mathcal{M}(A_u(Y,\mathcal{C}))\}$ approximates $P\{\sup_{\mathbf{t}\in\mathcal{C}} Y(\mathbf{t}) > u\}$ we get the following approximation

$$E\{\mathcal{M}(A_u(Y,\mathcal{C}))\} \approx \frac{\lambda(\mathcal{C})\det(\Lambda)^{\frac{1}{2}}2^{\frac{D+1}{4}}u^{\frac{D+1}{2}}}{(2\pi)^{\frac{D+1}{2}}}K_{\frac{D-3}{2}}(\sqrt{2}u). \tag{11}$$

This approximation is simpler than that given by (8).

5 Example

To check the validity of the approximations derived in the previous sections, we consider the Laplace random field ($D = 1$), when $X(t) = Z_1\sin(t) + Z_2\cos(t)$, $t \in [0, 2\pi]$ and Z_1 and Z_2 are standard normal random variables. The process $X(t)$ can be written as $X(t) = R\cos(t-\theta)$, where $R = \sqrt{Z_1^2 + Z_2^2}$ and θ is uniformly distributed in $(0, 2\pi)$. It is straightforward to see that R has the density

$$f(r) = r\exp(-\frac{r^2}{2}), \qquad r > 0.$$

and $\det(\Lambda) = 1$. This yields $V = \sup_{[0,2\pi]} Y(t) = \sqrt{Z}R$. So the density V is given by mixing over Z as follows

$$\begin{aligned} P\{V > u\} &= \int_0^\infty P\{R > \frac{u}{\sqrt{z}}\} \exp(-z) dz, \\ &= 2^{\frac{1}{2}} u K_1(\sqrt{2}u). \end{aligned}$$

For $D = 1$ and $\lambda(\mathcal{C}) = 2\pi$, (8) gives

$$E\{\chi(A_u(Y, \mathcal{C}))\} = 2^{\frac{1}{2}} u K_1(\sqrt{2}u)$$

while (11) gives

$$E\{\mathcal{M}(A_u(Y, \mathcal{C}))\} \approx 2^{\frac{1}{2}} u K_1(\sqrt{2}u)$$

Note that the approximations yield the same value for the one-dimensional case.

6 Mean Size of One Component

In this section, we derive an approximation to S, the mean size (Lebesgue measure) of one connected component of $A_u(Y, \mathcal{C})$ when u is large. The excursion set of the Laplace random field can be modelled as *mosaic process*, i.e., the connected components are random sets thrown at random according to a Poisson point process in $\mathcal{C}$. This way of modelling the excursion set is called the *Poisson clumping heuristic* (Aldous, 1989). By using the Poisson clumping heuristic, we get an approximation to the mean value of the size of one connected component as follows

$$E\{S\} \approx \frac{\lambda(\mathcal{C})(1 - F_Y(u))}{E\{\chi(A_u(Y, \mathcal{C}))\}}, \tag{12}$$

where

$$\begin{aligned} F_Y(y) &= \frac{1}{\sqrt{2}} \int_{-\infty}^{u} \exp(-\sqrt{2}|y|) dy, \\ &= 1 - \frac{1}{2} \exp(-\sqrt{2}u), \quad \text{for} \quad u > 0. \end{aligned} \tag{13}$$

Using (8), (12) and (13) we can get an approximation to $E\{S\}$.

7 Simulation of $Y(t)$

A Laplace random field can be simulated by simulating a multivariate Laplace distribution on a grid of $\mathcal{C}$. Here we propose more a simpler method. A Laplace random field, given by equation (2), can be simulated as follows:

- Simulate a Gaussian random field $X(\mathbf{t})$ with zero mean and variance equal to 1.
- Simulate exp(1) random variable R.
- $Y(\mathbf{t}) = \sqrt{R}X(\mathbf{t})$ is a realization of $Y(\mathbf{t})$.

8 Summary

In this paper, we introduced a new random field called the Laplace random field. This field has a motivation for detecting a breast cancer using mammogram images. We gave a theorem which characterizes this field. Then we derived a closed form for the expected Euler characteristic of its excursion set. This expectation is used to approximate the tail distribution of the supremum of the Laplace random field. We also derived an approximation to the mean of the size of one connected component. The results derived here can be extended to another random field, i.e., by replacing the random variable R by another one. For example, if we replace it by $\sqrt{\frac{\nu}{S}}$, where S is a chi square random variable with ν degrees of freedom, then we will have the student random field which is a generalization to the Gaussian random field. The example shows that the approximation works well.

Acknowledgements

The authors thank the two referees for their comments which helped us to improve the manuscript.This research is supported by a grant from Yarmouk University.

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