International Journal of Statistical Sciences *ISSN 1683–5603*
Vol. 6 (Special Issue), 2007, pp 19-28

Some Distributional Properties of Record Values From Generalized Extreme Value Distributions

M. Ahsanullah
Dept. of Management Sciences
Rider University
Lawrenceville, NJ 08648, USA

Chris P. Tsokos
Dept. of Mathematics and Statistics
University of South Florida
Tampa, Florida, 33620, USA

[Received April 12, 2005; Revised August 10, 2007; Accepted August 18, 2007]

Abstract

Several distributional properties of the record values from generalized extreme value distribution are discussed. Minimum variance unbiased estimates for location and scale parameters are presented.

Keywords and Phrases: Minimum variance unbiased estimates, record values, generalized extreme value distribution

AMS Classification: 62G05, 60E05.

1 Introduction

A random variable X is said to have the generalized extreme value (GEV) distribution if its cumulative distribution function (cdf) $F(x)$ is of the following form

$$F(x) = \exp\{-(1-\gamma\sigma^{-1}(x-\mu))^{1/\gamma}\} \tag{1}$$

Where $\gamma \neq 0$, and

$$x < \mu + \sigma\gamma^{-1}, \gamma > 0, \sigma > 0$$

$$x > \mu + \sigma\gamma^{-1}, \gamma < 0, \sigma > 0$$

GEV $(0, 1, -1)$ is unimodal. Gumbel (1958) has given various distributional properties of extreme value distributions. Fisher and Tippet (1925) introduced the extreme value distributions as the limiting distributions of largest and smallest observations. Extreme value distributions have been used in the analysis of data concerning floods, extreme sea levels and air pollution problems. For details see Gumbel (1958), Horwitz (1980), Jenkinson (1955) and Roberts (1979).

Suppose that $X_1, X_2, ...$ is a sequence of independent and identically distributed random variables with cumulative distribution function (*cdf*) $F(x)$ and the corresponding pdf $f(x)$.

Let Y_n =max (min) $\{X_1, X_2, ..., X_n\}$. We say X_j is an upper (lower) record value of $\{xn, n \geq 1\}$ if $Y_j > (<)\ Y_{j-1}, j > 1$. By definition X_1 is an upper as well as a lower record value. In this paper we will consider the lower record values of the generalized extreme value distribution. For theory of record values, see Ahsanullah (1995, 2004), Arnold, Balakrishnan and Nagaraja (1998) and Nevzorov (2001).

2 Distributional properties of mth lower record value

Let $X_{L(m)}$ be the mth lower record value of $\{X_{n1} n \geq 1\}$.
Lemma 2.1

$$P\left[X_{L(m)} \leq x\right) = \int_{-\infty}^{x} \frac{(h(u))^{m-1}}{\Gamma(m)} f(u)\, du \tag{3}$$

where $H(u) = -\ln F(u), 0 < F(u) < 1$.

Proof:

$$\begin{aligned}
F_1(x) = P\left[X_{L(1)} \leq x\right] &= P\left[X_1 \leq x\right] = F(x) \\
F_2(x) = P\left[X_{L(2)} \leq x\right] &= \textstyle\int_{-\infty}^{x} \int_{y}^{\infty} \sum_{j=1}^{\infty} (1 - F(u))^{j-1} f(u) f(y)\ dudy \\
&= \textstyle\int_{-\infty}^{x} \int_{-\infty}^{y} \frac{f(x)}{F(u)} f(u) f(y)\ du\, dy \\
&= \textstyle\int_{-\infty}^{x} H(y) f(y)\ dy \\
F_3(x) = P\left[X_{L(3)} \leq x\right] &= \textstyle\int_{-\infty}^{x} \int_{y}^{\infty} \sum_{j=1}^{\infty} (1 - F(u))^{j-1} H(u) f(u) f(y)\ dudy \\
&= \textstyle\int_{-\infty}^{x} \int_{y}^{\infty} H(u) \frac{f(u)}{F(u)} f(y)\ du\, dy \\
&= \textstyle\int_{-\infty}^{x} \frac{(H(y))^2}{2!} f(y)\ dy
\end{aligned}$$

The result follows by induction.

If $X \in GEV(\mu, \sigma, \gamma)$, then using (1) and (2) we can write the pdf $F_{(m)}(x)$ of the mth lower record value as

$$f_{(m)}(x) = \left\{1 - \gamma\sigma^{-1}(x - \mu)\right\}^{(m-1)/\gamma} f_{(m)}^{*}(x),$$

Where

$$f^{*}_{(m)}(x) = \frac{\left\{1-\gamma\sigma^{-1}(x-\mu)\right\}^{\frac{1-\gamma}{\gamma}}}{\sigma\,\Gamma(m)} \exp\left\{-\left(1-\gamma\sigma^{-1}\right)(x-\mu)^{1/2}\right\}, \gamma \neq 0$$

and

$$f_{(m)}(x) = \frac{e^{-m(x-\mu)/\sigma}}{\sigma\,\Gamma(m)} exp\left(-e^{-(x-\mu)/\sigma}\right), \qquad m = 1, 2, ..., \gamma = 0.$$

It can be shown that

$$X \underline{\underline{d}}_{L(m)} \mu + \sigma\gamma^{-1}\left\{1-(W_1+W_2+...+W_m)^{\gamma}\right\} ... \tag{4}$$

For $\gamma \neq 0$

$$X \underline{\underline{d}}_{L(m)} x - \sigma\left(W_1 + \frac{W_2}{2} + ... + \frac{W_m}{m}\right) ... \tag{5}$$

For $\gamma = 0$,

where $W_1, W_2, ..., W_m$ are independently distributed as exponential with cdf $F(x) = 1 - e^{-x}, x \geq 0$ and $X \underline{\underline{d}}$ GEV $(\mu, \sigma, 0)$.
Using (4), we obtain for $\gamma \neq 0$,

$$E\left(X_{L(i)}\right) = \mu + \sigma\gamma^{-1}[1 - \frac{\Gamma(i+\gamma)}{\Gamma(i)}] = \mu + \sigma\alpha, \;\; i > -\gamma, \tag{6}$$

$$Var\left(X_{L(m)}\right) = \mu^2\gamma^{-2}\{E(W_1+W_2+...+W_i)^{2\gamma} - (E(W_1+W_2+...+W_i)^{\gamma})^2$$

$$= \mu^2\gamma^{-2}[\frac{\Gamma(i+2\gamma)}{\Gamma(i)} - (\frac{\Gamma(i+\gamma)}{\Gamma(i)})^2], i > -2\gamma \tag{7}$$

For $r < s$

$$\gamma^2\sigma^{-2}\text{cov}\left(x_{L(\gamma)}x_{L(S)}\right) = E(\sum_{j=1}^{\gamma} w_j^{\gamma}) \sum_{j=1}^{s} w_j^{s} - E(\sum_{j=1}^{\gamma} w_j^{\gamma})(E \sum_{j=1}^{s} w_j^{\gamma})$$
$$= \int_0^{\infty}\int_0^{\infty} u^{\gamma}(u+v)^{\gamma} \frac{u^{\gamma-1}e^{-u}}{\Gamma(r)} . \frac{v^{S-\gamma-1}e^{-v}}{\Gamma(m-\gamma)} du\, dv - \frac{\Gamma(r+\gamma)}{\Gamma(r)} \frac{\Gamma(s+\gamma)}{\Gamma(s)}$$
$$= \frac{\Gamma(r+\gamma)}{\Gamma(\gamma)} . \frac{\Gamma(s+2\gamma)}{\Gamma(s)} - \frac{\Gamma(r+\gamma)}{\Gamma(r)} . \frac{\Gamma(s+\gamma)}{\Gamma(s)},$$

Since u and v are independent.
Thus

$$\text{cov}\left(X_{L(r)}X_{L(s)}\right) = \sigma_0^2 \text{a}_r b_s, r < s,$$

Where

$\sigma_0^2 = \frac{\sigma^2}{\gamma^2}$, $a_r = \frac{\Gamma(r+\gamma)}{\Gamma(r)}$ and $b_s = \frac{\Gamma(s+2\gamma)}{\Gamma(s/\gamma)} - \frac{\Gamma(s+\gamma)}{\Gamma(s)}$
For $\gamma = 0$, we obtain from (5)

$$\begin{aligned} &E\left(X_{L(r)}\right) = \mu + \alpha_r \sigma, \\ &Var\left(X_{L(r)}\right) = \sigma^2 V_{r,r}^* \\ &\text{cov}\left(X_{L(r)} X_{L(m)}\right) = Var\ \left(X_{L(m)}\right), r < m \end{aligned}$$

with

$\alpha_1^* = \nu$, the Euler's constant
$\alpha_j^* = \alpha_{j-1}^* - \frac{1}{j-1}$, $j \geq 2$
$V_{11}^* = \frac{x^2}{6}$
$V_{jj}^* = V_{j-1,j-1} - \frac{1}{(j-1)^2}, j \geq 2.$

3 Estimation of parameters

3.1 Minimum variance linear unbiased estimate (MVLUE) of μ and σ for known γ.

If $\gamma > 0$, then the variances of all the lower records are finite and we will consider the first m lower records. If $\gamma < 0$, then we have to consider record values greater than -2γ. Without any loss of generality, we will take $\gamma > 0$ and consider first m lower record values.

Theorem 3.1.
Suppose $\gamma > 0$, then the MVLUE $\hat{\mu}$ and $\hat{\sigma}$ based on the first m lower record values $r_1, r, ..., r_m$ are given by

$$\hat{\mu} = -\alpha' V^{-1} \left(1\alpha' - \alpha 1'\right) V^{-1} r/\Delta \tag{8}$$

and

$$\hat{\sigma} = -1 V^{-1} \left(1\alpha' - \alpha 1' \ V^{-1} r/\Delta, \tag{9}$$

where $\Delta = \left(1' V^{-1} 1\right)^{-1} \left(\alpha' V^{-1} \alpha\right) - \left(1' V^{-1} \alpha\right)^2$,
$V^{-1} = \left(V^{ij}\right)$,
$V^u = \left(\frac{1+\gamma}{\gamma^2}\right)^2 \frac{1}{\Gamma(1+2\gamma)}, V^{mm} = \frac{m-1+\gamma}{\gamma^2} \frac{b_{m-1}}{b_m} \frac{\Gamma(m)}{\Gamma(m-1+\gamma)}$
$V^{ii} = \frac{\Gamma(i)}{\Gamma(i+2\gamma)} \cdot \frac{(1+\gamma)^2 + (i-1)(i-1+2\gamma)}{\gamma^2}$, $j = i+1$, $i = 1, \ldots, m-1$
$V^{ij} = 0$, $|i-j| > 1$

$$1' = (1, 1, \ ..1)\,,\ r' = (\gamma_1, \gamma_2, ..., \gamma_m)\,,\ \alpha^1 = (\alpha_1, \alpha_2, ..., \alpha_m)\,.$$

and

$$\alpha_i = 1 - \frac{\Gamma(i+\gamma)}{\Gamma(i)}.$$

The variances and covariances are given by

$$\text{Var}(\hat{\mu}) = \frac{\sigma_0^2}{\Delta}(\alpha' V^{-1}\alpha)$$

$$\text{Var}(\hat{\sigma}) = \frac{\sigma_0^2}{\Delta}(1' V^{-1} 1)$$

and

$$\text{Cov}(\hat{\mu},\hat{\sigma}) = -\frac{\sigma_0^2}{\Delta}(1' V^{-1}\alpha).$$

Proof:

Let $R = \left(X_{L(1)}, X_{L(2)}, ..., X_{L(m)}\right)$, then we can write

$$E(R) = \mu 1 + \sigma_o \alpha$$

and

$$V(R) = \sigma_0^2 V$$

where

$$\alpha' = (a_1, ..., a_m)\,, \alpha_i = 1 - \frac{\Gamma\,(i+\gamma)}{\Gamma\,(i)}, \quad 1' = (1, , , , , 1)\,, V = (v_{ij})\,, V^{-1} = (V^{ij}).$$

$v_{ij} = a_i b_j, 1 \leq i, j \leq m.$

$$\begin{aligned}
V^{11} &= \tfrac{a_2}{a_1(a_2 b_1 - a_1 b_2)} = \tfrac{(1+\gamma)^2}{\gamma^2}\,\tfrac{1}{\Gamma(1+2\gamma)},\\
V^{mm} &= \tfrac{b_{m-1}}{b_m}\,\tfrac{1}{a_m b_{m-1} - a_{m-1} b_m} = \tfrac{b_{m-1}(m-1+\gamma)}{b_m \gamma^2}.\tfrac{\Gamma(m)}{\Gamma(m-1+\gamma)},\\
V^{ii} &= \tfrac{a_{i+1} b_{i-1} - a_{i-1} b_{i+1}}{(a_i b_{i-1} - a_{i-1} b_i)(a_{i+1} b_i - a_i b_{i+1})}\\
&= \tfrac{\Gamma(i)}{\Gamma(i+2\gamma)}.\tfrac{(i+\gamma)^2 + (i-1)(i-1+2\gamma)}{\gamma^2},\ \ i = 2...m-1,\\
V^{ij} &= V^{ji} = -\tfrac{1}{a_{i+1} b_i - a_i b_{i+1}} = -\tfrac{\Gamma(i)}{\Gamma(i+2\gamma)}.\tfrac{i+\gamma}{\gamma^2},\ j = i+1,\ i = 1, ..., m-1
\end{aligned}$$

and $V^{ij} = 0, |i-j| > 1.$

Using the method of Lloyd (1952) the MVLUEs $\hat{\mu}$ and $\hat{\sigma}$ of μ and σ based on m lower record values $r_1, r_2, ..., r_m$ are given by

$$\hat{\mu} = -a'V^{-1}\left(1a' - a1'\right)V^{-1}r/\Delta$$

and

$$\hat{\sigma} = -1'V^{-1}\left(1a' - a1'\right)V^{-1}r/\Delta,$$

where $\Delta = \left(1'V^{-1}1\right)\left(a'V^{-1}a\right) - \left(1'V^{-1}a\right)^2$.
$Var\left(\hat{\mu}\right) = \frac{\sigma_0^2}{\Delta}\left[a'V^{-1}a\right]$,
$Var\left(\hat{\sigma}\right) = \frac{\sigma_0^2}{\Delta}\left(1'V^{-1}1\right)$
and $Cov\left(\hat{\mu}, \hat{\sigma}\right) = -\frac{\sigma_0^2 1'V^{-1}a}{\Delta}$

The following table gives the coefficients of the MVLUEs $\hat{\mu}$ and $\hat{\sigma}$ for $2 \leq m \leq 6$ and $\gamma = 1$.

Table 3.1. the coefficients of $\hat{\mu}$, $\hat{\sigma}$ for γ=2

m		$X_{L(1)}$	$X_{L(2)}$	$X_{L(3)}$	$X_{L(4)}$	$X_{L(5)}$	$X_{L(6)}$
2	$\hat{\mu}$	1.25	-0.25				
2	$\hat{\sigma}$	0.25	-0.25				
3	$\hat{\mu}$	1.1458	-0.0764	-0.0972			
3	$\hat{\sigma}$	0.10410	-0.0069	-0.0972			
4	$\hat{\mu}$	1.125	-0.075	-0.025	-0.025		
4	$\hat{\sigma}$	0.0592	-0.0040	-0.0013	-0.0540		
5	$\hat{\mu}$	1.1195	-0.0746	-0.0249	-0.0107	-0.0093	
5	$\hat{\sigma}$	0.0386	-0.0026	-0.0009	-0.0004	-0.0348	
6	$\hat{\mu}$	1.1182	-0.0746	-0.0249	-0.0107	-0.0053	-0.0028
6	$\hat{\sigma}$	0.0273	-0.0018	-0.0006	-0.0003	-0.0001	-0.0245

3.2 MVLUEs $\hat{\mu}$ and $\hat{\sigma}$ of μ and σ when $\gamma = 0$

The following Theorem gives MVLUEs $\hat{\mu}$ and $\hat{\sigma}$.

Theorem 3.2

Suppose that $\gamma = 0$, then the MVLUEs $\hat{\mu}$ and $\hat{\sigma}$ of μ and σ based on the m observed lower records, $r_1, ..., r_n$, are respectively

$$\hat{\mu} = r_m - a_m^* \hat{\sigma} \tag{10}$$

and

$$\hat{\sigma} = \frac{1}{m-1} \sum_{i=1}^{m-1} r_1 - r_m, \tag{11}$$

where $a_1^* = v$, the Euler constant ($= 0.57722$)

and

$$a_j^* = a_{j-1}^* - \frac{1}{j-1}, \quad j \leqq 2.$$

Their corresponding variances and covariance are

$$Var\left(\hat{\mu}\right) = \sigma^2 \left\{ V_{mm}^* + \frac{\left(V_m^*\right)^2}{m-1} \right\}$$

$$Var\left(\hat{\sigma}\right) = \frac{\sigma^2}{m-1}$$

and

$$Cov\left(\hat{\mu}, \hat{\sigma}\right) = -\frac{\sigma^2 V_m^*}{m-1}.$$

Proof:

For $\gamma = 0$, $E\left(X_{L(j)}\right) = \mu + \sigma\, a_j^*$,
$a_1^* = v$, the Euler constant ($= 0.57722$)
and

$$a_j^* = a_{j-1}^* - \tfrac{1}{j-1}, \; j \leq 2.$$
$$Var\left(X_{L(j)}\right) = \sigma^2 V_{jj}^*, \; j = 1, 2, ...$$
$$Cov\left(X_{L(i)} X_{L(j)}\right) = Var\left(X_{L(j)}\right), \; 1 \leq i \leq j,$$
$$V_{11}^* = \tfrac{\pi^2}{6}, V_{jj}^* = V_{j-1j-1}^* - \tfrac{1}{(j-1)^2}.$$

Let $V = \left(V_{ij}^*\right)$ and $V^{-1} = \left(V^{ij}\right)$, then

$$1'V^{-1} = (0, 0, ..., 1/V^*_{mm}),$$
$$a'V^{-1} = \left(1, 1, ..., \frac{a_m}{V^*_{mm}} - m - 1\right)$$
$$a'V^{-1}1 = a_m/V^*_{mm}$$
$$a'V^{-1}a = \frac{a_m^2}{V^*_{mm}}$$

and

$$\Delta = \frac{m-1}{V^*_{mm}}.$$

Substituting these values in

$$\hat{\mu} = -a'V^{-1}\left(1a' - a1'\right)V^{-1}r/\Delta$$

and

$$\hat{\sigma} = -1'V^{-1}\left(1a' - a1'\right)V^{-1}r/\Delta,$$

the result follows.

Table 3.2 gives the coefficients of the MVLUEs $\hat{\mu}$ and $\hat{\sigma}$ for $2 \leq m \leq 6$ and $\gamma = 0$.

Table 3.2. The coefficients of the MVLUEs $\hat{\mu}$ and $\hat{\sigma}$ for $\gamma = 0$.

m		$X_{L(1)}$	$X_{L(2)}$	$X_{L(3)}$	$X_{L(4)}$	$X_{L(5)}$	$X_{L(6)}$
2	$\hat{\mu}$	0.42278	0.57722				
2	$\hat{\sigma}$	1	-1				
3	$\hat{\mu}$	0.46139	0.461239	0.07728			
3	$\hat{\sigma}$	$\frac{1}{2}$	$\frac{1}{2}$	-1			
4	$\hat{\mu}$	0.41403	0.41403	0.41403	-0.25611		
4	$\hat{\sigma}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	-1		
5	$\hat{\mu}$	0.37653	0.37653	0.37653	0.37653	-0.50611	
5	$\hat{\sigma}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	-1	
6	$\hat{\mu}$	0.32122	0.32122	0.32122	0.32122	0.32122	0.60610
6	$\hat{\sigma}$	0.2	0.2	0.2	0.2	0.2	-1

3.3 Estimate of γ for known values of μ and σ.

We will assume without any loss of generality $\mu = 0$ and $\sigma = 1$.

Let $Y_i = i\left(X_{L(i)} - X_{L(i+1)}\right)$, then
$\mathrm{E}(\mathrm{Y}_i) = 1\text{-}\gamma\mathrm{E}(\mathrm{X}_{L(i)})$ and $\mathrm{E}(\Sigma_{i=1}^{m-1} Y_i) = m - 1 - \gamma\mathrm{E}(\Sigma_{i=1}^{m-1} X_{L(i)})$.
Thus a moment estimator γ^* of γ is
$\gamma^* = \frac{1-\bar{\mathrm{y}}}{\bar{\mathrm{r}}}$,
where
$\bar{\mathrm{y}} = \Sigma_{\mathrm{i=1}}^{\mathrm{m-1}} \frac{Y_i}{m-1}$ and $\bar{\mathrm{r}} = \Sigma_{\mathrm{i=1}}^{\mathrm{m-1}} \frac{X_{L(i)}}{m-1}$.

References

1. Ahsanullah, M. (1995): Record Statistics, Nova Science Publishers Inc., New York, NY, USA

2. Ahsanullah, M. (2004): Record Values – Theory and Applications, University Press of America Inc., MD, USA

3. Fisher, R.A. and Tippett, L.H.C. (1928): Limiting Forms of the Frequency Distribution of the Largest or Smallest Member of a Sample, *Proceedings of the Cambridge Philosophical Society*, 24, 180-190.

4. Gumbel, E.J. (1958): Statistics of Extremes, Columbia University Press, New York, NY, USA.

5. Horwitz, J. (1980): Extreme Values from a Non Stationary Stochastic Process: an Application to Air

6. Quality Analysis (with Discussion), *Technometrics* 22, 469-482.

7. Jenkinson, A.F. (1955): The Frequency Distribution of Annual Maximum (or Minimum) Values of Meterological Elements, *Quarterly Journal of Meterology*, 87, 158-171.

8. Lloyd, E.H. (1952): Least Squares Estimation of Location and Scale Parameters Using Order Statistics, *Biometrika*, 39, 88-95.

9. Nevzorov, V.B. (2001): Records: Mathematical Theory, *Translation of Mathematical Monograph*, Volume 194, Providence, RI, American Mathematical Society.

10. Roberts, E.M. (1979): Review of Statistics of Extreme Values with Applications to Air Quality Data Part II Application, *J. Air Pollution Control Assoc.*, 29, 733-740.