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Detection of Boundaries in Regression Data in the Presence of Spatial Correlation

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Abstract

A regression model defined by a single set of parameters may be suitable for fitting a set of spatial data. However the data may be divided into subsets each of which is modelled using a different set of regression parameters. Statistics for detecting the presence of boundaries of such subsets of spatial data are available provided the data do not exhibit spatial correlation. Since the distribution of a change-boundary statistic is modified by the presence of spatial correlation, distributional results are presented to deal with the problem. Detecting boundaries in higher dimensional data is discussed.

Keywords and Phrases: Change boundaries; Spatial correlation; Residual processes.

AMS Classification: Primary 62M10; Secondary 62E20, 62G10, 62J05, 62M15.

1 Introduction

We consider linear regression of a random variable against general non-stochastic functions of a matrix array, but with error variables that form a stationary spatial process. We then examine the large sample properties of the stochastic process defined by the matrix array of the partial sums of regression residuals. After introducing the problem in section 2 we derive, in section 3, the residual processes for stationary spatial series satisfying a moment condition. These processes are used in section 4 to obtain the residual processes for regression against general nonstochastic regression functions of a matrix array when the errors form a stationary spatial series. We then discuss in section 5 the effect of spatial correlation on change boundary statistics and large sample adjustments to account for this spatial correlation. Extension to d-dimensional space is presented in section 6.

2 Regression Models and Error Process Structure

We first define the basic 2-dimensional model. Let $X(n,m)$ $(n,m=0,\pm1,\ldots)$ be a zero mean, stationary spatial series defined on a lattice with covariance function

$$R(u,v)=E\{X(t,s)X(t+u,s+v)\},\quad |u|,\ |v|<\infty\ .$$

If the covariance function is absolutely summable, i.e.,

$$\sum_{|u|<\infty}\sum_{|v|<\infty}|R(u,v)|<\infty\ , \tag{1}$$

then the spectral density function,

$$f(\lambda_1,\lambda_2)=\frac{1}{4\pi^2}\sum_{|u|<\infty}\sum_{|v|<\infty}e^{-i\lambda_1 u-i\lambda_2 v}R(u,v)\ ,\quad \lambda_1,\lambda_2\in[-\pi,\pi]\ ,$$

exists.

In the sequel we require a central limit theorem for spatially correlated series. Brillinger (1970) defined the cumulant functions for stationary spatial series as follows:

$$C_{k+1}(\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_k)=Cum\{X(\mathbf{n}+\mathbf{v}_1),\ldots,X(\mathbf{n}+\mathbf{v}_k),X(\mathbf{n})\}\ ,$$

where $\mathbf{v}_j=(v_{1j},v_{2j})$, $\mathbf{n}=(n_1,n_2)$. Stationarity to order $k+1$ is implicit in this definition. Note that the first two cumulants are $E\{X(n_1,n_2)\}$ and $R(v_1,v_2),\ |v_1|,|v_2|<\infty$. When necessary we assume the cumulants exist and satisfy what we call the Brillinger condition for spatial data namely,

$$C_{k+1}(\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_k)<\frac{L_k}{\prod_{j=1}^{k}(1+v_{1j}^2)(1+v_{2j}^2)}\quad , \tag{2}$$

for some finite L_k, where $\mathbf{v}_j=(v_{1j},v_{2j}),\ j=1,2,\ldots,k$.

If (1) and (2) are satisfied, the results of Brillinger (1973) become as follows: For $t,s\in[0,1]$ and with $[x]$ denoting the largest integer in x,

$$\frac{1}{n}\sum_{i=1}^{[nt]}\sum_{j=1}^{[ns]}X(i,j)\ ,$$

converges in distribution to the normal with zero mean and variance $\{4\pi^2 f(0,0)ts\}$. We now consider the regression model

$$Y_n(i,j) = \sum_{k=0}^{p} \beta_k g_k(i/n, j/n) + X(i,j) \ ,$$

where $\{g_k(\cdot,\cdot),\ 0 \le k \le p\}$ is a collection of regressor functions defined on the unit square.

If we denote the vector of regression coeffients by $\beta = (\beta_{\mathbf{0}}, \ldots, \beta_{\mathbf{p}})'$, the design matrix by $\mathbf{A}_n$, the stacked vector of observations by $\mathbf{Y}_n$ and the stacked vector of stationary spatial series by $\mathbf{X}_n$, then the model may be written in matrix form as

$$\mathbf{Y}_n = \mathbf{A}_n \beta + \mathbf{X}_n \ .$$

The regression parameter estimators are

$$\hat{\beta} = (\mathbf{A}_n' \mathbf{A}_n)^{-1} \mathbf{A}_n' \mathbf{Y}_n \ .$$

The matrix array of partial sums of regression residuals are defined as

$$S_{g_n}(k,l) = \sum_{i=1}^{k} \sum_{j=1}^{l} \{Y_n(i,j) - \hat{Y}_n(i,j)\} \ , \quad 1 \le k, l \le n,$$

where

$$\hat{Y}(i,j) = \hat{\beta}' \mathbf{g}(i/n, j/n)$$

and

$$\mathbf{g}(i/n, j/n)' = (g_0(i/n, j/n), \ldots, g_p(i/n, j/n)) \ .$$

Since we shall be concerned with weak convergence on the space of functions continuous on the unit square, $C[0,1]^2$, we use these matrix arrays of partial sums to define a sequence of stochastic processes $\{Z_{g_n}(t,s), t, s \in [0,1]\}$ $(n \ge 1)$ possessing continuous sample paths as follows (see Kuelbs (1968)):

$$\begin{aligned} nZ_{g_n}(t,s) \quad = \quad & S_{g_n}([nt],[ns]) + (nt - [nt])\{S_{g_n}([nt]+1,[ns]) \\ & - S_{g_n}([nt],[ns])\} \\ & + (ns - [ns])\{S_{g_n}([nt],[ns]+1) - S_{g_n}([nt],[ns])\} \\ & + n(nt - [nt])(ns - [ns])\{Y_n([nt]+1,[ns]+1) \\ & - \hat{Y}_n([nt]+1,[ns]+1)\} \ . \end{aligned}$$

If we let $\mathbf{e}_{nt,ns}$ denote the n^2-dimensional vector that has: 1 for components where $\mathbf{X}_n$ has as its component $X_n(i,j)$ with $i \le [nt]$ and $j \le [ns]$, $nt - [nt]$ with $i = [nt]+1$ and $j \le [ns]$, $ns - [ns]$ with $i \le [nt]$ and $j = [ns]+1$, $n(nt - [nt])(ns - [ns])$ with $i = [nt]+1$ and $j = [ns]+1$, and 0 otherwise, then we can write

$$nZ_{g_n}(t,s) = \mathbf{e}_{nt,ns}' \{\mathbf{I} - \mathbf{A}_n(\mathbf{A}_n' \mathbf{A}_n)^{-1} \mathbf{A}_n'\} \mathbf{X}_n \ .$$

3 The Partial Sum Limit Process for Stationary Spatial Series

To establish the limit process for $\{Z_{g_n}(t,s),\ t,s \in [0,1]\}$ we need first to examine the properties of the matrix array of partial sums of the error process $X(n,m)$ $(n,m = 0,\pm1,\ldots)$. Hence, we let $S_{X_n}(k,l) = \sum_{i=1}^{k}\sum_{j=1}^{l} X(i,j)$ and define another sequence of stochastic processes $\{Z_{X_n}(t,s),\ \ t,s \in [0,1]\}$ $(n \geq 1)$ possessing continuous sample paths by

$$\begin{aligned} nZ_{X_n}(t,s) \;&=\; S_{X_n}([nt],[ns]) \\ &\quad +(nt-[nt])\{S_{X_n}([nt]+1,[ns]) - S_{X_n}([nt],[ns])\} \\ &\quad +(ns-[ns])\{S_{X_n}([nt],[ns]+1) - S_{X_n}([nt],[ns])\} \\ &\quad +n(nt-[nt])(ns-[ns])X_n([nt]+1,[ns]+1)\ . \end{aligned}$$

We note first that

$$Z_{X_n}(0,0) = E\{Z_{X_n}(t,s)\} = 0$$

and consider next the covariance kernel of the process

$$K_n(t_1,s_1;t_2,s_2) = E\{Z_{X_n}(t_1,s_1)Z_{X_n}(t_2,s_2)\}\ .$$

We assume $t_1 = \min(t_1,t_2)$, $s_1 = \min(s_1,s_2)$, $k_i = [nt_i]$ and $l_i = [ns_i]$, $i = 1,2$. For sufficiently large n, we have

$$\left| K_n(t_1,s_1;t_2,s_2) - E\left\{ Z_{X_n}(\frac{k_1}{n},\frac{l_1}{n})Z_{X_n}(\frac{k_2}{n},\frac{l_2}{n})\right\}\right| \leq \frac{c}{n}\ ,$$

where $c > 0$ is independent of t_1, s_1, t_2, s_2 and n. Therefore, for large samples, we need only consider $K_n\left(\frac{k_1}{n},\frac{l_1}{n};\frac{k_2}{n},\frac{l_2}{n}\right)$. Then

$$\begin{aligned} K_n\left(\frac{k_1}{n},\frac{l_1}{n};\frac{k_2}{n},\frac{l_2}{n}\right) \;&=\; \frac{1}{n^2}E\{S_{X_n}(k_1,l_1)S_{X_n}(k_2,l_2)\} \\ &=\; \frac{1}{n^2}\sum_{t_1=1}^{l_1}\sum_{s_1=1}^{k_1}\sum_{t_2=1}^{l_2}\sum_{s_2=1}^{k_2} E\{X(t_1,s_1)X(t_2,s_2)\} \\ &=\; \frac{1}{n^2}\sum_{t_1=1}^{l_1}\sum_{s_1=1}^{k_1}\sum_{t_2=1}^{l_2}\sum_{s_2=1}^{k_2} R(t_2-t_1,s_2-s_1)\ . \end{aligned}$$

Conditions (1) and (2) imply

$$\begin{aligned} \frac{1}{n^2}\sum_{t_1=1}^{l_1}\sum_{t_2=1}^{l_2}\sum_{s_1=1}^{k_1}\sum_{s_2=1}^{k_2} R(t_2-t_1,s_2-s_1) \;&=\; \frac{1}{n^2}\sum_{|u|<l_1}\sum_{|v|<k_1}(l_1-|u|)(k_1-|v|)R(u,v) \\ &\quad +O(n^{-1})\ . \end{aligned}$$

We have shown that

$$\frac{1}{n^2}K_n(t_1,s_1;t_2,s_2) \to t_1 s_1 \sum_{u=-\infty}^{\infty}\sum_{v=-\infty}^{\infty} R(u,v) \ ,$$

where $t_1 = \min(t_1,t_2)$ and $s_1 = \min(s_1,s_2)$. That is,

$$\frac{1}{n^2}K_n(t_1,s_1;t_2,s_2) \to 4\pi^2 f(0,0)(t_1 \wedge t_2)(s_1 \wedge s_2) \ ,$$

where $t_1 \wedge t_2 = \min(t_1,t_2)$.

We adapt the methods of De Gooijer and MacNeill (1999) to establish asymptotic normality in the following theorem.

Theorem 1: Under assumptions (1) and (2), the p-vector $\{Z_{X_n}(t_1,s_1),\ldots,\ Z_{X_n}(t_p,s_p)\}$ has a non-trivial asymptotic probability distribution that is normal with zero mean and covariance matrix $4\pi^2 f(0,0)(t_i \wedge t_j)(s_i \wedge s_j)$.

Proof: The Brillinger condition for spatial data (2) can be used to demonstrate that the spatial cumulants of orders higher than two of a vector component of $Z_{X_n}(t_i,s_i)$ are $O(n^{-1})$ or smaller and hence that $Z_{X_n}(t_i,s_i)$ converges in distribution to the normal with zero mean and variance $4\pi^2 f(0,0)t_i s_i$. The Cramér-Wold device of demonstrating asymptotic multivariate normality by showing asymptotic normality with zero mean and variance

$$4\pi^2 f(0,0)\sum_{i=1}^{p}\sum_{j=1}^{p}\lambda_i\lambda_j(t_i \wedge t_j)(s_i \wedge s_j)$$

of $\sum_{i=1}^{p}\lambda_i Z_{X_n}(t_i,s_i)$, where the λ_i are arbitrary real numbers, can be used to complete the proof for the p-dimensional case.

We next show tightness of the sequence of measures P_{X_n} $(n=1,2,\ldots)$ generated in $C[0,1]^2$ by $\{Z_{X_n}(t,s),\ t,s \in [0,1]\}$. The arguments used above to derive the covariance kernel for these processes can be used to show the existence of a constant C such that for $t_1,s_1,t_2,s_2 \in [0,1]$,

$$E\{Z_{X_n}(t_1,s_1) - Z_{X_n}(t_2,s_2)\}^4 \le C\{(t_1-t_2)^2 + (s_1-s_2)^2\} \tag{3}$$

where C is not dependent on t_1,s_1,t_2,s_2 and n. We only discuss the case of $t_2 \ge t_1$ and $s_2 \ge s_1$ (the same argument holds for the other cases).

Let $[nt_1]=k_1,\ [ns_1]=l_1,\ [nt_2]=k_2,\ [ns_2]=l_2$, then

$$\begin{aligned}
& E\left\{\frac{1}{n}S_{X_n}(k_1,l_1) - \frac{1}{n}S_{X_n}(k_2,l_2)\right\}^4 \\
= \ & \frac{1}{n^4}E\left\{\left(\sum_{i=k_1+1}^{k_2}\sum_{j=l_1+1}^{l_2} + \sum_{i=1}^{k_1}\sum_{j=l_1+1}^{l_2} + \sum_{i=k_1+1}^{k_2}\sum_{j=1}^{l_1}\right)X_n(i,j)\right\}^4 \\
= \ & \frac{1}{n^4}\left(\sum_{A_1} + \sum_{A_2} + \sum_{A_3}\right)E\{X_n(i_1,j_1)X_n(i_2,j_2)X_n(i_3,j_3)X_n(i_4,j_4)\} \ ,
\end{aligned}$$

where

$$A_1 = \{(i_h, j_h): \quad k_1 < i_h \le k_2,\ l_1 < i_h \le l_2, \quad h = 1,2,3,4\}\ ,$$
$$A_2 = \{(i_h, j_h): \quad 1 \le i_h \le k_1,\ l_1 < i_h \le l_2, \quad h = 1,2,3,4\}\ ,$$
$$A_3 = \{(i_h, j_h): \quad k_1 < i_h \le k_2,\ 1 \le i_h \le l_1, \quad h = 1,2,3,4\}\ .$$

These fourth order moments can be expressed in terms of the corresponding fourth order cumulants and products of pairs of elements from the covariance function. Hence,

$$\begin{aligned} & \frac{1}{n^4}\sum_{A_i} E\{X_n(i_1,j_1)X_n(i_2,j_2)X_n(i_3,j_3)X_n(i_4,j_4)\} \\ = \;& \frac{1}{n^4}\sum_{A_i} Cum\{X_n(i_1,j_1),X_n(i_2,j_2),X_n(i_3,j_3),X_n(i_4,j_4)\} \\ & +\frac{1}{n^4}\sum_{A_i}\{R(i_1-i_2,j_1-j_2)R(i_3-i_4,j_3-j_4) \\ & +R(i_1-i_3,j_1-j_3)R(i_2-i_4,j_2-j_4) \\ & +R(i_1-i_4,j_1-j_4)R(i_2-i_3,j_2-j_3)\}\ , \end{aligned}$$

where $A_i,\ i = 1,2,3.$
Using the Brillinger condition for spatial data (2) we can obtain

$$\begin{aligned} & \frac{1}{n^4}\sum_{A_1} E\{X_n(i_1,j_1)X_n(i_2,j_2)X_n(i_3,j_3)X_n(i_4,j_4)\} \\ & \le C_1'\left(\frac{k_1}{n}-\frac{k_2}{n}\right)^2\left(\frac{l_1}{n}-\frac{l_2}{n}\right)^2 \\ & \le C_1\left\{\left(\frac{k_1}{n}-\frac{k_2}{n}\right)^2+\left(\frac{l_1}{n}-\frac{l_2}{n}\right)^2\right\}\ , \end{aligned}$$

$$\frac{1}{n^4}\sum_{A_2} E\{X_n(i_1,j_1)X_n(i_2,j_2)X_n(i_3,j_3)X_n(i_4,j_4)\} \le C_2\left(\frac{l_1}{n}-\frac{l_2}{n}\right)^2\ ,$$
$$\frac{1}{n^4}\sum_{A_3} E\{X_n(i_1,j_1)X_n(i_2,j_2)X_n(i_3,j_3)X_n(i_4,j_4)\} \le C_3\left(\frac{k_1}{n}-\frac{k_2}{n}\right)^2\ ,$$

where C_1, C_2 and C_3 are not dependent on t_1, s_1, t_2, s_2 and n.
Therefore, we can choose C independent of $t_1,\ s_1,\ t_2,\ s_2$ and n such that

$$E\left\{\frac{1}{n}S_{X_n}(k_1,l_1)-\frac{1}{n}S_{X_n}(k_2,l_2)\right\}^4 \le C\left\{\left(\frac{k_1}{n}-\frac{k_2}{n}\right)^2+\left(\frac{l_1}{n}-\frac{l_2}{n}\right)^2\right\}\ .$$

If the process $\{Z_X(t,s),\ t,s\in[0,1]\}$ is defined by

$$Z_X(t,s) = \{4\pi^2 f(0,0)\}^{\frac{1}{2}} Z(t,s)\ ,$$

where $Z(t,s)$ is a Brownian sheet and if W_X is the measure in $C[0,1]^2$ corresponding to $Z_X(\cdot,\cdot)$, then we have the following result.

Theorem 2: Under assumption (1) and (2),

$$P_{X_n} \Longrightarrow W_X \ .$$

Proof: Theorem 1 assures us that the finite dimensional distributions of P_{X_n} converge to those of W_X, and (3) implies that the sequence P_{X_n} $(n = 1, 2, \ldots)$ is tight. The proof is completed by applying Theorem 12.3 of Billingsley (1968).

4 The Regression Residual Process for Stationary Spatial Error Structure

We now consider the matrix array of partial sums of regression residuals when the error process is a stationary spatial series. The vector of regressor functions evaluated at (t, s) is denoted by $\mathbf{g}(t, s) = (g_0(t, s), \ldots, g_p(t, s))'$. It may be seen that the matrix

$$\lim_{n\to\infty} \frac{1}{n^2}(\mathbf{A}_n'\mathbf{A}_n) \equiv G$$

has as its (i, j)th component

$$\int_0^1 \int_0^1 g_i(t, s)g_j(t, s)dtds \ .$$

The inverse of G exists provided the regressor functions are linearly independent and square integrable; with this proviso, we define a multilinear form, $g(t_1, s_1; t_2, s_2)$, as follows:

$$g(t_1, s_1; t_2, s_2) = \mathbf{g}'(t_1, s_1)G^{-1}\mathbf{g}(t_2, s_2) \ .$$

Then we define a limit process $\{Z_{Xg}(t, s),\ t, s \in [0, 1]\}$ by

$$Z_{Xg}(t, s) = Z_X(t, s) - \int_0^t \int_0^s \int_0^1 \int_0^1 g(t_1, s_1; t_2, s_2)dZ_X(t_2, s_2)dt_1ds_1 \ ,$$

where $Z_X(t, s) = \sqrt{4\pi^2 f(0, 0)}Z(t, s)$ and $Z(t, s)$ is the Brownian sheet.
The partial sum process of regression residuals is given by

$$nZ_{Xg_n}(t, s) = \mathbf{e}_{nt,ns}'\{\mathbf{I} - \mathbf{A}_n(\mathbf{A}_n'\mathbf{A}_n)^{-1}\mathbf{A}_n'\}\mathbf{X}_n \ .$$

Theorem 3: Assume conditions (1) and (2). Further assume $g_k(t, s)$ $(k = 0, 1, \ldots, p)$ are linearly independent non-stochastic regressor functions that are continuously differentiable on $[0, 1]^2$. Then

$$Z_{Xg_n}(t, s) \Longrightarrow Z_{Xg}(t, s) \ .$$

It can be shown that

$$E\{Z_{Xg}(t,s)\} = Z_{Xg}(0,0) = 0,\ \ t,s \in [0,1]$$

and that the covariance kernel for any $t_1, s_1, t_2, s_2 \in [0,1]$ is

$$\begin{aligned} K(t_1,s_1;t_2,s_2) &= E\{Z_g(t_1,s_1)Z_g(t_2,s_2)\} \\ &= 4\pi^2 f(0,0)\{(t_1 \wedge t_2)(s_1 \wedge s_2) \\ &\quad - \int_0^{t_1}\int_0^{s_1}\int_0^{t_2}\int_0^{s_2} g(t_1,s_1;t_2,s_2)dt_1 ds_1 dt_2 ds_2\} \ . \end{aligned}$$

5 Effect of Spatial Autocorrelation on Change Detection Statistics

The class of possible boundaries we use to illustrate the methodology consists of those of rectangular shape with sides parallel to the sides of the unit square and with one corner having coordinates $(0,0)$. For the case of i.i.d error structure with $\sigma^2 < \infty$, a statistic for detecting change at one of these unknown boundaries in regression parameters is shown to be

$$Q_{g_n} = \frac{1}{n^4\sigma^2}\sum_{l=1}^{n}\sum_{k=1}^{n}\left\{\sum_{i=1}^{l}\sum_{j=1}^{k}[Y_n(i,j) - \hat{Y}_n(i,j)]\right\}^2 . \tag{4}$$

Other classes of boundaries result in change detection statistics defined by other functionals of the partial sums of the regression residuals; see Xie and MacNeill (2005) for further discussion. To make the statistic both operational and effective it is necessary to estimate σ^2 with an estimator that is consistent under both null and alternative hypotheses. Now assume the spatial error process is not i.i.d and $R(0,0)$ is used in place of σ^2. Note that

$$R(0,0) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2 \ .$$

Then, if $Z_g(t,s) = \{4\pi^2 f(0,0)\}^{-1/2} Z_{Xg}(t,s)$,

$$Q_{g_n} \to \frac{4\pi^2 f(0,0)}{\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2}\int_0^1\int_0^1 Z_g^2(t,s)dtds \ . \tag{5}$$

The distribution of $\int_0^1\int_0^1 Z_g^2(t,s)dtds$ is tabulated by Xie and MacNeill (2005). The above results indicate that the large sample effects of spatial correlation on Q_{g_n} can be adjusted for precisely by multiplying the quantiles of distributions for the i.i.d case by

$$\frac{4\pi^2 f(0,0)}{\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2} \ .$$

The same adjustment may be applied to any change-boundary statistic that is defined in terms of the squares of the partial sums of the residuals.

Example: If the noise process is a spatial unilateral multiplicative first-order autoregression (Martin (1979)), that is, for $|a| < 1$ and $|b| < 1$ and $\{\epsilon(t,s), s, t = 0, \pm 1, \ldots\}$ iid,

$$X(t,s) + aX(t-1,s) + bX(t,s-1) + abX(t-1,s-1) = \epsilon(t,s) \ ,$$

then we have

$$f(\lambda_1,\lambda_2) = \frac{\sigma^2}{4\pi^2}(1 + 2a\cos\lambda_1 + a^2)^{-1}(1 + 2b\cos\lambda_2 + b^2)^{-1}$$

and

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2 \ = \frac{\sigma^2}{(1-a^2)(1-b^2)} \ .$$

Hence

$$\frac{4\pi^2 f(0,0)}{\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2} = \frac{(1-a)(1-b)}{(1+a)(1+b)} \ .$$

Note that as $|a|$ approaches 1 and/or $|b|$ approaches 1 the adjustment to change-boundary statistics required to account for spatial correlation becomes highly significant and should not be ignored.

6 Extension to Higher Dimensional Spaces

We first define the basic model for d-dimensional spaces. Let $X_{n_1,\ldots,n_d}(n_1,\ldots,n_d = 0,\pm 1,\ldots)$ be a zero mean, stationary spatial series defined on a lattice with covariance function

$$R(u_1,\ldots,u_d) = E\{X(t_1,\ldots,t_d)X(t_1+u_1,\ldots,t_d+u_d)\}, \quad |u_i| < \infty \ .$$

If the covariance function is absolutely summable, i.e.,

$$\sum_{u_1=-\infty}^{\infty}\cdots\sum_{u_d=-\infty}^{\infty}|R(u_1,\ldots,u_d)| < \infty \ ,$$

then the spectral density function,

$$f(\lambda_1,\ldots,\lambda_d) = \frac{1}{(2\pi)^d}\sum_{|u_1|<\infty}\cdots\sum_{|u_d|<\infty} e^{-i\sum_{i=1}^{d}\lambda_i u_i}R(u_1,\ldots,u_d) \ , \quad \lambda_i \in [-\pi,\pi] \ ,$$

exists. Also

$$R(0,\ldots,0)=\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}f(\lambda_1,\ldots,\lambda_d)\prod_{i=1}^{d}d\lambda_d\ .$$

If, analogous to (4), we have the d-dimensional case with

$$Z_{gd}(t_1,\ldots,t_d)=\{(2\pi)^d f(0,\ldots,0)\}^{-1/2}Z_{X_{gd}}(t_1,\ldots,t_d)\ ,$$

then the change-boundary quadratic form converges as follows,

$$Q_{gd_n}\to\frac{(2\pi)^d f(0,\ldots,0)}{\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}f(\lambda_1,\ldots,\lambda_d)\prod_{i=1}^{d}d\lambda_i}\int_0^1\cdots\int_0^1 Z_{gd}^2(t_1,\ldots,t_d)\prod_{i=1}^{d}dt_i$$

where Z_{gk} is a d-dimensional Brownian sheet. Thus, the distributional results for the stationary case can be found from the i.i.d. case by a simple precise adjustment.

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