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On the Theory of Inversion

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Abstract

Various authors have discussed both continuous and discrete inverted distributions and its applications to various disciplines. The applications of inverted distributions extend from social sciences to geological, engineering, environmental and medical sciences. This paper is an attempt to review various results, to report some new properties of the inverted distributions and to develop a theory of inversion.

Keywords and Phrases: Theory of Inversion, Inverse Continuous Distribution, Inverse Discrete Distribution, Recurrence Relation, Characterizations, Hyper Poisson Distribution.

AMS Classification: 62E10.

1 Introduction

The irreversible damage to manufacturing materials is generally caused by a damage process such as fatigue, creep, fracture, corrosion and wear. The probability models corresponding to the reciprocal transformation arise in these different types of stresses. A number of authors have studied the moments of reciprocals of random variables and negative moments of positive random variables [See Jones and Zhigljavsky (2004), Rempala (2004) and Royer (2003)]. Virtuosa and Vieira (2004) have recently discussed the creep, shrinkage, cracking and deformation of concrete flange on the basis of negative moments. Mendenhall and Lehman (1960) while investigating the properties of the maximum likelihood estimator of the scale parameter of the Weibull distribution from a sample censored at time T, observe that the mean square error of the estimator depends on the negative moments of the positive binomial variate.

Random quantities arising in modeling, simulation, data analysis, life testing, and in decision-making lead to various kinds of distributional problems and quest for solutions. The concept of a model is familiar in applied science works. The unique feature of a model is that it describes a measurement variable in mathematical terms. The probability density function or a cumulative distribution function serves as a statistical model of a random phenomenon. A vast majority of well-known distributions, discrete as well as continuous, has been derived analytically from other models independent of their relevance to particular phenomena.

The applications of inverted probability distributions extend from social sciences to biological, engineering, environmental and medical sciences. In continuous distributions, Baysians have extensively made use of the inverted probability functions as priors. The commonly used continuous inverted distributions are half normal, gamma, Weibull, inverse gaussian etc. In discrete distributions, almost all well known distributions have been either inverted or their negative moments have been obtained. These are binomial, Poisson, negative binomial, geometric, logarithmic, hyper Poisson etc.

In spite of the adequacy of these probability functions in many important areas, no attempt has been made in comprehensively compiling their statistical properties and characteristics. In this paper, an attempt has been made to compile the statistical properties of inverted distributions. Some new results are derived. These results may lead to the development of theory of inversion.

2 Inverse Distributions

If $g(x) = 1/X$, then it is described by inverted random variable. There are many inverted distributions discussed in literature. Stephan (1945) is one of the earlier authors who discuss negative moments of binomial and hyper geometric functions. Grab and Savage (1954) have constructed the tables for negative moments of the binomial and Poisson distributions. Mendenhall and Lehman (1960), Govendarajulu (1962, 1963), Tiku (1964), Vijsokousku (1966), Stancu (1968), Skibusky (1970), Kabe (1976) Shahnbhag and Busawa (1971), Chao and Strawderman (1972), Gupta (1974, 1984), Lepage (1978), Kumar and Cousul (1979), Cressie et al (1986), Cressie et al (1981), Ahmad and Sheikh (1981, 1983, 1984, 1987), Cressie and Borkent (1986), Jones (1987), are some of the early authors who have discussed various aspects of inverted distributions or negative moments. More recently, Roohi (2002), Jones and Zhiglijavsky (2004), Rempala (2004) and Ahmad and Roohi (2004a, 2004b, 2004c, 2004d), have also discussed both continuous and discrete inverted distributions.

3 Inverse Continuous Distributions

Inverted gamma distribution has been discussed first in literature, however, we start with a popular and extensively used inversion of normal probability distribution also known as Bernstein distribution and alpha distribution. It is applied in engineering and other sciences [See Druzhinin (1963), Kordonsky and Faridman (1976), Vysokovskii (1966, 1970) and Pronikov (1973a & 1973b)]. In several cases when identical machine parts (such as cams, splines, cutting tools, machine tools etc.) are tested to determine the wear on the contact surfaces, the wear process $W(x)$ obtained as a function of time X is linear as shown in Figure 1. In Figure 1 each realization (line) represents the wear on the surface of a single part. The random wear process in this case can be expressed by the random function $W(x)$, given by $W(x) = ax + b$, where a and b are random variables such that $a = \frac{dW(x)}{dx}$ = rate of wear and $b = W(o)$ = initial value of wear. Suppose that the machine part fails if $W(x) \geq W_\ell$, where W_ℓ is the permissible wear limit. The random variable X defining the life of the component is given by $X = (W_\ell - b)/a$, a > 0. If $a \sim N\left[\mu_a, \sigma_a^2\right]$, Ahmad and Sheikh (1983) have derived the distribution of X as a function of a^{-1} called inverted normal distribution and discussed its various properties. Because of its bimodality and problems at zero, the distribution has not been studied thoroughly, though many authors have found its various applications [See Ahmad and Sheikh (1981, 1984) and Druzhinin (1963)].

The probability density function, $f(x)$ of X is $B(C, \alpha)$ where $B(C, \alpha)$ is inverted normal distribution where α and $C > 0$ are parameters. [See Ahmad and Sheikh (1987), Gertshakh and Kordons (1969), Kordonsky and Faridman (1976)]. Some of the known properties of the density are (See Ahmad and Sheikh, 1984 and Sheikh and Ahmad, 1982, 1983).

(i) $f(x)$ is bimodal and modes are $= 2c\left[1 \pm \sqrt{1+8\alpha}\,\right]$

(ii) Median $= c$

(iii) Suppose X has $B(c, \alpha)$ inverted normal random variable, then

- (a) aX has a density given by $B(ac, \alpha)$, $a > 0$,
- (b) $Y = bX/(aX + d)$ has a density function given by $B\left(bc/d + ac, ad^2/\left(d + ac\right)^2\right)$,
- (c) $Y = aX_1\, X_2/(X_1 + b\, X_2)$ has a density function given by $B\left(ac/b + 1,\ b^2 + 1\big/(b+1)^2\right)$ when X_1 and X_2 are independent random variables each having inverted normal distribution,
- (d) $Y_n = \left(\frac{1}{n}\sum\limits_{i=1}^{n} . (1/X_i)\right)^{-1}$ has a density function given by $B\left(c, \alpha/n\right)$ provided X_i, $i = 1, 2, \cdots, n$ are independent inverted random variables where Y_n is harmonic mean of X,
- (e) $\frac{1}{2\alpha}\sum\limits_{i=1}^{n}\left(1 - c/x_i\right)^2$ has $\chi^2_{(n)}$ with n degrees of freedom and
- (f) $Y = \exp\left(1/X\right)$ has log-normal distribution.

The inverted distribution can further be explored and its applications traced in various disciplines.

Figure 5: Sample function of linear non-stationary random wear process.

4 Inverse Discrete Distributions

Early authors give approximate results for the negative moments of some discrete distributions, mainly binomial and Poisson truncated at the point zero along with bounds for the cumulative error. Chao and Strawderman (1972) using the probability generating function of the random variable $(X + A - 1) > 0$, give a technique of obtaining negative moments of the form $E\left[(X + A)^{-k}\right]$, and derive the first order negative moments of the binomial and Poisson distributions. Kumar and Consul (1979) obtain negative moments of the form $E\left[(X + A)^{-k}\right]$ for the Lagrangian binomial and Lagrangian Poisson distributions. Govindarajulu (1963) obtains a recurrence relation for the kth negative moment of the positive binomial distribution. Skibinsky (1970) and Shahnbhag and Busawa (1971) using a probability property obtain characterizations of the hyper-geometric and binomial distributions respectively. In the following sections, we find recurrence relations involving negative moments and obtain characterizations of some discrete distributions using these recurrence relations.

4.1 Recurrence Relations

In this section, a series of recurrence relations involving negative and negative factorial moments of some discrete distributions have been given [See Ahmad and Roohi (2004a,

2004b), and Roohi (2002)].
Theorem 1: Suppose X has binomial probability distribution with parameter n $(n = 1, 2, \cdots)$ and p $(0 \leq p < 1)$, then for $A > 1$, the following relation holds

$$p\,(A+n)\;\; E\left(\frac{1}{X+A}\right) = 1-(A-1)\;(1-p)\;\; E\left(\frac{1}{X+A-1}\right),\;\; n=1,2,\cdots$$

Theorem 2: Suppose X is a binomial random variable with parameters n and p, then

$$\mu'_{-[k]} = E\left[\prod_{i=1}^{k}(X+i)\right]^{-1} = 1-p\,(k!)^{-1}\;{}_2F_1\left[1,-n;k+1;-\frac{p}{1-p}\right],$$

$k = 1, 2, \cdots$, and $0 < p < 1$, where $\mu'_{-[k]}$ is the negative ascending factorial moment of X, $\Gamma_{-[k]} = E\,[X\,(X+1)\ldots.(X+k-1)]^{-1}$.
Theorem 3: Suppose X has a binomial probability distribution with parameters n and p. Suppose $\mu'_{-[k]}$ is the kth negative ascending factorial moment of X. Then the relation

$$(k+1+n)\;pk\;\mu'_{-[k+1]} = [k\,(1-p)+np]\;\mu'_{-[k]} - \mu'_{-[k-1]}$$

holds, for k = 2, 3, 4, ... and 0 < p < 1.
Theorem 4: Suppose X is negative binomially distributed random variable with parameters r and p. Then

$$\mathrm{E(X\;+\;A)}^{-1} \;=\; p^r\;A^{-1}\;{}_2F_1\;[A,\;r;\;A+1;\,q]\,,$$

where $A > 0,\; q \;=\; 1-p$ and ${}_2F_1\,[a,b;c;z]$ is the hypergeometric series function.
Theorem 5: Suppose X is negative binomially distributed with parameters r and p. Then the relation

$$\mathrm{q\,(A\text{-}r)\;\;E(X\;+\;A)}^{-1} \;=\; (A-1)\,\mathrm{E\,(X+A\text{-}1)}^{-1}-p,$$ holds for A>1 and A≠r.

Theorem 6: Suppose X has a negative binomial probability distribution with parameters r and p. Suppose $\mu'_{-[k]}$ is the kth negative factorial moment of X. Then the relation

$$(k+1-r)\,qk\;\mu'_{-[k+1]} \;=\; [k\,(q-p)-qr]\;\;\mu'_{-[k]}\;+\;p\,\mu'_{-[k-1]},\; \mathrm{k}>1$$

holds.
Theorem 7: Suppose the random variable X has a logarithmic series distribution with parameter θ and probability function

$$P\,(X=x) = \frac{\theta^{x-1}}{x}\,[\theta\ln(1-\theta)]^{-1}\,,\; x=1,2,...,$$

then $A^2\theta\;E\left[(X+A)^{-1}\right] = A\,(A-1)\,E\left[(X+A-1)^{-1}\right]-A\,(1-\theta)+P_1$ holds where $P_1 = P\,(X=1)\,, 0<\theta<1$ and $A>1$.

Theorem 8: Suppose the random variable X has a logarithmic series distribution with parameter θ and $\mu'_{-[k]}$ is the kth negative ascending factorial moment of X. Then the relation

$$(k+1)^2 \theta \mu'_{-[k+1]} = [(k+1)(\theta-1) + k\theta] \mu'_{-[k]} + (1-\theta) \mu'_{-[k-1]}, \quad 0 < \theta < 1, \ for \ k = 2, 3, ...$$

holds

Theorem 9: Suppose the random variable X has a hyper-Poisson distribution with parameters θ and λ as:

$$P(X = x) = C_{\theta,\lambda} \frac{\theta^x}{\lambda^{[x]}}, x = 0, 1, ...; \theta > 0, \ \lambda > 0,$$

where $C_{\theta,\lambda} = \{{}_1F_1[1; \lambda; \theta]\}^{-1}$, $\lambda^{[0]} = 1$ and $\lambda^{[x]} = \lambda(\lambda+1) ... (\lambda+x-1)$. then $\theta E(X+A)^{-1} = 1 + \frac{1-\lambda}{A-1} P_0 + (\lambda - A) \ E(X+A-1)^{-1}$ for A > 1 and $P_0 = P(X = 0)$.

4.2 Characterizations

Characterization based on negative moments of some discrete distributions have been given below (Ahmad and Roohi, 2004 b,c):

Theorem 10: X has a binomial probability function P_x(n, p) if and only if for A > 1 and $p(n-x) P_x - (1-p)(x+1) P_{x+1} \geq 0$

$$p(A+n) E\left[(X+A)^{-1}\right] = 1 - (A-1)(1-p) \ E\left[(X+A-1)^{-1}\right], \quad 0 \leq p \leq 1 \text{ and x = 0, 1, 2, 3, ... n},$$

holds.

Theorem 11: X has a negative binomial probability function P_x(r, p) if and only if for A>1 and $P_{x+1} > \frac{q(x+r)}{x+1}$,

$$(A-r) qE\left[(X+A)^{-1}\right] = (A-1) E\left[(X+A-1)^{-1}\right] - p \ \text{ holds},$$

where $0 \leq p \leq 1$, q=1-p and x = 0, 1, 2, ...

Theorem 12: X has a logarithmic probability function P_x with parameter θ if and only if for A > 1 and $(1+x) P_{x+1} - x\theta P_x \geq 0$,

$$A\theta \ E\left[(X+A)^{-1}\right] = (A-1) E\left[(X+A-1)^{-1}\right] - 1 + \theta + \frac{1}{A} P_1$$

holds for $0 < \theta < 1$, x = 1, 2, 3, ... and P_1= P(X=1).

Theorem 13: X has a hyper-Poisson probability function $P_x(\theta, \lambda)$ if and only if for A > 1 and $\theta P_x - (x+\lambda) P_{x+1} \geq 0, \ \theta, \lambda > 0$

$$E(X+A)^{-1} = \frac{1}{\theta}\left\{1 - \frac{1-\lambda}{1-A} P_0 + (\lambda - A) \ E(X+A-1)^{-1}\right\}$$

holds, where x = 0, 1, 2, ...; P_0 = P(X = 0), $\theta > 0$ and $\lambda > 0$,

5 Conway-Maxwell-Hyper Poisson Distribution

Shmueli et al (2005) have generalized one-parameter Poisson distribution to a two parameters distribution called Conway-Maxwell-Poisson (CMP) distribution, discussed some of its properties and have fitted the CMP distribution to non-Poisson count data. In this section, we have found, a natural extension of two-parameter CMP to three-parameter distribution, which may be called Conway-Maxwell-Hyper-Poisson distribution (CMHP). Hyper-Poisson distribution has been in literature for some time. It is a natural extension of the Poisson distribution providing information on super-Poisson, and sub-Poisson depending on one of its parameters (Bardwell and Crow, 1964).

Let the CMP distribution be shifted to $X = a$ and let $y = x - a$, then

$$P\left(Y = y\right) = \frac{1}{Z\left(\lambda, \nu, a\right)} . \frac{\lambda^{y+a}}{\left[(y+a)!\right]^{\nu}}, \quad y = 0, 1, 2, ..., \ a \geq 0, \ \nu \geq 0 \, and \ \ \lambda > 0, \qquad (1)$$

where $Z\left(\lambda, \nu, a\right) \quad = \left[\sum_{i=0}^{\infty} \frac{\lambda^{i+a}}{\left[(i+a)!\right]^{\nu}}\right]$. $Z\left(\lambda, \nu, a\right)$ converges for any $\lambda > 0$ and $\nu > 0$ except for $\nu = 0$ and $\lambda \geq 1$. The equation holds for any $a \geq 0$. If $a = 0$, P(Y = y) is CMP distribution. $P\left(Y = y\right)$ is also true for any positive real value of a. Further, it may be noted that a truncated CMHP is again CMHP. Following Shmueli et al (2005) procedures, we have:

$$E\left(Y + a\right)^{r+1} = \begin{cases} \lambda \, E\left(Y + a - 1\right)^{1-\nu} + \frac{a\,\lambda^{a-1}}{Z(\lambda,\nu,a)\,(a!)^{\nu}}, r = 0, \\ \lambda \, \frac{d}{d\lambda} E\left(Y + a\right)^{r} + E\left(Y + a\right)^{r} \, E\left(Y + a\right), r > 0. \end{cases}$$

We may also find the rth negative moment of Y + A, $A \quad > 0$ of CMHP

$$E\left(Y + A\right)^{-r} \quad = \frac{1}{Z\left(\lambda, \, \nu, \, a\right)} \sum_{y=0}^{\infty} \frac{1}{\left(y + a\right)^{r}} \, \frac{\lambda^{y+a}}{\left[(y+a)!\right]^{\nu}}$$

Following Shmueli et al (2005) procedure, we have

$$E\left(Y + A\right)^{-r} \quad = \left[E\left(Y + A\right)^{1-r} - \lambda \frac{d}{d\lambda} \, E\left(Y + A\right)^{-r}\right] \left[E\left(Y + A\right)\right]^{-1}, \, r = 1, 2, ...$$

If $a = 0$, then we have rth negative moment of CMP distribution.

5.1 Characterization of CMHP and CMP

Theorem 14: Y is a CMHP random variable if and only if

$$\frac{P\left(Y = y\right)}{P\left(Y = y - 1\right)} = \lambda / (y + a)^{\nu}, \nu, \, a \geq 0 \ \ and \ \ \lambda > 0 \qquad (2)$$

Proof: Suppose Y is CMHP random variable. The relation (2) is trivial.
Suppose the relation (2) holds. Let $P_y = P(Y = y)$. If the proportion (2) holds, then $P_y = \frac{\lambda}{(y+a)^\nu} P_{y-1}$. For y = 1, 2, …, we have

$$P_y = \frac{\lambda^y}{[(a+1)...(a+y)]^\nu} P_0 = \frac{\lambda^y \Gamma^\nu(a)}{\Gamma^\nu(y+a)} P_0.$$

Since $\sum P_y = 1$ then $P_0 = \left(\Gamma^\nu(a) \sum_{y=0}^{\infty} \frac{\lambda^y}{\Gamma^\nu(y+a)} \right)^{-1}$.
On simplification, we get (1).
If a = 0, Y is a CMP random variable. The characterization theorem holds for CMP and under conditions stated by Shmueli et al (2005), the characterization holds for ordinary Poisson, Bernoulli and geometric distributions.

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