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Optimal and Partially Efficiency Balanced Designs

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Abstract

In this investigation, we have shown that most of the optimal designs for comparing test treatments with a control are partially efficiency balanced (PEB) designs. We prove it using the $\mathbf{M_o}$-matrix of the design. Further, we established that some of the PEB designs are simple PEB designs. We also point out that some of the optimal S-type BTIB designs available in the literature are proper efficiency balanced designs.

Keywords and Phrases: Partially Efficiency Balanced (PEB) Design, Efficiency Balanced (EB) Design, Simple PEB Design, Balanced Treatment Incomplete Block (BTIB) Design, Group Divisible Treatment (GDT) Design.

AMS Classification: 62K05, 62K10.

1 Introduction

In the literature of design of experiments the concept of balancing designs has been put forth differently in different contexts. In the case of incomplete block design, a design is said to be balanced if the variance of the estimate of each of the possible elementary treatment contrast is the same. This concept of balance is known as variance balance. A design is said to be efficiency balanced if the variance of the estimate of an elementary contrast of two treatment effects is proportional to the sum of reciprocal of the replication of the concerned two treatments in the design.

In factorial designs the concept of balance is related to confounded interactions. If the loss of information of each of the confounded interactions of a given order is the same, say i, though i may differ from order to order of the interactions, then such confounded factorial designs are known as balanced factorial designs. Again, in confounded asymmetrical factorial designs, if the loss of information on each degree of freedom of each of the confounded interactions is the same, say j, where j may differ from interaction to interaction, then such confounded asymmetrical factorial designs are called balanced. Balanced incomplete block designs available in the literature are either variance-balanced or efficiency-balanced. The variance-balanced designs can have both equal and unequal number of replications and block sizes.

With the introduction of efficiency-balanced designs through the work of Calinski (1971), Puri and Nigam (1975), Williams (1975), Kageyama (1981, 82) and others, the concept of balance has undergone a change. Once such a change gains ground, further modifications are likely to follow. Das and Ghosh (1985) introduced a more general definition of balance in designs such that all the existing concepts of balance of incomplete block design become its special cases. They showed that the efficiency balanced designs as reinforced incomplete block designs.

Blocking is an experimental technique commonly used in agricultural, industrial, and biological experiments to eliminate heterogeneity in one direction. In any experimental situation requiring usage of a block design, it is desirable to maximize the amount of information gained on the treatments being studied by using an optimal block design. Let d be a block design having v treatments arranged in b blocks of size $k(v > k)$ called an incomplete block design. Then d has associated with it a $v \times b$ matrix $\mathbf{N_d}$ whose entries n_{dij} give the number of times the ith treatment occurs in the jth block. When $n_{dij} = 1$ or 0 for all i, j, the design is said to be binary. The ith row of $\mathbf{N_d}$ is denoted by r_{di} and represents the number of times treatment i is replicated in the design. The matrix $\mathbf{N_d}\mathbf{N_d^{'}}$ where $\mathbf{N_d^{'}}$ is the transpose of $\mathbf{N_d}$ is referred as the concurrence matrix of d, and its entries are denoted by λ_{dij}.

The mathematical model which is usually used to analyze the data obtained from d is the two-way additive model. This model specifies that all observations y_{mn} (the observation obtained after applying the m^{th} treatment to a unit occurring in the n^{th} block) are uncorrelated, have constant variance, and have expectation $\alpha_m + \beta_n$, where α_m and β_n are unknown parameters representing the effects of the m^{th} treatment and n^{th} block, respectively. Let $\mathbf{T_d}$ and $\mathbf{B_d}$ denote vectors of treatment and block totals respectively, then the reduced normal equation for estimating the treatment effects in d can be written in matrix form as

$$\mathbf{C_d}\alpha = \mathbf{T_d} \text{ -}(1/k)\mathbf{N_d}\mathbf{B_d}$$

where

$$\mathbf{C_d} = \text{diag}(r_{d1}, ..., r_{dv}) \text{ - } \mathbf{N_d} \text{ diag}(1/k_1, 1/k_2, ..., 1/k_b)\mathbf{N_d}^{'}$$

$\alpha' = (\alpha_1, ..., \alpha_v)$ and diag$(r_{d1}, ..., r_{dv})$ denotes a $v \times v$ diagonal matrix. The matrix $\mathbf{C_d}$ is called the information matrix or $\mathbf{C}$-matrix of d and is positive semi-definite with zero row sums. In the subsequent sections, we need to find the $\mathbf{M_o}$-matrix where;

$$\mathbf{M_o} = \mathbf{M} - \mathbf{J} \ (r_o, r_1, ..., r_v)^{'}/n,$$

where $\mathbf{M} = \text{diag}(1/r_o, ..., 1/r_v)$ - $(\mathbf{N_d} \text{ diag}(1/k_1, 1/k_2, ..., 1/k_b)\mathbf{N_d}^{'})$.
Under the two-way additive model given above for d, it is well known that a necessary condition for a linear combination $\sum_{i=1}^{v}\mathbf{c_i}\ \alpha_i$ of the treatment effects to be estimable is that $\sum_{i=1}^{v}\mathbf{c_i} = 0$. Such a linear combination of the treatment effects is called a treatment contrast. A contrast of the form $\alpha_i - \alpha_j$ is called a treatment difference. A design is said to be connected provided all possible treatment differences are estimable. Alternatively, it can be shown that a design d is connected if and only if its $\mathbf{C}$-matrix has rank $v - 1$. Since connectedness is a desirable property for most block designs to have, only such designs are considered in this investigation. Let $D(v, b, k)$ denotes the class of all connected block designs having v treatments arranged in b blocks of size k.

In this investigation, we classify the optimal designs into three categories namely (i) $r_o = bt$ $(R - type)$ (ii) $r_o > bt$ $(S - type)$ and (iii) $r_o < bt$ $(S - type)$. Again we have shown that most of the optimal designs for comparing test treatments with a control are partially efficiency balanced (PEB) designs. We prove it using the $\mathbf{M_o}$-matrix of the design. Further, we established that some of the PEB designs are simple PEB designs. We also point out that some of the optimal $(S-)$ type BTIB designs available in the literature are proper efficiency balanced designs.

2 Optimal PEB Designs

Block designs are widely used in many fields of research. A wide range of "balanced" and "partially balanced" incomplete block designs are available in the literature. However, most of the known designs are restricted to equal numbers of replications and block sizes. The practical considerations often dictate the use of varying replicate and varying block-sized designs. We shall here consider a class of incomplete block designs called PEB designs introduced by Puri and Nigam (1977). Their designs are available in varying replicates and/or varying block sizes, and thus give experimenter's more freedom in designing experiments in unconventional circumstances.

Definition 2.1. A design $d(v,b,k,r)$ is said to be a partially efficiency balanced (PEB) design with m-efficiency classes if

(i) there exists a set of $(v-1)$ linearly independent contrasts s_i, $i = 1,2,...,m$ such that ρ_i of them satisfy the equation

$$\mathbf{M_o} s_{ij} = \mu_i s_{ij},\ i = 1,2,...,m; j = 1,2,..,\rho_i$$

so that the efficiency factor associated with every contrast of the i^{th} class is $(1-\mu_i)$ where $\mu_i(i = 1,2,...,m)$ are eigen values of $\mathbf{M_o}$ with multiplicities $\rho_i(\Sigma\rho_i = v-1)$, and

(ii) there exists mutually orthogonal idempotent matrices $L_i(i = 1,2,...,m)$ of ranks ρ_i such that

$$\mathbf{M_o}{=}\Sigma_{i=1}^{m}\mu_i L_i, \text{ and } \Sigma_{i=1}^{m} L_i{=}\mathbf{I}-\mathbf{Jr}'/n$$

The parameters of PEB design with m-efficiency classes may now be written as $v,b,r,k,\mu_i,\rho_i,L_i(i = 1,2,...,m)$.
This definition was given by Puri and Nigam (1977). The efficiency balanced (EB) design may be regarded as trivial PEB design with only one efficiency class and with the parameters $v,b,r,k,\mu,\rho = v-1$, $\mathbf{L} = (\mathbf{I}-\mathbf{Jr}'/n)$.
The balanced incomplete block (BIB) designs are also EB with $\mu = (r-\lambda)/rk$ and $\mathbf{L} = (\mathbf{I}-\mathbf{JJ}'/v)$.

Definition 2.2. A connected block design $d(v+1,b,k)$ having $v+1$ treatments arranged in b blocks of size k, binary in the test treatments, is called a balanced treatment incomplete block (BTIB) design with parameters r_o, r, λ_0 and λ_1 if:

(i) $\lambda_{01} = ...= \lambda_{0v} = \lambda_0$

(ii) $\lambda_{12} = \lambda_{13} = ...{=}\lambda_{v-1,v} = \lambda_1$

where $\lambda_{ij} = \sum_{p=1}^{b} n_{ip}n_{jp}; i,j = 0,1,2,...,v$.
This definition is due to Bechhofer and Tamhane (1981).

Definition 2.3. A connected block design $d(v+1,b,k)$ having $v+1$ treatments arranged in b blocks of size k, binary in the test treatments, is called a group divisible treatment (GDT) design with parameters m, n, λ_0, λ_1 and λ_2, if the treatments 0, 1,..., v can be partitioned into $m+1$ disjoint groups V_0, V_1,..., V_m of sizes v_0, v_1,..., v_m such that the following conditions hold:

(i) $V_0 = \{0\}$

(ii) v_1=...= $v_m = n$

(iii)$\lambda_{0j} = \lambda_0$ for j = 1,..., v.

(iv) For $p, q \in V_i$ $(p \neq q;\ i = 1, ..., m)$, $\lambda_{pq} = \lambda_1$.

(v) For $p \in V_i$, $q \in V_j$ $(i, j = 1, ..., m; i \neq j)$, $\lambda_{pq} = \lambda_2$.

This definition is due to Jacroux (1989).

Definition 2.4. Let the design $d(v+1, b, k; t, s)$ be a BTIB design, where integers $t \in \{0, 1, ..., k-1\}$ and $s \in \{0, 1, ..., b-1\}$, such that

(i) $n_{ij} \in \{0, 1\}$, $i = 1, 2, ..., v$; $j = 1, ..., b$;

(ii) $n_{01} = ... = n_{0s} = t + 1$;

(iii) $n_{0,s+1} = ... = n_{0b} = t$

when $s > 0$, the BTIB $(v, b, k; t, s)$ is called a Step $(S-)$ type $(r_o > bt$ and $r_o < bt)$ design.
This definition is due to Cheng et al.(1988).

Definition 2.5. A design $d(v + 1, b, k; t, s)$ is called a $(R-)$ type (Rectangular Type: $r_o = bt$) design when $s = 0$. In $(R-)$ type design, control treatment is replicated same number of times in all the blocks.

For the evaluation of the eigen values of the $\mathbf{M_o}$-matrix, the following Lemma (Mukerjee and Kageyama, 1990) is useful.

Lemma 2.1. *Let $u, s_1, s_2, ..., s_u$ be positive integers, and consider the $s \times s$ matrix*

$$\mathbf{A} = \begin{pmatrix} a_1\mathbf{I}_{s_1} + b_{11}\mathbf{J}_{s_1s_1} & b_{12}\mathbf{J}_{s_1s_2} & \cdots & b_{1u}\mathbf{J}_{s_1s_u} \\ b_{21}\mathbf{J}_{s_2s_1} & a_2\mathbf{I}_{s_2} + b_{22}\mathbf{J}_{s_2s_2} & \cdots & b_{2u}\mathbf{J}_{s_2s_u} \\ \vdots & \vdots & \cdots & \vdots \\ b_{u1}\mathbf{J}_{s_us_1} & b_{u2}\mathbf{J}_{s_us_2} & \cdots & a_u\mathbf{I}_{s_u} + b_{uu}\mathbf{J}s_us_u \end{pmatrix}$$

where $s = s_1 + s_2 + ... + s_u$ and $u \times u$ matrix $\mathbf{B}$= (b_{ij}) is symmetric. Then the eigen values of $\mathbf{A}$ are a_i with multiplicity $s_i - 1(1 \leq i \leq u)$ and $\mu_1^*, ..., \mu_u^*$, where $\mu_1^*, ..., \mu_u^*$ are the eigen values of $\mathbf{\Delta} = \mathbf{D_a} + \mathbf{D_s^{1/2}BD_s^{1/2}}$, $\mathbf{D_a}$ = diag $(a_1, ..., a_u)$, $\mathbf{D_s}$ = diag $(s_1, s_2, ..., s_u)$, $\mathbf{D_s^{1/2}}$ = diag $(s_1^{1/2}, s_2^{1/2}, ..., s_u^{1/2})$.

Theorem 2.1. *A Rectangular ($R-$) type A-optimal or MV- optimal BTIB design is a PEB design, provided the design d has two types of eigen values for its* $\mathbf{M_o}$*-matrix.*

Proof. For any ($R-$) type design $d(v+1, b, k; t, 0)$, the $\mathbf{N_d N'_d}$ matrix is given by the expression:

$$\mathbf{N_d N'_d} = \left(\begin{array}{ccc} r_o t & | & \lambda_o \mathbf{J}_{1\times v} \\ --- & - & --- \\ \lambda_o \mathbf{J}_{v\times 1} & | & \lambda_1 \mathbf{J}_{vv} + (r-\lambda_1)\mathbf{I}_v \end{array} \right)$$

where r_o and r denote the replications of the control treatment and the test treatments respectively in $d(v+1, b, k+t)$, t the replication of the control treatment in each block, $\mathbf{J}$ a vector with all unity and $\mathbf{I}$ an identity matrix. Also λ_o and λ_1 are defined in the definition of BTIB designs.
The $\mathbf{M}$-matrix of the design is given by the formula:
$\mathbf{M}$ = diag $(1/r_o, 1/r_1, ..., 1/r_v)$ ($\mathbf{N}$ diag $(1/k_1, 1/k_2, ..., 1/k_b)\mathbf{N}'$), and it is of the form:

$$M = \left(\begin{array}{ccc} \frac{t}{k+t} & | & \frac{\lambda_o}{r_o(k+t)}\mathbf{J}_{1\times v} \\ ---- & - & -------- \\ \frac{\lambda_o}{r(k+t)}\mathbf{J}_{v\times 1} & | & \frac{r-\lambda_1}{r(k+t)}\mathbf{I}_v + \frac{\lambda_1}{r(k+t)}\mathbf{J}_{vv} \end{array} \right)$$

The $\mathbf{M_o}$- matrix is given by the expression:
$\mathbf{M_o} = \mathbf{M}$ - $\mathbf{J}$ $(r_o, r_1, ..., r_v)'/n$ where $\mathbf{J}$ is a column vector of one's of order $v+1$ and n is the total number of units, and it is of the form:

$$\mathbf{M_o} = \left(\begin{array}{ccc} \frac{t}{k+t} - \frac{r_o}{b(k+t)} & | & (\frac{\lambda_o}{r_o(k+t)} - \frac{r}{b(k+t)})\mathbf{J}_{1\times v} \\ ------- & - & ------------ \\ (\frac{\lambda_o}{r(k+t)} - \frac{r_o}{b(k+t)})\mathbf{J}_{v\times 1} & | & \frac{r-\lambda_1}{r(k+t)}\mathbf{I}_v + (\frac{\lambda_1}{r(k+t)} - \frac{r}{b(k+t)})\mathbf{J}_{vv} \end{array} \right)$$

Using Lemma 2.1, the eigen values of $\mathbf{M_o}$-matrix are $\mu_1 = \frac{r-\lambda_1}{r(k+t)}$ with multiplicity $(v-1)$, and μ_2=trace($\mathbf{M_o}$) - $(v-1)\mu_1$ with multiplicity one.

So, any ($R-$) type BTIB design is partially efficiency balanced (PEB) with efficiency factor $(1-\mu_1)$ with multiplicity $(v-1)$ and $(1-\mu_2)$ with multiplicity one. Therefore, we conclude that a Rectangular ($R-$) type A-optimal or MV- optimal BTIB design is a PEB design. This completes the proof.

Example 2.1. Consider the A-optimal BTIB design (8, 7, 4)given by Hedayat, Jacroux, and Majumdar (1988):

Table 2.1: $(R-)$ type A-optimal BTIB design with blocks shown in columns

$$d = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 5 & 6 & 7 & 1 \\ 4 & 5 & 6 & 7 & 1 & 2 & 3 \end{pmatrix}$$

In this design $r_o = 7$, $r = 3$, $\lambda_o = 3$, $\lambda_1 = 1$ and $t = 1$.
The concurrence matrix $\mathbf{N_d N_d'}$ is given by:

$$\mathbf{N_d N_d'} = \begin{pmatrix} 7 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 1 & 1 & 1 & 1 & 1 & 1 \\ 3 & 1 & 3 & 1 & 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 3 & 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 1 & 3 & 1 & 1 & 1 \\ 3 & 1 & 1 & 1 & 1 & 3 & 1 & 1 \\ 3 & 1 & 1 & 1 & 1 & 1 & 3 & 1 \\ 3 & 1 & 1 & 1 & 1 & 1 & 1 & 3 \end{pmatrix}$$

And the $\mathbf{M_o}$-matrix is given by:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.1429 & -.0238 & -.0238 & -.0238 & -.0238 & -.0238 & -.0238 \\ 0 & -.0238 & 0.1429 & -.0238 & -.0238 & -.0238 & -.0238 & -.0238 \\ 0 & -.0238 & -.0238 & 0.1429 & -.0238 & -.0238 & -.0238 & -.0238 \\ 0 & -.0238 & -.0238 & -.0238 & 0.1429 & -.0238 & -.0238 & -.0238 \\ 0 & -.0238 & -.0238 & -.0238 & -.0238 & 0.1429 & -.0238 & -.0238 \\ 0 & -.0238 & -.0238 & -.0238 & -.0238 & -.0238 & 0.1429 & -.0238 \\ 0 & -.0238 & -.0238 & -.0238 & -.0238 & -.0238 & -.0238 & 0.1429 \end{pmatrix}$$

The eigen values of $\mathbf{M_o}$-matrix of the design d are $\mu_1 = 0.1667$ with multiplicity 6 $(= v - 1)$ and $\mu_2 = 0$ with multiplicity one.
So, the design is partially efficiency balanced (PEB) with efficiency factors:

(i) $1 - \mu_1 = 0.8333$ with multiplicity $(v - 1) = 6$ and

(ii)$1 - \mu_2 = 1.0$ with multiplicity 1.

Theorem 2.2. *A Step $(S-)$ type $(r_o < bt)$ A-optimal or $MV-$ optimal BTIB design is a PEB design, provided the design d has two types of eigen values for its $\mathbf{M_o}$-matrix.*

Proof. For any $(S-)$ type $(r_o < bt$ design $d(v + 1, b, k; t, s)$, the concurrence matrix $\mathbf{N_d N_d'}$ is given by the expression:

$$\mathbf{N_d N_d'} = \begin{pmatrix} r_o & | & \lambda_o \mathbf{J}_{1\times v} \\ --- & - & --- \\ \lambda_o \mathbf{J}_{v\times 1} & | & \lambda_1 \mathbf{J}_{vv} + (r-\lambda_1)\mathbf{I}_v \end{pmatrix}$$

where r_o and r denote the replications of the control treatment and the test treatments respectively in $d(v+1,b,k)$, $\mathbf{J}$ a vector with all unity and $\mathbf{I}$ an identity matrix. Also λ_o and λ_1 are defined in the definition of BTIB designs.
The $\mathbf{M}$-matrix of the design is given by the formula:
$\mathbf{M}$ = diag $(1/r_o, 1/r_1, ..., 1/r_v)$ ($\mathbf{N}$ diag $(1/k_1, 1/k_2, ..., 1/k_b)\mathbf{N}'$), and it is of the form:

$$\mathbf{M} = \begin{pmatrix} \frac{1}{k} & | & \frac{\lambda_o}{r_o k}\mathbf{J}_{1\times v} \\ ---- & - & ------- \\ \frac{\lambda_o}{rk}\mathbf{J}_{v\times 1} & | & \frac{r-\lambda_1}{rk}\mathbf{I}_v + \frac{\lambda_1}{rk}\mathbf{J}_{vv} \end{pmatrix}$$

The $\mathbf{M_o}$-matrix is given by the expression:
$\mathbf{M_o} = \mathbf{M}$ - $\mathbf{J}$ $(r_o, r_1, ..., r_v)'/n$ where $\mathbf{J}$ is a column vector of one's of order $v+1$ and n is the total number of units, and it is of the form:

$$\mathbf{M_o} = \begin{pmatrix} \frac{1}{k} - \frac{r_o}{bk} & | & (\frac{\lambda_o}{r_o k} - \frac{r}{bk})\mathbf{J}_{1\times v} \\ ----- & - & ---------- \\ (\frac{\lambda_o}{rk} - \frac{r_o}{bk})\mathbf{J}_{v\times 1} & | & \frac{r-\lambda_1}{rk}\mathbf{I}_v + (\frac{\lambda_1}{rk} - \frac{r}{bk})\mathbf{J}_{vv} \end{pmatrix}$$

Using Lemma 2.1, the eigen values of $\mathbf{M_o}$-matrix are $\mu_1 = \frac{r-\lambda_1}{rk}$ with multiplicity $(v-1)$, and μ_2=trace($\mathbf{M_o}$) - $(v-1)\mu_1$ with multiplicity one.

So, any $(S-)$ type BTIB design is partially efficiency balanced (PEB) with efficiency factor $(1-\mu_1)$ with multiplicity $(v-1)$; and $(1-\mu_2)$ with multiplicity one. Therefore, we conclude that $(S-)$ type A-optimal or MV- optimal BTIB design is a PEB design. This completes the proof.

Example 2.2. Consider the MV-optimal $(S-)$ type BTIB design (7,11,3; 0, 9) given by Hedayat, Jacroux and Majumdar (1988):

Table 2.2: $MV-$optimal $(S-)$ type BTIB design with blocks shown in columns

$$d = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 \\ 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 2 & 5 \\ 4 & 5 & 6 & 4 & 5 & 6 & 4 & 5 & 6 & 3 & 6 \end{pmatrix}$$

For this design $b = 11$, $r_o = 9$, k = 3, $r = 4$, $t = 0, \lambda_o = 3$, $\lambda_1 = 1$, and $r_o < b$.
The concurrence matrix $\mathbf{N_d N_d'}$ is given by:

$$\mathbf{N_d N_d'} = \begin{pmatrix} 9 & 3 & 3 & 3 & 3 & 3 & 3 \\ 3 & 4 & 1 & 1 & 1 & 1 & 1 \\ 3 & 1 & 4 & 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 4 & 1 & 1 & 1 \\ 3 & 1 & 1 & 1 & 4 & 1 & 1 \\ 3 & 1 & 1 & 1 & 1 & 4 & 1 \\ 3 & 1 & 1 & 1 & 1 & 1 & 4 \end{pmatrix}$$

The $\mathbf{M_o}$- matrix is given by:

$$\mathbf{M_o} = \begin{pmatrix} 0.0606 & -.0101 & -.0101 & -.0101 & -.0101 & -.0101 & -.0101 \\ -.0227 & 0.2121 & -.0379 & -.0379 & -.0379 & -.0379 & -.0379 \\ -.0227 & -.0379 & 0.2121 & -.0379 & -.0379 & -.0379 & -.0379 \\ -.0227 & -.0379 & -.0379 & 0.2121 & -.0379 & -.0379 & -.0379 \\ -.0227 & -.0379 & -.0379 & -.0379 & 0.2121 & -.0379 & -.0379 \\ -.0227 & -.0379 & -.0379 & -.0379 & -.0379 & 0.2121 & -.0379 \\ -.0227 & -.0379 & -.0379 & -.0379 & -.0379 & -.0379 & 0.2121 \end{pmatrix}$$

The eigen values of $\mathbf{M_o}$-matrix of the design d are $\mu_1 = 0.25$ with multiplicity 5 $(= v - 1)$ and $\mu_2 = 0.083$ with multiplicity one.

So, the design is partially efficiency balanced (PEB) with efficiency factors:

(i) $1 - \mu_1 = 0.75$ with multiplicity $(v - 1) = 5$ and

(ii) $1 - \mu_2 = 0.917$ with multiplicity 1.

Theorem 2.3. *A Step $(S-)$ type $(r_o > bt)$ $A-$optimal or $MV-$ optimal BTIB design is a PEB design, provided the design d has two types of eigen values for its $\mathbf{M_o}$-matrix.*

Proof. For any $(S-)$ type $(r_o > bt)$ design $d(v + 1, b, k; t, s)$, the concurrence matrix $\mathbf{N_d N_d'}$ is given by the expression:

$$\mathbf{N_d N_d'} = \begin{pmatrix} s(t+1)^2 - (b-s)t^2 & | & \lambda_o \mathbf{J}_{1\times v} \\ ---------- & - & -------- \\ \lambda_o \mathbf{J}_{v\times 1} & | & (r - \lambda_1)\mathbf{I}_v + \lambda_1 \mathbf{J}_{vv} \end{pmatrix}$$

where r_o and r denote the replications of the control treatment and the test treatments respectively in $d(v + 1, b, k)$, $\mathbf{J}$ a vector with all unity and $\mathbf{I}$ an identity matrix. Also $\lambda_o$and λ_1 are defined in the definition of BTIB design.
The $\mathbf{M}-$matrix of the design is given by the formula:
$\mathbf{M} =$ diag $(1/r_o, 1/r_1, ..., 1/r_v)$ ($\mathbf{N}$ diag $(1/k_1, 1/k_2, ..., 1/k_b)\mathbf{N}'$), and it is of the form:

$$\mathbf{M} = \left(\begin{array}{ccc} \frac{s(t+1)^2-(b-s)t^2}{kr_o} & | & \frac{\lambda_o}{r_o k}\mathbf{J}_{1\times v} \\ ------- & - & ------- \\ \frac{\lambda_o}{rk}\mathbf{J}_{v\times 1} & | & \frac{r-\lambda_1}{rk}\mathbf{I}_v + \frac{\lambda_1}{rk}\mathbf{J}_{vv} \end{array} \right)$$

The $\mathbf{M_o}$-matrix is given by the expression:
$\mathbf{M_o} = \mathbf{M}$ - $\mathbf{J}$ $(r_o, r_1, ..., r_v)'/n$ where $\mathbf{J}$ is a column vector of one's of order $v+1$ and n is the total number of units, and it is of the form:

$$\mathbf{M_o} = \left(\begin{array}{ccc} \frac{s(t+1)^2-(b-s)t^2}{kr_o} - \frac{r_o}{bk} & | & (\frac{\lambda_o}{r_o k} - \frac{r}{bk})\mathbf{J}_{1\times v} \\ --------- & - & --------- \\ (\frac{\lambda_o}{rk} - \frac{r_o}{bk})\mathbf{J}_{v\times 1} & | & \frac{r-\lambda_1}{rk}\mathbf{I}_v + (\frac{\lambda_1}{rk} - \frac{r}{bk})\mathbf{J}_{vv} \end{array} \right)$$

Using Lemma 2.1, the eigen values of $\mathbf{M_o}$-matrix are $\mu_1 = \frac{r-\lambda_1}{rk}$ with multiplicity $(v-1)$, and μ_2=trace($\mathbf{M_o}$) - $(v-1)\mu_1$ with multiplicity one.

So, any $(S-)$ type BTIB design is partially efficiency balanced (PEB) with efficiency factor $(1-\mu_1)$ with multiplicity $(v-1)$; and $(1-\mu_2)$ with multiplicity one. Therefore, we conclude that step $(S-)$ type $(r_o > bt)$ A-optimal or MV-optimal BTIB design is a PEB design. This completes the proof.

Example 2.3. Consider the Optimal $(S-)$ type BTB design(4, 9, 4; 1, 3) given by Jacroux and Majumdar (1989):

Table 2.3: Optimal $(S-)$ type BTB design with blocks shown in columns

$$d = \left(\begin{array}{ccccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \end{array} \right)$$

For this design $b = 9$, $r_o = 12$, $k = 4$, $r = 8$, $t = 1$, $s = 3$, $\lambda_o = 10$ and $\lambda_1 = 7$; so $r_o > bt$ is satisfied.
The Concurrence matrix $\mathbf{N_d N_d'}$ of the design d is given by:

$$\mathbf{N_d N_d'} = \left(\begin{array}{cccc} 18 & 10 & 10 & 10 \\ 10 & 8 & 7 & 7 \\ 10 & 7 & 8 & 7 \\ 10 & 7 & 7 & 8 \end{array} \right)$$

The $\mathbf{M_o}$-matrix is given by:

$$\mathbf{M_o} = \begin{pmatrix} 0.0417 & -.0139 & -.0139 & -.0139 \\ -.0208 & 0.0278 & -.0035 & -.0035 \\ -.0208 & -.0035 & 0.0278 & -.0035 \\ -.0208 & -.0035 & -.0035 & 0.0278 \end{pmatrix}$$

The eigen values of $\mathbf{M_o}$-matrix of the design d are $\mu_1 = 0.0313$ with multiplicity 2 $(= v - 1)$ and $\mu_2 = 0.0625$ with multiplicity one.

So, the design is partially efficiency balanced (PEB) with efficiency factors:

(i) $1 - \mu_1 = 0.9687$ with multiplicity $(v - 1) = 2$ and

(ii) $1 - \mu_2 = 0.9375$ with multiplicity 1.

Theorem 2.4. *A Rectangular $(R-)$ type or Step $(S-)$ type A-optimal or MV- optimal GDT design is a PEB design provided d has three types of eigen values for its* $\mathbf{M_o}$*-matrix.*

(a) The Concurrence matrix and the C-matrix of optimal $(R-)$ type GDT design ($r_o = bt$ and $r_o < bt$)

$$\mathbf{N_d N_d'} = \begin{pmatrix} r_o t & \lambda_o \mathbf{J}_{1\times v} \\ \lambda_o \mathbf{J}_{v\times 1} & \mathbf{N_{\bar{d}} N_{\bar{d}}'} \end{pmatrix}$$

where $\mathbf{N_{\bar{d}}}$ denotes the incidence matrix of GD design.
The concurrence matrix can also be expressed in another form:

$$\mathbf{N_d N_d'} = \mathbf{D_o} + \lambda_\mathbf{o}\mathbf{D_1} + \lambda_\mathbf{1}\mathbf{D_2} + \lambda_\mathbf{2}\mathbf{D_3}$$

where $\mathbf{D_o} = \begin{pmatrix} r_o t & \lambda_o \mathbf{J}_{1\times v} \\ \lambda_o \mathbf{J}_{v\times 1} & \mathbf{O}_{v\times v} \end{pmatrix}$; $\mathbf{D_1} = \begin{pmatrix} \mathbf{O} & \mathbf{O}_{1\times v} \\ \mathbf{O}_{v\times 1} & \mathbf{B}_o \end{pmatrix}$

$$\mathbf{D_2} = \begin{pmatrix} \mathbf{O} & \mathbf{O}_{1\times v} \\ \mathbf{O}_{v\times 1} & \mathbf{B}_1 \end{pmatrix}; \ \mathbf{D_3} = \begin{pmatrix} \mathbf{O} & \mathbf{O}_{1\times v} \\ \mathbf{O}_{v\times 1} & \mathbf{B}_2 \end{pmatrix}$$

where λ_o, λ_1 and λ_2 are defined in the definition of GDT design, the matrices $\mathbf{B}_i$'s $(i = 0, 1, 2)$ are of order $(v \times v)$ called the association matrices of the association schemes of GD designs with $\mathbf{B_o} = \mathbf{I_v}$ and $\sum_{i=0}^{2} \mathbf{B}_i = \mathbf{J}_{vv}$ and all $\mathbf{D}_i$'s $(i = 0, 1, 2, 3)$ are matrices of order $(v + 1) \times (v + 1)$.
The $\mathbf{C}$-matrix is of the form:

$$\mathbf{C_d} = \begin{pmatrix} r_o(k - t)/k & -\lambda_o \mathbf{J}_{1\times v}/k \\ -\lambda_o \mathbf{J}_{v\times 1}/k & (kr_d \mathbf{I}_v - \mathbf{N_{\bar{d}} N_{\bar{d}}'})/k \end{pmatrix}$$

Example 2.4. Consider the $A-$optimal GDT design (7, 16, 4) given by Jacroux (1989), and having parameters $m = 3$, $n = 2$, $\lambda_o = 8$, $\lambda_1 = 4$ and $\lambda_2 = 3$:

Table 2.4: A-optimal $(R-)$ type GDT design with blocks shown in columns

$$d = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 4 \\ 2 & 2 & 2 & 2 & 3 & 3 & 4 & 4 & 3 & 3 & 5 & 5 & 4 & 4 & 5 & 5 \\ 3 & 4 & 5 & 6 & 5 & 6 & 5 & 6 & 4 & 4 & 6 & 6 & 5 & 6 & 6 & 6 \end{pmatrix}$$

For this design $r_o = 16$, $t = 1$, $b = 16$; so $r_o = bt$.

The eigen values of $\mathbf{M_o}$-matrix are $\mu_1 = 0.1875$ with multiplicity 2, $\mu_2 = 0.125$ with multiplicity 3 and $\mu_3 = 0$ with multiplicity 1.
So, the design is partially efficiency balanced (PEB) with efficiency factors:

(i) $1 - \mu_1 = 0.8125$ with multiplicity 2

(ii) $1 - \mu_2 = 0.875$ with multiplicity 3 and

(iii) $1 - \mu_3 = 1$ with multiplicity 1.

(b) The $\mathbf{N_d N'_d}$ matrix and the C-matrix of $(S-)$ type GDT design:

$$\mathbf{N_d N'_d} = \begin{pmatrix} s_o & \lambda_o \mathbf{J}_{1\times v} \\ \lambda_o \mathbf{J}_{v\times 1} & \mathbf{N_{\bar{d}} N'_{\bar{d}}} \end{pmatrix}$$

The **C**-matrix is of the form:

$$\mathbf{C_d} = \begin{pmatrix} s_o(k-1)/k & -\lambda_o \mathbf{J}_{1\times v}/k \\ -\lambda_o \mathbf{J}_{v\times 1}/k & (kr_d \mathbf{I}_v - \mathbf{N_{\bar{d}} N'_{\bar{d}}})/k \end{pmatrix}$$

3 Optimal Simple PEB Designs

Definition 3.1. A particular class of two-efficiency class PEB designs having $\mu_1 \neq 0$ and $\mu_2 = 0$ with multiplicities ρ_1 and $\rho_2 = v - \rho_1 - 1$ may be of special interest because of their simple analysis. Such a class of designs is termed as simple PEB or PEB(S). For such designs $\mathbf{M_o}$ is given by:

$$\mathbf{M_o} = \mu_1 \mathbf{L_1}$$

Orthogonally supplemented balanced designs considered by Calinski (1971), affine resolvable PBIB designs, semi-regular and singular group divisible PBIB designs, PBIB designs obtained through partial geometry (r, k, t), simple lattice designs and linked block designs are all PEB(S). Hence the analysis of all such PBIB designs can be

greatly simplified.
This definition is due to Puri and Nigam (1977).

Theorem 3.1. *A* $(R-)$ *type A-optimal or* MV*- optimal BTIB design is a S-PEB provided* $\mu_1 \neq 0$ *and* $\mu_2 = 0$.

Proof. Consider the Theorem 2.1.
The $\mathbf{M_o}$-matrix is given by the expression:

$$\mathbf{M_o} = \begin{pmatrix} \frac{t}{k+t} - \frac{r_o}{b(k+t)} & | & (\frac{\lambda_o}{r_o(k+t)} - \frac{r}{b(k+t)})\mathbf{J}_{1\times v} \\ ------ & - & ------------ \\ (\frac{\lambda_o}{r(k+t)} - \frac{r_o}{b(k+t)})\mathbf{J}_{v\times 1} & | & \frac{r-\lambda_1}{r(k+t)}\mathbf{I}_v + (\frac{\lambda_1}{r(k+t)} - \frac{r}{b(k+t)})\mathbf{J}_{vv} \end{pmatrix}$$

Using Lemma 2.1, the eigen values of $\mathbf{M_o}$-matrix are $\mu_1 = \frac{r-\lambda_1}{r(k+t)}$ with multiplicity $(v-1)$, and μ_2=trace($\mathbf{M_o}$) - $(v-1)\mu_1$ with multiplicity one. Here the eigen values can assume values $\mu_1 \neq 0$ and $\mu_2 = 0$. When this condition is satisfied this BTIB design is a S-PEB. This completes the proof.

Example 3.1. Consider the following A-optimal $(R-)$ type BTIB design (8, 7, 5) given by Jacroux (1989):

Table 3.1: A-optimal $(R-)$ type BTIB design with blocks shown in columns

$$d = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 2 & 2 & 1 & 1 & 1 & 1 \\ 5 & 3 & 3 & 3 & 3 & 2 & 2 \\ 6 & 6 & 4 & 5 & 4 & 5 & 4 \\ 7 & 7 & 5 & 7 & 6 & 6 & 7 \end{pmatrix}$$

For this design $v = 7$, $b = 7$, $k = 4$, $r_o = 7$, $r = 4$, $t = 1$, λ_o= 4 and λ_1= 2.
The Concurrence Matrix $\mathbf{N_d N_d'}$ is given by:

$$\mathbf{N_d N_d'} = \begin{pmatrix} 7 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\ 4 & 4 & 2 & 2 & 2 & 2 & 2 & 2 \\ 4 & 2 & 4 & 2 & 2 & 2 & 2 & 2 \\ 4 & 2 & 2 & 4 & 2 & 2 & 2 & 2 \\ 4 & 2 & 2 & 2 & 4 & 2 & 2 & 2 \\ 4 & 2 & 2 & 2 & 2 & 4 & 2 & 2 \\ 4 & 2 & 2 & 2 & 2 & 2 & 4 & 2 \\ 4 & 2 & 2 & 2 & 2 & 2 & 2 & 4 \end{pmatrix}$$

The $\mathbf{M_o}$-matrix is given by:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.0857 & -.0143 & -.0143 & -.0143 & -.0143 & -.0143 & -.0143 \\ 0 & -.0143 & 0.0857 & -.0143 & -.0143 & -.0143 & -.0143 & -.0143 \\ 0 & -.0143 & -.0143 & 0.0857 & -.0143 & -.0143 & -.0143 & -.0143 \\ 0 & -.0143 & -.0143 & -.0143 & 0.0857 & -.0143 & -.0143 & -.0143 \\ 0 & -.0143 & -.0143 & -.0143 & -.0143 & 0.0857 & -.0143 & -.0143 \\ 0 & -.0143 & -.0143 & -.0143 & -.0143 & -.0143 & 0.0857 & -.0143 \\ 0 & -.0143 & -.0143 & -.0143 & -.0143 & -.0143 & -.0143 & 0.0857 \end{pmatrix}$$

The eigen values of $\mathbf{M_o}$-matrix of the design d are $\mu_1 = 0.10$ with multiplicity 6 and $\mu_2 = 0$ with multiplicity one.
So, the A-optimal $(R-)$ type BTIB design is a simple PEB design.

Theorem 3.2. *A $(R-)$ type A-optimal or MV- optimal GDT design is a S-PEB provided $\mu_1 \neq 0$ and $\mu_2 = 0$.*

Proof. For any $(R-)$ type GDT design $d(v+1, b, k; t, 0)$ the concurrence matrix is given by the expression:

$$\mathbf{N_d N_d'} = \begin{pmatrix} r_o t & \lambda_o \mathbf{J}_{1\times v} \\ \lambda_o \mathbf{J}_{v\times 1} & \mathbf{N_{\bar{d}} N_{\bar{d}}'} \end{pmatrix}$$

where $\mathbf{N_{\bar{d}}}$ denotes the incidence matrix of a semi-regular GD design.
The concurrence matrix of a semi-regular GD design is of the form:

$$\mathbf{N_{\bar{d}} N_{\bar{d}}'} = \mathbf{r I}_v + \lambda_1 \mathbf{B}_1 + \lambda_2 \mathbf{B}_2$$

The $\mathbf{M_o}$-matrix is given by the expression:
$\mathbf{M_o} = \mathbf{M}$ - $\mathbf{J}$ $(r_o, r_1, ..., r_v)'/n$ where $\mathbf{J}$ is a column vector of one's of order $v+1$ and n is the total number of units, and it is of the form:

$$\mathbf{M_o} = \begin{pmatrix} \frac{t}{k+t} - \frac{r_o}{n} & | & (\frac{\lambda_o}{r_o(k+t)} - \frac{r}{n})\mathbf{J}_{1\times v} \\ ------- & - & -------- \\ (\frac{\lambda_o}{r(k+t)} - \frac{r_o}{n})\mathbf{J}_{v\times 1} & | & \frac{r\mathbf{I_v}+\lambda_1\mathbf{B_1}+\lambda_2\mathbf{B_2}}{r(k+t)} - \frac{r}{n}\mathbf{J}_{vv} \end{pmatrix}$$

Using Lemma 2.1, the eigen values of $\mathbf{M_o}$-matrix are:

(i) $\mu_1 = \frac{r-\lambda_1}{r(k+t)}$ with multiplicity $m(n-1)$;

(ii) $\mu_2 = \frac{t}{k+t} - \frac{r_o}{n}$ with multiplicity one, and

(iii) $\mu_3 = \frac{rk - v\lambda_2}{r(k+t)}$ with multiplicity $(m-1)$.

Now, $\mu_2 = \frac{t}{k+t} - \frac{r_o}{n}$
$= \frac{t}{k+t} - \frac{r_o}{(k+t)b}$
$= 0$(Since $r_o = b$ and $t = 1$)
And, $\mu_3 = \frac{rk - v\lambda_2}{r(k+t)}$
$= 0$ (Since, for a semi-regular GD design, $rk = v\lambda_2$)

That is, the eigen values are $\mu_1 \neq 0$, $\mu_2 = 0$ and $\mu_3 = 0$. Hence, the $(R-)$ type $A-$optimal or $MV-$optimal GDT design is a S-PEB design.
This completes the proof.

Example 3.2. Consider the A-optimal $(R-)$ type GDT design (10, 9, 4) given by Jacroux (1989), and having parameters $m = 3$, $n = 3$, $\lambda_o = 3$, $\lambda_1 = 0$ and $\lambda_2 = 1$.

Table 3.2: A-optimal $(R-)$ type GDT design with blocks shown in columns

$$d = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 \\ 4 & 5 & 6 & 4 & 5 & 6 & 4 & 5 & 6 \\ 7 & 8 & 9 & 8 & 9 & 7 & 9 & 7 & 8 \end{pmatrix}$$

For this design $v = 9$, $b = 9$, $k = 3$, $r_o = 9$, $r = 3$ and $t = 1$.
The Concurrence Matrix $\mathbf{N_d N'_d}$ is given by:

$$\mathbf{N_d N'_d} = \begin{pmatrix} 9 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 3 & 0 & 3 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 3 & 0 & 0 & 3 & 1 & 1 & 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 1 & 3 & 0 & 0 & 1 & 1 & 1 \\ 3 & 1 & 1 & 1 & 0 & 3 & 0 & 1 & 1 & 1 \\ 3 & 1 & 1 & 1 & 0 & 0 & 3 & 1 & 1 & 1 \\ 3 & 1 & 1 & 1 & 1 & 1 & 1 & 3 & 0 & 0 \\ 3 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 3 & 0 \\ 3 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 3 \end{pmatrix}$$

The $\mathbf{M_o}$-matrix is given by:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.1667 & -.0833 & -.0833 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -.0833 & 0.1667 & -.0833 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -.0833 & -.0833 & 0.1667 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.1667 & -.0833 & -.0833 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -.0833 & 0.1667 & -.0833 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -.0833 & -.0833 & 0.1667 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.1667 & -.0833 & -.0833 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -.0833 & 0.1667 & -.0833 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -.0833 & -.0833 & 0.1667 \end{pmatrix}$$

The eigen values of $\mathbf{M_o}$-matrix of the design d are $\mu_1 = 0.25$ with multiplicity 6 and $\mu_2 = 0$ with multiplicity 1 and $\mu_3 = 0$ with multiplicity 2. So, the $(R-)$ type GDT design is an optimal S-PEB design.

4 Optimal Proper Efficiency Balanced Designs

In this section, we show that some of the optimal BTIB designs are proper efficiency balanced designs. Some of the optimal non-binary BTIB designs $(r_o > b)$ or some optimal BTIB designs with $\lambda_o = \lambda_1$ $(r_o < b)$ are optimal proper efficiency balanced designs.

Definition 4.1. A design $d(v, b, k, r)$ is said to be efficiency balanced (EB) if for all treatment contrasts $s^{'}T$

$$\mathbf{M_o} s = \mu s$$

where μ is the unique nonzero eigen value of $\mathbf{M_o}$-matrix with multiplicity $(v-1)$ and

$$\mathbf{M_o} = \mathbf{R^{-1}P} - \mathbf{Jr}^{'}/n, \text{ and } \mathbf{P} = \mathbf{NK^{-1}N}^{'},$$

where $\mathbf{N}$ is the $v \times b$ incidence matrix, $\mathbf{r}$ is the $v \times 1$ vector of treatment replications, $\mathbf{k}$ is the $b \times 1$ vector of block sizes, $\mathbf{R}$ and $\mathbf{K}$ denote the diagonal matrices with diagonal elements as r and k, and $\mathbf{R^{-1}}$ and $\mathbf{K^{-1}}$ are their inverses, and n denotes the total number of units.
It follows from Jones (1959) and Calinski (1971) that if there exists a set of $(v-1)$ linearly independent contrasts (s_{ij}) such that ρ_i of them satisfy the equation

$$\mathbf{M_o} s_{ij} = \mu_i s_{ij},\ i = 1, 2, ..., m; j = 1, 2, .., \rho_i,$$

where μ_i's are the distinct eigen values of $\mathbf{M_o}$ with multiplicities ρ_i, then effective information obtained on all the ρ_i contrasts is $(1 - \mu_i)$.
This definition is given by Puri and Nigam (1975).

Theorem 4.1. *An optimal BTIB design is a proper efficiency balanced design, provided the design d has only one type of eigen values for its* $\mathbf{M_o}$*-matrix.*

Proof. For a $(S-)$ type BTIB design, the concurrence matrix $\mathbf{N_d N_d'}$ is given by the expression:

$$\mathbf{N_d N_d'} = \begin{pmatrix} s_o(t+1)-(b'-s)t & | & \lambda_o \mathbf{J}_{1\times v} \\ ---------- & - & -------- \\ \lambda_o \mathbf{J}_{v\times 1} & | & (r-\lambda_1)\mathbf{I}_v + \lambda_1 \mathbf{J}_{vv} \end{pmatrix}$$

where s_o and r denote the replications of the control treatment and the test treatments respectively in $d(v,b,k)$, b' the number of blocks containing the control treatment at t times, $\mathbf{J}$ a vector with all unity and $\mathbf{I}$ an identity matrix. Also λ_o and λ_1 are defined in the definition of BTIB design.
The $\mathbf{M}$-matrix of the design is given by the formula:
$\mathbf{M}$ = diag $(1/r_o, 1/r_1, ..., 1/r_v)$ ($\mathbf{N}$ diag $(1/k_1, 1/k_2, ..., 1/k_b)\mathbf{N}'$), and it is of the form:

$$\mathbf{M} = \begin{pmatrix} \frac{s_o(t+1)-(b'-s)t}{ks_o} & | & \frac{\lambda_o}{s_o k}\mathbf{J}_{1\times v} \\ ------ & - & ------ \\ \frac{\lambda_o}{rk}\mathbf{J}_{v\times 1} & | & \frac{r-\lambda_1}{rk}\mathbf{I}_v + \frac{\lambda_1}{rk}\mathbf{J}_{vv} \end{pmatrix}$$

The $\mathbf{M_o}$-matrix is given by the expression:
$\mathbf{M_o} = \mathbf{M}$ - $\mathbf{J}$ $(r_o, r_1, ..., r_v)'/n$ where $\mathbf{J}$ is a column vector of one's of order $v+1$ and n is the total number of units, and it is of the form:

$$\mathbf{M_o} = \begin{pmatrix} \frac{s_o(t+1)-(b'-s)t}{ks_o} - \frac{s_o}{bk} & | & (\frac{\lambda_o}{s_o k} - \frac{r}{bk})\mathbf{J}_{1\times v} \\ --------- & - & -------- \\ (\frac{\lambda_o}{rk} - \frac{s_o}{bk})\mathbf{J}_{v\times 1} & | & \frac{r-\lambda_1}{rk}\mathbf{I}_v + (\frac{\lambda_1}{rk} - \frac{r}{bk})\mathbf{J}_{vv} \end{pmatrix}$$

Using Lemma 2.1, the eigen values of $\mathbf{M_o}$-matrix are $\mu_1 = \frac{r-\lambda_1}{rk}$ with multiplicity $(v-1)$, and μ_2=trace($\mathbf{M_o}$) - $(v-1)\mu_1$ with multiplicity one.

Here μ_2 can also be equal to μ_1. When this condition is satisfied this $(S-)$ type BTIB design is efficiency balanced (EB) with efficiency factor $(1-\mu_1)$ with multiplicity v. Therefore, we conclude that $(S-)$ type BTIB designs are EB designs. This completes the proof.

Example 4.1. Consider an optimal BTIB design (6,7,4) given by Hedayat, Jacroux and Majumdar (1988).

Table 4.1: Optimal BTIB design with blocks shown in columns

$$d = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 2 & 2 & 2 \\ 1 & 3 & 3 & 4 & 3 & 4 & 3 \\ 2 & 4 & 5 & 5 & 5 & 5 & 4 \end{pmatrix}$$

For this design $b = 7$, $b' = 6$, $s_o = 8$, $s = 2$, $r = 4$, $k = 4$, $\lambda_o = 4$, $\lambda_1 = 2$ and $t = 1$. The concurrence matrix is given by:

$$\mathbf{N_d N_d'} = \begin{pmatrix} 12 & 4 & 4 & 4 & 4 & 4 \\ 4 & 4 & 2 & 2 & 2 & 2 \\ 4 & 2 & 4 & 2 & 2 & 2 \\ 4 & 2 & 2 & 4 & 2 & 2 \\ 4 & 2 & 2 & 2 & 4 & 2 \\ 4 & 2 & 2 & 2 & 2 & 4 \end{pmatrix}$$

The $\mathbf{M_o}$-matrix is given by:

$$\mathbf{M_o} = \begin{pmatrix} 0.0893 & -.0179 & -.0179 & -.0179 & -.0179 & -.0179 \\ -.0357 & 0.1071 & -.0179 & -.0179 & -.0179 & -.0179 \\ -.0357 & -.0179 & 0.1071 & -.0179 & -.0179 & -.0179 \\ -.0357 & -.0179 & -.0179 & 0.1071 & -.0179 & -.0179 \\ -.0357 & -.0179 & -.0179 & -.0179 & 0.1071 & -.0179 \\ -.0357 & -.0179 & -.0179 & -.0179 & -.0179 & 0.1071 \end{pmatrix}$$

That is, the eigen values of $\mathbf{M_o}$-matrix are $\mu_1 = 0.125$ with multiplicity 5. So, the design is efficiency balanced (EB) with efficiency factor 1 - 0.125 = 0.875.

Example 4.2. Consider an optimal BTIB design (5,7,3) given by Bechhofer and Tamhane (1981).

Table 4.2: Optimal BTIB design with blocks shown in columns

$$d = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 2 & 3 & 0 & 0 & 0 & 2 \\ 3 & 3 & 4 & 1 & 2 & 4 & 4 \end{pmatrix}$$

In this design $r_o > b$. The eigen values of $\mathbf{M_o}$-matrix are $\mu_1 = 0.22$ with multiplicity 4. So, the design is efficiency balanced (EB) with efficiency factor 1 - 0.22 = 0.78.

Example 4.3. Consider an optimal BTIB design (7,7,3) given by Bechhofer and Tamhane (1981).

Table 4.3: Optimal BTIB design with blocks shown in columns

$$d = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 2 & 3 \\ 1 & 2 & 4 & 2 & 5 & 3 & 4 \\ 3 & 6 & 5 & 4 & 6 & 5 & 6 \end{pmatrix}$$

In this design $r_o < b$ with $\lambda_o = \lambda_1$. The eigen values of $\mathbf{M_o}$-matrix are $\mu_1 = 0.22$ with multiplicity 6. So, the design is efficiency balanced (EB) with efficiency factor 1 - 0.22 = 0.78.

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