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Erlang Mixture of Normal Moment Distribution

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Abstract

In this paper, Erlang mixture of normal moment distribution has been defined. Then different moments, characteristic function, shape characteristics and estimators of the parameters have been provided. Computer Program has been developed to compute the ordinates of the distribution for different values of the parameters, and the corresponding curves have been presented

Keywords and Phrases: Mixture of distribution, Erlang distribution, normal moment distribution, characteristic function.

AMS Classification: $74F_{xx}$, $60E_{xx}$, $78M_{xx}$, 60E10.

1 Introduction

A mixture of distributions is nothing but a weighted average of probability distribution having total positive weights equal to one Blischke (1963). The distributions that mixed are called the component of the mixture. The weight themselves include a probability distribution called the mixing distribution. Thus a mixture is a probability distribution as a property of weights.

Probability distribution of this kind arises when observed phenomenon can be the consequence of two or more related but usually unobserved phenomena, each of which leads to a different probability distribution. Mixtures and related structures often occur in the construction of probabilistic model like factor analysis model.

The components of a mixture may be discrete or continuous or both discrete and continuous. Thus mixtures are classified in accordance with the numbers of their components as finite or uncountably infinite.

Pearson (1894) was the pioneer in the field of mixture of distributions who considered the mixture of two normal distributions. After a long gap Robbins (1948) studied some basic properties of mixture distributions. Rider (1961)and. Wasilewiski (1967) studied the problems of estimation for mixtures of two exponential distributions, of two binomial distributions and of two generalized gamma distributions.

Jewell (1982) has discussed non-parametric mixtures of exponential and Weibull distributions. Roy et al (1992,93,94,2003) considered Poisson, binomial, negative binomial and Chi-square mixtures on some standard distributions. In the light of these distributions, we have defined Erlang mixture of normal moment distribution and then different characteristics of the distribution have been presented.

A random variable X is said to have a normal moment distribution if its density function is defined by

$$g\left(x;k\right)=\frac{x^{2k}e^{-\frac{1}{2}x^{2}}}{2^{k+\frac{1}{2}}\Gamma\left(k+\frac{1}{2}\right)};\qquad -\infty\ < x <\ \infty\ , \tag{1}$$

where k is an integer.

The name 'normal moment distribution' is due to the fact that the form of the distribution is a density proportional to the factor

$$x^{2k}e^{-\frac{1}{2}x^{2}}.$$

The main characteristics of the distribution (1) are Mean=0, Variance=$2k+1$, Skewness=0 and Kurtosis=$\frac{(2k+3)}{(2k+1)}$.

Note that, when $k=0$, the distribution (1) becomes a standard normal distribution. The distribution is also symmetrical and bimodal. As the value of k increases, the modes of the distribution shift to both sides of the mean.

2 Preliminaries

Mixtures mostly occur when the parameter θ of a family of distributions, given by the density function $f\left(x;\theta\right)$, is itself subjected to the chance variation. The mixing distribution, say, $g\left(\theta\right)$ is then a probability distribution on the parameter of the distribution.

The general formula for the finite mixture is

$$\sum_{i=1}^{k} f\left(x;\theta_i\right)g\left(\theta_i\right). \tag{2}$$

and the infinite analogue in which g is a density function is as follows

$$\int f\left(x;\theta\right)g\left(\theta\right)d\theta. \tag{3}$$

Roy et al. (1992,1993,1994 and 2003) defined Poisson, binomial, negative binomial and chi-square mixtures of distributions as follows.

Definition 2.1. A random variable X is said to have a Poisson mixture of distributions if its density function is given by

$$\sum_{r=0}^{\infty} \frac{e^{-\theta}\theta^r}{r!} g\left(x; r, \theta\right). \tag{4}$$

Definition 2.2. A random variable X is said to have a binomial mixture of distributions if its density function is given by

$$\sum_{r=0}^{N} \binom{N}{r} p^r \left(1-p\right)^{N-r} g\left(x; r, \theta\right). \tag{5}$$

Definition 2.3. A random variable X is said to have a negative binomial mixture of distributions if its density function is given by

$$\sum_{k=0}^{n} \binom{k+r-1}{r-1} p^r q^k g\left(x; r, \theta\right). \tag{6}$$

Definition 2.4. A random variable X is said to have a negative binomial mixture of normal moment distribution if its density function is given by

$$\sum_{k=0}^{\infty} \binom{k+r-1}{r-1} p^r q^k \frac{e^{-\frac{1}{2}x^2} x^{2k}}{2^{k+\frac{1}{2}} \Gamma\left(k+\frac{1}{2}\right)} g\left(x; r, \theta\right); \quad -\infty < x < \infty. \tag{7}$$

Definition 2.5. A random variable X is said to have a chi-square mixture of distributions if its density function is given by

$$\int_{0}^{\infty} \frac{e^{-\frac{1}{2}\chi^2} \left(\chi^2\right)^{\frac{k}{2}-1}}{2^{\frac{k}{2}} \Gamma\left(\frac{k}{2}\right)} g\left(x; r, \theta\right), \tag{8}$$

where $g\left(x; r, \theta\right)$ is a density function.

3 Main Results

In this paper, we first define the Erlang mixture of distributions and Erlang mixture of normal moment distributions. The main results of this study are furnished in the form of some definitions and theorems.

Definition 3.1. A random variable X is said to have an Erlang mixtured distributions if its pdf is defined by

$$f(x;\lambda,k,n,p)=\int_0^\infty \frac{(\lambda k)^k e^{-\lambda kr} r^{k-1}}{\Gamma(k)} g(x;n,p)\,dr, \tag{9}$$

where $g(x;n,p)$ is a probability function or probability density function. The name Erlang mixture of distributions is due to the fact that the derived distribution in (9) is the weighted average of $g(x;n,p)$ with weights equal to the probability density functions of the Erlang distribution.

Definition 3.2. A random variable X having the density function

$$f(x;\lambda,k)=\int_0^\infty \frac{(\lambda k)^k e^{-\lambda kr} r^{k-1}}{\Gamma(k)} \frac{e^{-\frac{x^2}{2}} x^{2r}}{2^{r+\frac{1}{2}}\Gamma\left(r+\frac{1}{2}\right)} dr; \quad -\infty < x < \infty \tag{10}$$

is said to have an Erlang mixture of normal moment distributions with parameters λ and k. If $k=0$, then

$$f(x)=\frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}. \tag{11}$$

which is nothing but the probability density function of a standard normal distribution.We now state some characteristics of the distribution in the following theorem.

Theorem 3.1: If X follows an Erlang mixture of standard normal distributions with parameters λ and k then

$$\mu'_{2s+1}=\mu_{2s+1}=0,$$

$$\mu'_{2s}=\mu_{2s}=\int_0^\infty \frac{(\lambda k)^k e^{-\lambda kr} r^{k-1}}{\Gamma(k)} \frac{2^s\Gamma\left(r+s+\frac{1}{2}\right)}{\Gamma\left(r+\frac{1}{2}\right)} dr,$$

Mean=0, Variance=$\frac{2+\lambda}{\lambda}$, $\beta_1=0$ and $\beta_2=\frac{3\lambda^2 k+8\lambda k+4(k+1)}{k(\lambda+2)^2}$,

where μ'_s is the sth moment about the mean, β_1 and β_2 are the measures of skewness and kurtosis respectively.

Proof: The odd raw moment about origin is given by

$$\begin{aligned}
&\mu'_{2s+1}=E\left[X^{2s+1}\right]\\
&=\int_0^\infty \frac{(\lambda k)^k e^{-\lambda kr} r^{k-1}}{\Gamma(k)} \int_{-\infty}^\infty \frac{x^{2s+1} e^{-\frac{x^2}{2}} x^{2r}}{2^{r+\frac{1}{2}}\Gamma\left(r+\frac{1}{2}\right)} dx\,dr\\
&=\int_0^\infty \frac{(\lambda k)^k e^{-\lambda kr} r^{k-1}}{\Gamma(k)} \frac{1}{2^{r+\frac{1}{2}}\Gamma\left(r+\frac{1}{2}\right)} \int_{-\infty}^\infty x^{2(s+r)+1} e^{-\frac{x^2}{2}} x^{2r} dx\,dr
\end{aligned}$$

=0 [Since $x^{2(s+r)+1}e^{-\frac{x^2}{2}}$ is an odd function of x].

That is, all odd moments are equal to zero. Hence

$$\mu'_{2s+1} = \mu_{2s+1} = 0.$$

Then all moments about origin become central moments. The general even moments of order $2s$ is defined by

$$\mu_{2s} = \mu'_{2s} = E\left[X^{2s}\right]$$

$$= \int_0^\infty \frac{(\lambda k)^k e^{-\lambda k r} r^{k-1}}{\Gamma(k)} \frac{1}{2^{r+\frac{1}{2}}\Gamma\left(r+\frac{1}{2}\right)} \int_{-\infty}^\infty \left(x^2\right)^{s+r+\frac{1}{2}-1} e^{-\frac{x^2}{2}} dx^2 dr$$

$$= \int_0^\infty \frac{(\lambda k)^k e^{-\lambda k r} r^{k-1}}{\Gamma(k)} \frac{1}{2^{r+\frac{1}{2}}\Gamma\left(r+\frac{1}{2}\right)} \int_{-\infty}^\infty \left(x^2\right)^{s+r+\frac{1}{2}-1} e^{-\frac{x^2}{2}} dx^2 dr$$

$$= \int_0^\infty \frac{(\lambda k)^k e^{-\lambda k r} r^{k-1}}{\Gamma(k)} \frac{2^s\Gamma\left(r+s+\frac{1}{2}\right)}{\Gamma\left(r+\frac{1}{2}\right)} dr. \tag{12}$$

If $s=1$ in equation (12), then we have

$$\mu_2 = \mu'_2 = \int_0^\infty \frac{(\lambda k)^k e^{-\lambda k r} r^{k-1}}{\Gamma(k)} \frac{2^s\Gamma\left(r+\frac{3}{2}\right)}{\Gamma\left(r+\frac{1}{2}\right)} dr = \frac{2+\lambda}{\lambda}.$$

If $s=2$ in equation (12), we have

$$\mu_4 = \mu'_4 = \frac{3\lambda^2 k + 8\lambda k + 4(k+1)}{\lambda^2 k}$$

To find the skewness and kurtosis, we have

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = 0$$
$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3\lambda^2 k+8\lambda k+4(k+1)}{k(\lambda+2)^2}$$

This completes the proof.

Theorem 3.2: If X follows an Erlang mixture of standard normal distributions with parameters λ and k, then its characteristic function is given by

$$\varphi_x(t) = \int_0^\infty \frac{(\lambda k)^k e^{-\lambda k r} r^{k-1} e^{-\frac{t^2}{2}}}{\Gamma(k)} \sum_{u=0}^{2r} \binom{2r}{2u} (it)^{2u} \frac{\Gamma\left(r+\frac{1}{2}-u\right)}{2^u \Gamma\left(r+\frac{1}{2}\right)} dr.$$

Proof:

$$\phi_x(t) = E\left[e^{itX}\right]$$

$$= \int_0^\infty \frac{(\lambda k)^k e^{-\lambda k r} r^{k-1}}{\Gamma(k)} \int_{-\infty}^\infty \frac{e^{itx} x^{2r} e^{-\frac{x^2}{2}}}{2^{r+\frac{1}{2}} \Gamma\left(r+\frac{1}{2}\right)} dx\, dr$$

$$= \int_0^\infty \frac{(\lambda k)^k e^{-\lambda k r} r^{k-1}}{\Gamma(k)} \frac{1}{2^{r+\frac{1}{2}} \Gamma\left(r+\frac{1}{2}\right)} \int_{-\infty}^\infty e^{-\frac{1}{2}(x-it)^2} e^{-\frac{t^2}{2}} dx\, dr$$

$$= \int_0^\infty \frac{(\lambda k)^k e^{-\lambda k r} r^{k-1}}{\Gamma(k)} \frac{e^{-\frac{t^2}{2}}}{2^{r+\frac{1}{2}} \Gamma\left(r+\frac{1}{2}\right)} \int_{-\infty}^\infty e^{-\frac{y^2}{2}} (y+it)^{2r} dy\, dr \quad [Putting\ y = x - it]$$

$$= \int_0^\infty \frac{(\lambda k)^k e^{-\lambda k r} r^{k-1}}{\Gamma(k)} \frac{e^{-\frac{t^2}{2}}}{2^{r+\frac{1}{2}} \Gamma\left(r+\frac{1}{2}\right)} \int_{-\infty}^\infty e^{-\frac{y^2}{2}} \sum_{u=0}^{2r} \binom{2r}{u} (it^u)\, y^{2r-u} dy\, dr \qquad (13)$$

Now, if u is odd, the integral in (13) is an odd function of y and its value is zero. In order to make the function even, let us replace u by $2u$, then (13) takes the form

$$\phi_x(t) = \int_0^\infty \frac{(\lambda k)^k e^{-\lambda k r} r^{k-1}}{\Gamma(k)} \frac{e^{-\frac{t^2}{2}}}{2^{r+\frac{1}{2}}\Gamma(r+\frac{1}{2})} \sum_{u=0}^{2r} \binom{2r}{2u} (it)^{2u} \int_{-\infty}^\infty e^{-\frac{y^2}{2}} y^{2r-2u} dy dr$$

$$= \int_0^\infty \frac{(\lambda k)^k e^{-\lambda k r} r^{k-1}}{\Gamma(k)} \frac{e^{-\frac{t^2}{2}}}{2^{r+\frac{1}{2}}\Gamma(r+\frac{1}{2})} \sum_{u=0}^{2r} \binom{2r}{2u} (it)^{2u} \int_0^\infty e^{-\frac{y^2}{2}} \left(y^2\right)^{r-u+\frac{1}{2}} dy^2 dr$$

$$= \int_0^\infty \frac{(\lambda k)^k e^{-\lambda k r} r^{k-1} e^{-\frac{t^2}{2}}}{\Gamma(k)} \sum_{u=0}^{2r} \binom{2r}{2u} (it)^{2u} \frac{\Gamma\left(r+\frac{1}{2}-u\right)}{2^u \Gamma\left(r+\frac{1}{2}\right)} dr \qquad (14)$$

This completes the proof.

Remarks: If $\lambda \to \infty$, then all the values of $\phi_X^{(t)}$, μ_1', μ_2, μ_3, μ_4, β_1 and β_2 are true for normal distribution with mean zero and variance unity.

Computer program has been developed to find the ordinates of the Erlang mixture of normal moment distributions for various values of the parameters by using S-plus. The corresponding curves were presented in the appendix. It is observed that the distribution is bimodal and symmetrical. The shapes of the curves are more concentrated as the values of k and lambda increase.

Estimation of the parameters:

Let $X_1, X_2, X_3, \ldots, X_n$ be a random sample from the distribution (10). We assume that the parameter k is known. Then the distribution contains only one unknown parameter viz, λ.

The second raw sample moment is

$m'_2 = \frac{1}{n}\sum\limits_{i=1}^{n} X_i^2 = S^2$

We have already found

$$\mu'_2 = \mu_2 = \frac{2+\lambda}{\lambda}.$$

Hence by the method of moments, we get

$$S^2 = \frac{2+\lambda}{\lambda}.$$

$$\hat{\lambda} = \frac{2}{S^2 - 1}$$

Therefore,

$$\hat{\lambda} = \frac{2}{S^2 - 1}$$

Special Case: If $k = 1$, the distribution in (10) reduces to an exponential mixture of standard normal distributions. The density function of the exponential mixture of standard normal distribution is

$$f(x;\lambda) = \int\limits_0^\infty \lambda e^{-\lambda r} \frac{e^{-\frac{x^2}{2}} x^{2r}}{2^{r+\frac{1}{2}}\Gamma\left(r+\frac{1}{2}\right)} dr; \quad -\infty < x < \infty \tag{15}$$

If X follows the distribution (15), then

$$\phi_x(t) = \int\limits_0^\infty \lambda e^{-\lambda r} \sum_{u=0}^{r} \frac{e^{-\frac{t^2}{2}} \binom{2r}{2u} (it)^{2u}\, \Gamma\left(r+\frac{1}{2}-u\right)}{2^u \Gamma\left(r+\frac{1}{2}\right)} dr, \tag{16}$$

Mean=0, Variance=$\frac{2+\lambda}{\lambda}$, $\beta_1 = 0$, $\beta_2 = \frac{\left(8+8\lambda+3\lambda^2\right)}{(2+\lambda)^2}$.

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