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A Class of Minimum Variance Unbiased Estimators for Two-Parameter Pareto Distribution

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Abstract

In this paper, we consider a general class of UMVUE for functions of two parameters of the two-parameter Pareto distribution. We also obtain the variance of the UMVUE.

Keywords and Phrases: UMVUE; CV; Inverse Laplace Transform.

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1 Introduction

A two-parameter Pareto distribution is given by

$$f(x;\theta,\lambda) = \frac{\theta \cdot \lambda^{\theta}}{x^{\theta+1}},\ x > \lambda > 0,\ \theta > 0. \tag{1}$$

It is well-known that the Pareto distribution has been found to be suitable for approximating the right tails of distributions with positive skewness. It has been found to adapt to several socio-economic, physical, and biological phenomena. Johnson and Kotz (1970) have given a brief description of most of the research work in this area. Likes (1969) derived the uniform minimum variance unbiased estimator (UMVUE) of

†I take this opportunity to congratulate Professor Mir Masoom Ali on his 70th birthday anniversary. I am indeed very pleased to have been able to collaborate with him in many joint research work over more than a decade. I am also grateful to him for giving me the opportunity to go to Ball State University and work with him as a visiting professor during the 1993-94 academic year.

two parameters in two-parameter Pareto distribution and subsequently a simplified derivation and its variance for the UMVUE of the two parameters were derived by Baxter (1980). Saksena and Johnson (1984) showed that the MLE are jointly complete and obtained the best unbiased estimators of two parameters and their variances.

Here we consider a general class of UMVUE for functions of two parameters in the two-parameter Pareto distribution and also obtain the variance of the UMVUE. We then show that the results of Baxter (1980) and Saksena and Johnson (1984) are special cases of ours.

2 A Class of UMVUE

Let $X_1, X_2, \ldots, X_n$ be independent random variables having the density (1) and $X_{(1)}$, $X_{(2)}, \ldots, X_{(n)}$ be the corresponding order statistics. The following Lemma 1 is due to Epstein and Sobel (1954).k

Lemma 1. *Let $X_{(1)} = \min(X_1, X_2, \ldots, X_n)$ and $A \equiv n/\sum_{i=1}^{n} \ln(X_i/X_{(1)})$. Then $X_{(1)}$ and A are independent random variables with $X_{(1)}$ having a Pareto distribution with parameters $(\lambda, n\theta)$ and $T \equiv 2n\theta/A$ having a χ^2-distribution with $n-2$ degrees of freedom.*

It is well-known that the statistics $(X_{(1)}, A)$ are complete sufficient statistics for (λ, θ) (see Saksena and Johnson (1984)). Likes (1969) derived the UMVUE of λ and θ in terms of $X_{(1)}$ and A. A simplified proof of Lemma 1 with a compact derivation of the UMVUE was given by Baxter (1980). An alternative proof of Lemma 1 was also given by Malik (1970).

Theorem 3. *For any real number a, b, c, and d, and any integer $n > 1$ such that $n - b - c - 1 > 0$ and $n\theta > a$, the UMVUE of $\lambda^a \cdot \theta^c \cdot (\theta - d)^b$ in the two-parameter Pareto distribution with the density (1) is*

$$U \equiv \frac{\Gamma(n-1)}{\Gamma(n-b-c-1)} X_{(1)}^a (A/n)^{b+c} [F(-b; n-b-c-1; \frac{dn}{A}) - \frac{a}{(n-b-c-1)A} F(-b; n-b-c; \frac{dn}{A})],$$

here $F(a; b; x)$ is the Kummer's function and A is as defined in Lemma 1.
Proof: Since the statistics $(X_{(1)}, A)$ are complete sufficient statistics for (λ, θ), it is sufficient for us to find an unbiased estimator of $\lambda^a \cdot \theta^c \cdot\ (\theta - d)^b$ by using a statistic $X_{(1)}^a \cdot g(A; \theta)$ for a real-valued function $g(x)$. Since $X_{(1)}$ and A are independent random variables with $X_{(1)}$ having a Pareto$(\lambda, n\theta)$ and $T = 2\theta n/A$ having a chi-square distribution with $2n - 2$ degrees of freedom, the function g is determined by the following condition

$$E[X_{(1)}^a \cdot g(A; \theta)] = \frac{n\theta\lambda^a}{n\theta - a} E[g(A; \theta)] = \lambda^a \theta^c (\theta - d)^b. \tag{2}$$

Since $T = 2n\theta/A$ follows a chi-square distribution with $2n-2$ degrees of freedom

$$E[g(A;\theta)] = \frac{\theta^{n-1}}{\Gamma(n-1)} \int_0^\infty g(\frac{n}{t};\theta) t^{n-2} e^{-\theta t} dt, \text{ from (2).}$$

From the formula 4.18 of inverse Laplace transform in Oberhettinger and Badii (1973), we can obtain the following.

$$\begin{aligned} g(\frac{n}{t};\theta)\cdot t^{n-2} &= (n-2)![\frac{1}{\Gamma(n-b-c-1)} t^{n-b-c-2}\cdot F(-b;n-b-c-1;d\cdot t) \\ &\quad -\frac{a}{n}\frac{1}{\Gamma(n-b-c)} t^{n-b-c-1}\cdot F(-b;n-b-c;d\cdot t)] \text{ for } n-b-c>1. \end{aligned}$$

Hence, the proof is complete.

Next we derive the variance of the UMVUE of $\lambda^a\cdot\theta^c\cdot(\theta-d)^b$.

Theorem 4. *For any real numbers $a, b, c,$ and $d > 0$, and any integer $n > 1$ such that $n-2b-2c-1 > 0$, $n-b-c-1 > 0$ and $n\cdot\theta > 2a$, $\theta > 2d$, the variance of the UMVUE of $\lambda^a\cdot\theta^c\cdot(\theta-d)^b$ is given by*

$$\begin{aligned} Var(U) &= \lambda^{2a}\theta^{2c}[\frac{n\Gamma(n-1)\theta^{2b+1}\Gamma(n-2b-2c-1)}{\Gamma^2(n-b-c-1)(\theta n-2a)}(F(n-2b-2c-1;-b,-b;n-b-c-1;\frac{d}{\theta},\frac{d}{\theta}) \\ &\quad +\frac{a^2(n-2b-2c)(n-2b-2c-1)}{n^2(n-b-c-1)^2\theta^2}F(n-2b-2c+1;-b,-b;n-b-c,n-b-c;\frac{d}{\theta},\frac{d}{\theta}) \\ &\quad -\frac{2a(n-2b-2c-1)}{n(n-b-c-1)\theta}F(n-2b-2c;-b.-b;n-b-c-1,n-b-c;\frac{d}{\theta},\frac{d}{\theta}))-(\theta-d)^{2b}], \end{aligned}$$

where $F(z;p,q;r,s;u,w)$ is a generalized hypergeometric function of several variables in Oberhettinger and Badii (1973).

Proof: From relation between Kummer's function $F(a;b;c)$ and the Whittaker function $M_{a,b}(z)$ in Oberhettinger and Badii (1973), for $d > 0$, the UMVUE of Theorem 1 can be represented by the Whittaker's function as

$$\begin{aligned} U &= \frac{\Gamma(n-1)}{\Gamma(n-b-c-1)} d^{-(n-b-c)/2} X_{(1)}^a\cdot(A/n)^{(n+b+c)/2}\cdot e^{\frac{dn}{2A}} \\ &\quad \cdot[(\frac{dn}{A})^{1/2} M_{\lambda_1,\mu_1-\frac{1}{2}}(\frac{dn}{A}) - \frac{a}{(n-b-c-1)A}\cdot M_{\lambda_2,\mu_2-\frac{1}{2}}(\frac{dn}{A})], \end{aligned}$$

where $\lambda_1 = \frac{n+b-c-1}{2}$, $\lambda_2 = \frac{n+b-c}{2}$, $\mu_1 = \frac{n-b-c-1}{2}$, $\mu_2 = \frac{n-b-c}{2}$.

From the formula 7.622.3 in Gradshteyn and Ryzhik (1965) and Lemma 1, we can obtain $E(U^2)$ as follows.

$$
\begin{aligned}
E(U^2) &= \lambda^{2a}\theta^{2b+2c+1}\frac{n\Gamma(n-1)\Gamma(n-2b-2c-1)}{\Gamma^2(n-b-c-1)(\theta n-2a)}\cdot \\
&\quad (F(n-2b-2c-1;-b,-b;n-b-c-1,n-b-c-1;\frac{d}{\theta},\frac{d}{\theta}) \\
&\quad +\frac{a^2(n-2b-2c)(n-2b-2c-1)}{n^2(n-b-c-1)^2\theta^2}F(n-2b-2c+1;-b,-b;n-b-c,n-b-c;\frac{d}{\theta},\frac{d}{\theta}) \\
&\quad -\frac{2a(n-2b-2c-1)}{n(n-b-c-1)\theta}F(n-2b-2c;-b,-b;n-b-c-1,n-b-c;\frac{d}{\theta},\frac{d}{\theta})).
\end{aligned}
$$

Hence, we have the result.

Theorem 5. *For any real number a, b, c, and d, and $d < 0$, and any integer $n > 1$ such that $n-2b-2c-1 > 0$, $n-b-c-1 > 0$ and $n\theta > 2a$, $\theta > 2d$, the variance of the UMVUE of $\lambda^a\theta^c(\theta-d)^b$ is*

$$
\begin{aligned}
Var(U) &= \lambda^{2a}\theta^{2c}[\frac{n\Gamma(n-1)\theta^{n-2c}\Gamma(n-2b-2c-1)}{\Gamma^2(n-b-c-1)(\theta n-2a)(\theta-2d)^{n-2b-2c-1}} \\
&\quad (F(n-2b-2c-1;n-c-1,n-c-1;n-b-c-1,n-b-c-1;\frac{-d}{\theta-2d},\frac{-d}{\theta-2d}) \\
&\quad +\frac{a^2(n-2b-2c)(n-2b-2c-1)}{n^2(n-b-c-1)^2(\theta-2d)^2}F(n-2b-2c+1;n-c,n-c;n-b-c; \\
&\quad n-b-c;\frac{-d}{\theta-2d},\frac{-d}{\theta-2d})-\frac{2a(n-2b-2c-1)}{n(n-b-c-1)(\theta-2d)}F(n-2b-2c;n-c-1,n-c; \\
&\quad n-b-c-1,n-b-c;\frac{-d}{\theta-2d},\frac{-d}{\theta-2d}))-(\theta-d)^{2b}].
\end{aligned}
$$

Proof: For $d < 0$, from the well-known relation between the Kummer's function F and the Whittaker's function in Oberhettinger and Badii (1973), the UMVUE of Theorem 1 can be represented by

$$
\begin{aligned}
U &= \frac{\Gamma(n-1)}{\Gamma(n-b-c-1)}(-d)^{-(n-b-c)/2}\cdot X_{(1)}^a\cdot(A/n)^{(n+b+c)/2}\cdot e^{\frac{dn}{2A}} \\
&\quad \cdot[(-nd/A)^{1/2}M_{-\lambda_1,\mu_1-1/2}(-\frac{nd}{A})-\frac{aM_{-\lambda_2,\mu_2-1/2}(-nd/A)}{(n-b-c-1)A}],
\end{aligned}
$$

where $\lambda_1, \lambda_2, \mu_1$, and μ_2 are as defined earlier.

From the formula 7.622.3 in Gradshteyn and Ryzhik (1965) and Lemma 1, we can obtain the second moment of the UMVUE as follows.

$$
\begin{aligned}
E(U^2) \quad = \quad & \lambda^{2a}\theta^{n}\frac{n\Gamma(n-1)\Gamma(n-2b-2c-1)}{\Gamma^2(n-b-c-1)(\theta n-2a)(\theta-2d)^{n-2b-2c-1}} \\
& (F(n-2b-2c-1; n-c-1, n-c-1; n-b-c-1, n-b-c-1; \frac{-d}{\theta-2d}, \frac{-d}{\theta-2d}) \\
& + \frac{a^2(n-2b-2c)(n-2b-2c-1)}{n^2(n-b-c-1)^2(\theta-2d)^2} F(n-2b-2c-1; n-c, n-c; \\
& n-b-c, n-b-c; \frac{-d}{\theta-2d}, \frac{-d}{\theta-2d}) \\
& - \frac{2a(n-2b-2c-1)}{n(n-b-c-1)(\theta-2d)} F(n-2b-2c; n-c-1, n-c; \\
& n-b-c-1, n-b-c; \frac{-d}{\theta-2d}, \frac{-d}{\theta-2d})).
\end{aligned}
$$

Hence, we have the result.

From Theorems 1, 2, and 3 and properties of Kummer's function and generalized hypergeometric function in Grashteyn and Ryzhik (1965) and Oberhettinger and Badii (1973), we can derive the UMVUE and its variance for the special function of two parameters. Among the following, (1) and (2) are well-known UMVUE obtained by Baxter (1980) and Saksena and Johnson (1984).

(1) If $a = b = 0$ and $c = 1$, then $(n-2)A/n$ is the UMVUE of θ, which has variance $\frac{\theta^2}{n-3}$ for $n > 3$.

(2) If $a = -1, c = 1$, and $b = 0$, then $\frac{(n-2)A+1}{nX_{(1)}}$ is UMVUE of $\frac{\theta}{\lambda}$, which has the variance $\frac{\theta}{\lambda^2}\frac{n^2\theta^2+n-3}{n(n-3)(\theta n+2)}$ for $n > 3$.

(3) If $a = 1$ and $b = c = 0$, then $X_{(1)}(1 - \frac{1}{(n-1)A})$ is the UMVUE of λ which has variance $\frac{\lambda^2}{\theta(n-1)(\theta n-2)}$.

(4)If $a = b = 0$ and $c = -1$, then $\frac{n}{(n-1)A}$ is the UMVUE of $\frac{1}{\theta}$ which has the variance $\frac{1}{(n-1)\theta^2}$.

(5) If $a = -1$ and $b = c = 0$, then $\frac{1}{X_{(1)}}(1 + \frac{1}{(n-1)A})$ is the UMVUE of $\frac{1}{\lambda}$ which has the variance $\frac{1}{(n-1)(\theta n+2)\theta\lambda^2}$.

(6) If $a = b = 0$ and $c = -r$, then $\frac{\Gamma(n-1)}{\Gamma(n+r-1)}(\frac{n}{A})^r$ is the UMVUE of $(\frac{1}{\theta})^r$, which has the variance $[(\Gamma(n-1)\Gamma(n+2r-1)/\Gamma^2(n+r-1)) - 1]/\theta^{2r}$.

(7) The following UMVUE estimators and their variances can be represented by Kummer's function and generalized hypergeometric functions:

(a) When $a = 0, d = 2, b = c = -\frac{1}{2}$, it is the coefficient of variation of the Pareto distribution $\theta^{-1/2}(\theta - 2)^{-1/2}$.

(b) When $a = 0, b = c = -1/2$, and $d = -2$, it is the coefficient of variation of a power function distribution: $\theta^{-1/2}(\theta + 2)^{-1/2}$.

(c) When $a = d = r$ and $b = -1$, $c = 1$, it is the rth moment of the Pareto distribution,

(d) When $a = d = -r$ and $b = -1, c = 1$, it is the rth moment of the Power function distribution.

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