Imperfect Ranking Models for Ranked-Set Sampling

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Abstract[‡]

In this paper we propose and study models that can be used to evaluate the performance of ranked-set sampling when there are errors in the ranking process.

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1 Introduction

Ranked-set sampling is an alternative to simple random sampling for obtaining a sample of observations from a population. Compared to simple random sampling, ranked-set sampling can lead to improved estimators and tests with higher power in situations where taking an observation is expensive or destructive and there is a

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relatively inexpensive way to order the observations without taking an actual measurement. McIntyre (1952) introduced ranked-set sampling in the context of estimating total crop yields. Halls and Dell (1966) used it for estimating forage yields, Mode *et al.* (1999) estimated stream habitat area with ranked-set sampling. Estimation using ranked-set sampling was used by Cobby *et al.* (1985) for grazing effects on ryegrass and clovers, plutonium levels in soil by Gilbert (1995) and gasoline sampling by Nussbaum and Sinha (1997).

There is a large body of statistical research where the benefits of ranked-set sampling (RSS) have been investigated. For a partial review of parametric ranked-set sampling see Kaur *et al.* (1995). Bohn (1996) provides a nice review of ranked-set sampling for nonparametric procedures. The majority of the comparisons between ranked-set sampling and simple random sampling, however, have been performed under the assumption that the judgment ranking is without error. There are many situations where it is believed that ranked-set sampling is still beneficial, but perfect judgment ranking is not a realistic assumption.

In this paper we develop several models for the probability of judgment ranking errors. These models can be used to evaluate the gains of ranked-set sampling over simple random sampling when the judgment rankings are not perfect. Models for imperfect judgment rankings should accomplish several goals. First they should be flexible enough to represent different degrees of judgment error in the rankings. The models should also be able to capture the case of random rankings at one extreme and perfect rankings at the other extreme, as well as reasonably cover the range in between. In addition, they should be extendable to different set sizes so that they can be used to answer questions regarding optimal set size.

The new models developed in this paper will be extensions of the expected spacings model first developed by Bohn and Wolfe (1994) and they focus on alternative ways to specify the $\vec{P}$ matrix. These include a baseline model, modified baseline model, interval type models, and simulation models. Sections 2 and 3 describe the notation and give background on models that have been previously developed. Section 4 presents the development of the baseline and modified baseline models. Section 5 contains the development of several interval type models. This is followed by a section describing three simulation models. The last section contains a brief discussion of these models.

2 Preliminary Results

In this section we develop the notation for a ranked-set sample that is used throughout the paper. We also present the models that have already been developed in the literature for incorporating imperfect rankings into ranked-set samples.

Assume that the population of interest has cumulative distribution function (c.d.f.) F with location parameter θ (this could be the mean or the median). To obtain a ranked-set sample, the first step is to randomly select k observations from the population, where k is referred to as the set size. The k observations are completely ranked

from smallest to largest by any means other than actually measuring the observations. This is often done by judgment ordering, but could be done by other means, such as use of a concomitant, that do not involve measurement of the observations themselves. The unit believed to be the smallest, called the first judgment order statistic, is then measured and denoted by $X_{[1]}$. Then a second set of k observations is chosen randomly and the one that is judged to be second smallest is measured and denoted by $X_{[2]}$, the second judgment order statistic. This is repeated until at the kth step k observations are chosen and the one judged to be the largest is measured and denoted by $X_{[k]}$, the kth judgment order statistic. The resulting measured variables are $X_{[1]}, \ldots, X_{[k]}$. This cycle is repeated n times so that nk^2 observations have been examined and judgment ranked in groups of k each, but only nk of them have been measured. A subscript denoting the cycle is added so the first cycle results in $X_{[1]1}, \ldots, X_{[k]1}$ and the ith cycle results in $X_{[1]i}, \ldots, X_{[k]i}$. The complete, balanced ranked-set sample consisting of n cycles, each with set size k, is then: $\{X_{[r]i} : r = 1, \ldots, k; i = 1, \ldots, n\}$. Here $X_{[r]i}$, called the rth judgment order statistic from the ith cycle, is the observation judged to be the rth order statistic for one of the k sets in the ith cycle.

There have been a few previous attempts to account for imperfect rankings. Dell and Clutter (1972) developed a model based on adding random errors to the observations to represent judgment error. They let $Y_i = X_i + \epsilon_i$, where ϵ_i represents the error in the judgment ranking process. Then, with this model, $X_{[r]}$ is the X value corresponding to $Y_{(r)}$. They considered the case where $\epsilon_i \sim n(0, \sigma_\epsilon^2)$, but the distribution of the X's can be taken to be any distribution that might be reasonable in practice.

The problem with this model is that it is oversimplified for the setting of ranked-set sampling. Dell and Clutter mention that the assumption of independent, paired comparisons is not reasonable. They also note that judgment errors in the ordering process can be influenced by the set size, by the specific values of other elements in the set and the magnitudes of the elements themselves. None of these things can be taken into account via this model.

Stokes (1977) suggested that the ranking be based on a concomitant variable. If a variable that is concomitant to the variable of interest exists and the variables follow certain strict distribution assumptions, her model can be used to determine relative precisions under imperfect rankings. She assumes that a concomitant variable for the variable of interest is available and that it can be measured accurately with little or no cost. This concomitant can then be used to order the variable of interest. To illustrate, assume that the variable of interest is X, which has mean μ_X and variance σ_X^2, and the concomitant variable is Y with mean μ_Y and variance σ_Y^2. Further, assume that both $(X - \mu_X)/\sigma_X$ and $(Y - \mu_Y)/\sigma_Y$ have common c.d.f. F.

The good thing about this model is that, if it fits the situation well, there are nice theoretical results for the efficiency of ranked-set sampling relative to simple random sampling. Also, the model is able to directly incorporate information obtained through the concomitant variable into the properties of the concomitant ranked-set sampling estimator of the mean. The main problem with the model is that the assumptions

are very restrictive and it is unlikely that many practical situations will satisfy these assumptions. However, the model could be approximately true in some cases and, therefore, it may be useful.

Finally, another common model that has been developed is the expected spacings model due to Bohn and Wolfe (1994). Let p_{ij} be the probability that the item that has actual rank i in the set is chosen as the jth judgment order statistic. That is, $p_{ij} = P(X_{[j]} = X_{(i)})$. For perfect ranking $p_{ii} = 1$ and $p_{ij} = 0$ for $i \neq j$. If the ranking process is completely random then $p_{ij} = \frac{1}{k}$ for all i and j. However, in general, there is no specific structure for the p_{ij}'s. For each $j = 1, \ldots, k$, we have $\sum_{i=1}^{k} p_{ij} = 1$ since the jth judgment order statistic will equal $X_{(i)}$ for some $i = 1, \ldots, k$.

The p_{ij}'s are unknown parameters in the model and there is no way to estimate them based on a ranked-set sample, since the true order statistics will never be known. To fully specify their model, Bohn and Wolfe take p_{ij} to be inversely proportional to $|E[X_{(j)} - X_{(i)}]|$, the expected spacing between $X_{(j)}$ and $X_{(i)}$. Let $p_{ij} = \frac{a_j}{|E[X_{(j)} - X_{(i)}]|}$ and $p_{jj} = c_j$. This specification of the p_{ij}'s and a doubly stochastic restriction on the matrix $\vec{P}$ results in a fully specified model, but involves dependence on the underlying distribution through its expected spacings. Now, using the relation that $F_{[j]}(x) = \sum_{i=1}^{k} p_{ij} F_{(i)}(x)$, properties of various statistics (test statistics or estimators) can be found with this model for the case where judgment ranking is not perfect.

The Bohn and Wolfe model provides a general form for the relationship between the matrix $\vec{P}$ of probabilities representing the likelihood of misclassifying an observation and the c.d.f. of the judgment order statistics using the assumption that $F_{[j]}(t) = \sum_{i=1}^{k} p_{ij} F_{(i)}(t)$. This relationship can then be used to computate the relative precisions of various procedures.

The expected spacings model is intuitive in the sense that if the expected difference between any two order statistics is large then it is unlikely that the ranking in a set will result in one of these order statistics being mistakenly chosen as the other. This is not to say it can not happen, but only that it is less likely to occur than if the expected difference between them is small.

A drawback of the expected spacings model is that it depends on the independence assumption that has been criticized by Presnell and Bohn (1999). While this assumption is not true, in general, it does appear to be a reasonable approximation for most distributions likely to be encountered in practice. Also, although the Bohn-Wolfe model explicitly incorporates the expected spacings, it does not take into account closeness as measured by the variances of these spacings.

The expected spacings model is dependent on the underlying distribution and thus is not nonparametric. Finally, the model is difficult to extend to set sizes larger than five. Even the model derived by Bohn and Wolfe for $k = 4$ is difficult to work with. This makes it hard to use the model for such things as determining optimal set sizes or which order statistics to measure when the sample size is greater than four.

3   Properties of the $\vec{P}$ Matrix

The models developed in this paper revolve around the same relation (R^*) $F_{[j]}(x) = \sum_{i=1}^{k} p_{ij}F_{(i)}(x)$ that was used by Bohn and Wolfe for the expected spacings model, where p_{ij} is the probability that the ith order statistic from the set of k is judgment ranked as the jth smallest observation in the set. That is, $p_{ij} = P(X_{[j]} = X_{(i)})$. Let

$$\vec{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1k} \\ p_{21} & p_{22} & \cdots & p_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ p_{k1} & p_{k2} & \cdots & p_{kk} \end{bmatrix}.$$

For the relationship R^* to hold we must have

$$\sum_{i=1}^{k} p_{ij} = 1, \tag{1}$$

for each $j = 1, \ldots, k$. This assumption is implicitly satisfied by the very manner in which we obtain a ranked-set sample. It corresponds to the fact that the jth judgment order statistic is equal to exactly one true order statistic and corresponds to the assumption that the columns of $\vec{P}$ sum to one.

The related condition that

$$\sum_{j=1}^{k} p_{ij} = 1 \tag{2}$$

is not assumed. We view the matrix $\vec{P}$ as applying to an entire cycle of a ranked-set sample. For this interpretation, condition (2) is not necessary. It would imply that the ith order statistic could be chosen as a judgment order statistic only once within a cycle. This, however, is not necessarily the case for ranked-set sampling with imperfect rankings. For example, in the first set of k observations we could correctly have $X_{[1]} = X_{(1)}$, but in the second set the item judged to be second smallest is actually the smallest, namely, $X_{[2]} = X_{(1)}$. This is not to say that the same observation is chosen twice since the order statistics are from two independent samples of size k from F. However, in this situation the first order statistic is chosen twice with the consequence that some other order statistic will not be chosen at all in that cycle.

In addition, under the assumption that the underlying distribution is unimodal, it is reasonable to assume that $p_{11} > p_{22} > \ldots < p_{(k-1)(k-1)} < p_{kk}$. This corresponds to the condition that it is easier to judgment rank extremes than central observations for such distributions, which makes intuitive sense because observations that fall in the tails of a unimodal distribution will tend to be further apart than those falling nearer the center.

It is also assumed that $p_{jj} \geq p_{ij}$ for any diagonal element p_{jj} and all $i \neq j$. That is, the most likely occurrence is a correct judgment ranking. This is reasonable since it is unlikely that a ranked-set sampling approach would be useful for situations where this assumption is not appropriate.

When the underlying distribution is symmetric certain symmetries should also appear in the matrix $\vec{P}$, such as $p_{11} = p_{kk}, p_{22} = p_{(k-1)(k-1)}$, etc. In general, we would expect $p_{ij} = p_{(k+1-i)(k+1-j)}$ as well. However, we do not expect the $\vec{P}$ matrix to be totally symmetric. For example, the condition that $p_{ij} = p_{ji}$ would imply that the probability of judgment ranking the ith order statistic to be the jth is the same as the probability of judgment ranking the jth order statistic to be the ith. There is no reason to believe that this should be the case even in the underlying distribution is symmetric.

These conditions on $\vec{P}$, along with the ability to extend the model to different set sizes and having both perfect ranking and random ranking as special cases, are properties we require for our models.

4   Baseline Model

We first propose a baseline model that assumes that the judgment order statistics are related to the true order statistics in a way that is similar to how the true order statistics are related to the quantiles of the underlying distribution. Assume that the population has c.d.f. F with inverse F^{-1} and probability distribution function (p.d.f.) f. Let k be the set size. Then the model specifies that:

$$p_{ij} = P(X_{[j]} = X_{(i)}) \approx P\left(X_{(j)} = F^{-1}\left(\frac{i - \frac{1}{2}}{k}\right)\right).$$

Of course this does not directly result in a specification since

$$P\left(X_{(j)} = F^{-1}\left(\frac{i - \frac{1}{2}}{k}\right)\right) = 0$$

for all continuous distributions. A more appropriate description of the model would be:

$$p_{ij} = P(X_{[j]} = X_{(i)}) \approx P\left(X_{(j)} \in \mathcal{N}\left(F^{-1}\left(\frac{i - \frac{1}{2}}{k}\right)\right)\right),$$

where $\mathcal{N}(y)$ is a neighborhood around the point y. The question then is how to define the neighborhood. An even more intuitive way of defining p_{ij} would be:

$$p_{ij} = P(X_{[j]} = X_{(i)}) \approx P(X_{(j)} \in \mathcal{N}(E[X_{(i)}])),$$

and the definition of the neighborhood could depend on other properties of the underlying distribution, including moments of adjacent order statistics, the variance of the

i^{th} order statistic, the kurtosis or skewness of the distribution and/or the covariances between the ith order statistic and adjacent order statistics.

For the assumption in (1) to hold, the neighborhoods must be defined in such a way that they are non-overlapping intervals that partition the support of the underlying distribution. This, it turns out, is overly restrictive for the model. In situations when the neighborhoods do not partition the support of F the probabilities will be standardized so that they do sum to one and satisfy (1).

4.1   Specification of the Baseline Model

To implement this idea we must specify a way of defining the neighborhood around $E[X_{(i)}]$. Consider the following result:

Result 1. Let F be the c.d.f. of a continuous, symmetric distribution such that $\frac{1}{1-F(x)}$ is convex. Then,

$$F^{-1}\left(\frac{i-1}{k}\right) \leq E[X_{(i)}] \leq F^{-1}\left(\frac{i}{k}\right) \iff \frac{i-1}{k} \leq F(E[X_{(i)}]) \leq \frac{i}{k}.$$

For proof of this equivalence see David (1981). The condition that $\frac{1}{1-F(x)}$ is convex is rather mild and holds for all standard exponential families and distributions discussed in this paper. Also, it is known that $E[X_{(i)}] \approx F^{-1}(\frac{i}{k+1})$, based on a first order Taylor series expansion of $X_{(i)}$. These inequalities suggest partitioning the support of f into pieces of equal probability, which leads to the following definition of the p_{ij}'s:

$$p_{ij} = P(X_{[j]} = X_{(i)}) \equiv P\left(F^{-1}\left(\frac{i-1}{k}\right) \leq X_{(j)} \leq F^{-1}\left(\frac{i}{k}\right)\right),$$

which is simply

$$p_{ij} = P\left(\frac{i-1}{k} \leq F(X_{(j)}) \leq \frac{i}{k}\right).$$

In the current setup, since the distribution is continuous, the jth order statistic from a simple random sample of size k has p.d.f.:

$$f_{X_{(j)}}(t) = \frac{k!}{(j-1)!(k-j)!}f(t)[F(t)]^{j-1}[1-F(t)]^{k-j}. \tag{3}$$

Then, it is well-known that $F(X_{(j)}) \overset{d}{=} U_{(j)}$, where $U_{(j)}$ is the jth order statistic for a sample of size k from a uniform $(0,1)$ distribution. Moreover, this common distribution is Beta with parameters j and $k-j+1$. Thus, when the support of the underlying distribution is divided into equal probability intervals this model, hereafter referred to as the baseline model, is the same for all distributions.

Consider the case of $k = 4$. For the baseline model the matrix $\vec{P}$ of p_{ij}'s is:

$$\vec{P} = \begin{bmatrix} 0.684 & 0.262 & 0.051 & 0.004 \\ 0.254 & 0.426 & 0.262 & 0.059 \\ 0.059 & 0.262 & 0.426 & 0.254 \\ 0.004 & 0.051 & 0.262 & 0.684 \end{bmatrix}.$$

For $k = 5$ the matrix is:

$$\vec{P} = \begin{bmatrix} 0.672 & 0.263 & 0.058 & 0.007 & 0.000 \\ 0.250 & 0.400 & 0.260 & 0.080 & 0.010 \\ 0.068 & 0.250 & 0.365 & 0.250 & 0.068 \\ 0.010 & 0.080 & 0.260 & 0.400 & 0.250 \\ 0.000 & 0.007 & 0.058 & 0.263 & 0.672 \end{bmatrix}.$$

The biggest advantage of the baseline model is that it does not depend on the underlying distribution and it meets all the conditions that we wished to impose on the $\vec{P}$ matrix. In addition, $\vec{P}$ is doubly stochastic (i.e., satisfies both (1) and (2)), and it is easy to compute and use for any set size. The biggest drawback of the model is that it is fixed and has no parameter to adjust the probabilities to reflect that judgment ranking may be easier in some situations than in others. This limitation is addressed in the next section by modifying the baseline model.

4.2   Modified Baseline Model

The ability to correctly judgment rank elements in a set may be easier for some situations than others. This could be because the underlying distributions are different or due to the fact that for certain problems or objects the ranking process may be more visually distinct than for others. For instance, circumference and height of two objects may have the same distribution, but it would be easier to judgment rank heights than it would be to judgment rank circumferences. Another example would be when the objects that are to be judgment ranked are separated as opposed to being all in one place for the person doing the judgment ranking. To reflect such features the baseline model must be generalized, since it assigns the same probabilities in every situation for which the underlying distribution is the same.

For the baseline model $p_{ij} = P(\frac{i-1}{k} \leq U_{(j)} \leq \frac{i}{k})$, where $U_{(j)}$ is the jth order statistic from a uniform simple random sample of size k. One way to modify this model is to adjust it according to the variance of the distribution of the order statistic while keeping its mean constant at $E[U_{(j)}] = j/(k+1)$. This allows $p_{jj} = P(X_{[j]} = X_{(j)})$ to be adjusted for each j. By doing this for each j we provide added flexibility for specifying the diagonal elements for the matrix $\vec{P}$.

Consider the case $k = 5$ where it is believed that $p_{11} = 0.9$. It is known that $U_{(1)} \sim \text{Beta}(1, 5)$, so that $E[U_{(1)}] = \frac{1}{6}$. Let $Y \sim \text{Beta}(\alpha, \beta)$, where α and β are chosen

such that $\frac{\alpha}{\alpha+\beta} = \frac{1}{6}$ and $P(Y \leq \frac{1}{5}) = 0.9$. The first condition implies that $\beta = 5\alpha$, from which it follows that the second condition requires us to solve

$$\int_0^{1/5} \frac{\Gamma(6\alpha)}{\Gamma(\alpha)\Gamma(5\alpha)} y^{\alpha-1}(1-y)^{5\alpha-1} dy = 0.9$$

for α. Doing so yields $\alpha = 35.4$ and $\beta = 177$. The following table shows the necessary α's and β's for the other judgment order statistics when $k = 5$ under the conditions that $p_{22} = 0.8$, $p_{33} = 0.7$, $p_{44} = 0.8$ and $p_{55} = 0.9$.

j	p_{jj}	$E[X_{(j)}]$	α	β
1	0.9	1/6	35.4	177
2	0.8	2/6	14.2	28.4
3	0.7	3/6	13.4	13.4
4	0.8	4/6	28.4	14.2
5	0.9	5/6	177	35.4

Using these α's and β's with this modified baseline model gives:

$$\vec{P} = \begin{bmatrix} 0.900 & 0.023 & 0.000 & 0.000 & 0.000 \\ 0.100 & 0.800 & 0.150 & 0.000 & 0.000 \\ 0.000 & 0.176 & 0.700 & 0.176 & 0.000 \\ 0.000 & 0.000 & 0.150 & 0.800 & 0.100 \\ 0.000 & 0.000 & 0.000 & 0.023 & 0.900 \end{bmatrix}.$$

This approach will allow the diagonal probabilities to be as high as desired, but not necessarily as small as we might want. When its mean is fixed, there is an implied maximum variance for the distribution of the associated order statistic. This is a consequence of the finite support for the Beta distribution. With the mean fixed at $\frac{1}{6}$ as above, for example, the expression for the variance is

$$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}.$$

Now for the mean to be $\frac{1}{6}$, $\beta = 5\alpha$ so the variance becomes

$$\frac{5\alpha^2}{(6\alpha)^2(6\alpha+1)} = \frac{5}{36(6\alpha+1)}.$$

This converges to 0 as $\alpha \to \infty$ and to $\frac{5}{36}$ as $\alpha \to 0$. For the smoothing modification of the baseline model the restriction that $\max(\alpha, \beta) \geq 1$ should be met so that the distribution is not U-shaped which would lead to a violation of the assumption that $p_{jj} \geq p_{ij}$ for $i \neq j$. There are, however, some situations where it may be appropriate to relax this assumption. For $k = 5$, the lowest attainable p_{11} is 0.674 corresponding to $\alpha = 1.16312$ and $\beta = 5.8156$. It is interesting to note that this does not occur at

the minimum variance. Similarly, the lowest p_{55} is 0.674. Even though the central order statistics have larger variances, the modified baseline model is able to obtain very small p_{jj} values for them. The following table presents the lowest attainable p_{jj}'s and the corresponding α's and β's for $k = 5$.

j	p_{jj}	$E[X_{(j)}]$	α	β
1	0.67	1/6	1.163	5.816
2	0.19	2/6	0.000001	0.000002
3	0.20	3/6	0.000001	0.000001
4	0.19	4/6	0.000002	0.000001
5	0.67	5/6	5.816	1.163

For a uniform distribution (or on the uniform scale using the probability integral transformation for other distributions) the effect of changing α and β can be interpreted in a way similar to a Bayesian prior. Consider the 2nd order statistic in the case of $k = 5$, namely, $U_{(2)} \sim \text{Beta}(2, 4)$. When the α and β parameters are changed, as in the first example, to $\alpha = 14.2$ and $\beta = 28.4$ the mean stays the same by design. The interesting thing to note is that this is like the distribution of the 14.2^{th} order statistic for a sample of size 41.6 from a uniform distribution. (For an integer sample size this is similar to the technical device of fractional order statistics introduced by Stigler, 1977.) In general it is like the distribution of the αth order statistic for a sample of size $\alpha + \beta - 1$ from a uniform distribution. Note that because the means are held constant the relation between the two settings is: $\frac{2}{5+1} = \frac{14.2}{41.6+1}$. So in a proportional sense they are similar order statistics relative to different sample sizes.

Note that $U_{(j)} \sim \text{Beta}(j, k - j + 1)$. Letting $U_{[j]} \sim \text{Beta}(\alpha, \beta)$ has the same interpretation on the scale of F. Let $x = F^{-1}(u)$ so that $u = F(x)$ and $J = |\frac{\partial}{\partial u} F(u)| = f(u)$. Then

$$f_{X_{[j]}}(x) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)}[F(x)]^{\alpha-1}(1 - F(x))^{\beta-1}f(x),$$

which is the distribution of the αth order statistic from a random sample of size $\alpha + \beta - 1$ from F. Thus, the smoothing of the baseline model is doing similar things on both the uniform scale and the scale of the original distribution.

This smoothed baseline model has the same advantages as the baseline model, although because of its added flexibility it is slightly harder to implement. However, since it is able to represent different degrees of judgment error, it does not have that drawback possessed by the baseline model. In addition, to use the smoothed baseline model we must specify the entire diagonal of the $\vec{P}$ matrix. If a one-number model were desired we could use the same relationship on the diagonal of this model as we have for the baseline model. For $k = 5$ the baseline model has $p_{11} = p_{55} = 0.673$, $p_{22} = p_{44} = 0.400$ and $p_{33} = 0.365$. Thus, for a smoothed baseline model we could let $p_{22} = p_{44} = \frac{0.400}{0.672}p_{11}$, $p_{33} = \frac{0.365}{0.672}p_{11}$, and $p_{55} = p_{11}$. With these stipulations the only parameter is p_{11}, as with the Bohn-Wolfe expected spacings model. Another way to achieve this would be to let $p_{22} = p_{44} = p_{11} - (0.672 - 0.400)$, $p_{33} = p_{11} - (0.672 - 0.365)$, and $p_{55} = p_{11}$.

5 Variance and Covariance Interval Models

In addition to the baseline and modified baseline models we propose another group of models that use the means and variances of order statistics and covariances of adjacent order statistics to define the intervals that specify the probabilities for the $\vec{P}$ matrix.

5.1 Univariate Variance Model

For this approach we define the intervals in terms of the expected value of the order statistic plus or minus a constant times its standard deviation. The motivation for this model is that as the variance of an order statistic gets larger relative to the difference in the means of adjacent order statistics the order statistics will be harder to judgment rank correctly. For the normal distribution, for example, the differences between the means of adjacent order statistics are greater for both the largest and smallest sets of order statistics but the variances of these order statistics are also larger. In fact, relative to the standard deviations of the order statistics, the expected differences between the means are smaller for these extreme order statistics. On the other hand, for the uniform distribution the expected differences in the means are the same for all order statistics. The variances, however, are lower for the extreme order statistics. Based on this information we might expect that the accuracy in judgment ranking the extreme order statistics as opposed to the central order statistics should be relatively better for the uniform distribution than for the normal distribution. To reflect this, we define

$$q_{ij} = P\left(E[X_{(i)}] - z\sqrt{\mathrm{Var}(X_{(i)})} \le X_{(j)} \le E[X_{(i)}] + z\sqrt{\mathrm{Var}(X_{(i)})}\right),$$

where z is a parameter that can be varied, and set $p_{ij} = \frac{q_{ij}}{q_{\cdot j}}$, where $q_{\cdot j} = \sum_{i=1}^{k} q_{ij}$.

For the uniform distribution with $k = 5$ and $z = 0.5$, we get the following for $E[X_{(i)}] \pm z\sqrt{\mathrm{Var}(X_{(i)})}$:

i	Lower	Upper
1	0.096	0.237
2	0.244	0.422
3	0.406	0.594
4	0.578	0.756
5	0.763	0.904

resulting in the following $\vec{P}$ matrix:

$$\vec{P} = \begin{bmatrix} 0.571 & 0.278 & 0.080 & 0.013 & 0.001 \\ 0.302 & 0.364 & 0.252 & 0.096 & 0.021 \\ 0.105 & 0.249 & 0.336 & 0.249 & 0.105 \\ 0.021 & 0.096 & 0.252 & 0.364 & 0.302 \\ 0.001 & 0.013 & 0.080 & 0.278 & 0.571 \end{bmatrix}.$$

For the normal distribution with $k = 5$ and $z = 0.5$, we get the following for $E[X_{(i)}] \pm z\sqrt{\text{Var}(X_{(i)})}$:

i	Lower	Upper
1	0.067	0.204
2	0.219	0.415
3	0.394	0.606
4	0.585	0.781
5	0.796	0.933

resulting in the following $\vec{P}$ matrix:

$$\vec{P} = \begin{bmatrix} 0.559 & 0.237 & 0.056 & 0.007 & 0.001 \\ 0.320 & 0.394 & 0.259 & 0.091 & 0.017 \\ 0.104 & 0.270 & 0.370 & 0.270 & 0.104 \\ 0.017 & 0.091 & 0.259 & 0.394 & 0.320 \\ 0.001 & 0.007 & 0.056 & 0.237 & 0.559 \end{bmatrix}.$$

The $\vec{P}$ matrix for the normal distribution does not indicate that the probability of judgment ranking the extremes correctly is lower than for the central order statistics. However, comparing it to the $\vec{P}$ matrix for the uniform distribution we see that the extreme order statistics are harder to judgment rank relative to the central order statistics for the normal distrubiton than for the uniform distribution.

One drawback of this univariate variance model is that it does not incorporate any information about adjacent order statistics. It is reasonable to believe that the probability of judging the ith order statistic correctly will at least depend on the distributions of the $(i-1)^{th}$ and $(i+1)^{th}$ order statistics as well. One advantage of the univariate variance model is that it can be extended easily to other distributions.

5.2   Midpoints Models

Another approach to modeling the p_{ij}'s is to start on the uniform scale and let

$$q_{ij} = P(E[(U_{(i-1)} + U_{(i)})/2] \leq U_{(j)} \leq E[(U_{(i)} + U_{(i+1)})/2]), \tag{4}$$

where $U_{(0)} \equiv 0$, $U_{(k+1)} \equiv 1$ and $U_{(1)} \leq \ldots \leq U_{(k)}$ are the order statistics for a random sample of size k from a Uniform$(0, 1)$ distribution. This approach uses the expected midpoints between adjacent order statistics to partition the interval $[0, 1]$. Thus, the resulting matrix automatically has columns that sum to one.

For $k = 5$ this gives:

$$\vec{P} = \begin{bmatrix} 0.633 & 0.328 & 0.099 & 0.016 & 0.002 \\ 0.262 & 0.344 & 0.246 & 0.090 & 0.018 \\ 0.085 & 0.221 & 0.310 & 0.221 & 0.085 \\ 0.018 & 0.090 & 0.246 & 0.344 & 0.262 \\ 0.002 & 0.016 & 0.099 & 0.328 & 0.633 \end{bmatrix}.$$

This method can be generalized by adjusting the expected midpoints by factors depending on the covariances between $U_{(i-1)}$ and $U_{(i)}$ and between $U_{(i)}$ and $U_{(i+1)}$. First, let $W_{(i,j)} = \frac{U_{(i)}+U_{(j)}}{2}$. The generalization is to let

$$q_{ij} = P\Big(E[W_{(i-1,i)}] \quad - \quad z\sqrt{\mathrm{Cov}(U_{(i-1)},U_{(i)})} \leq U_{(j)} \leq E[W_{(i,i+1)}]$$
$$+ \quad z\sqrt{\mathrm{Cov}(U_{(i)},U_{(i+1)})}\Big),$$

where once again $U_{(0)} \equiv 0$ and $U_{(k+1)} \equiv 1$. Then we set $p_{ij} = \frac{q_{ij}}{q_{\cdot j}}$, where $q_{\cdot j} = \sum_{i=1}^{k} q_{ij}$. The covariance between uniform order statistics from a sample of size k is

$$\mathrm{Cov}(U_{(i)},U_{(j)}) = \frac{i(k+1-j)}{(k+1)^2(k+2)}.$$

This generalized midpoints model will be referred to as the midpoints covariance model or simply the covariance model. With $k = 5$ and $z = 0.5$, we obtain the following lower and upper endpoints for the intervals:

i	Lower	Upper
1	0.013	0.313
2	0.187	0.494
3	0.340	0.660
4	0.506	0.813
5	0.687	0.987

yielding the $\vec{P}$ matrix:

$$\vec{P} = \begin{bmatrix} 0.623 & 0.296 & 0.100 & 0.021 & 0.002 \\ 0.256 & 0.339 & 0.244 & 0.104 & 0.023 \\ 0.096 & 0.239 & 0.312 & 0.239 & 0.096 \\ 0.023 & 0.104 & 0.244 & 0.339 & 0.256 \\ 0.002 & 0.021 & 0.100 & 0.296 & 0.623 \end{bmatrix}.$$

An alternative way to generalize (4) is to use the variance of the midpoint instead of the covariance between the two adjacent order statistics. Again, let $W_{(i,j)} = \frac{U_{(i)}+U_{(j)}}{2}$. For this model let

$$q_{ij} = P\Big(E[W_{(i-1,i)}] \quad - \quad z\sqrt{\mathrm{Var}(W_{(i-1,i)})} \leq U_{(j)} \leq E[W_{(i,i+1)}]$$
$$+ \quad z\sqrt{\mathrm{Var}(W_{(i,i+1)})}\Big),$$

where once again $U_{(0)} \equiv 0$ and $U_{(k+1)} \equiv 1$. Then take $p_{ij} = \frac{q_{ij}}{q_{\cdot j}}$, where $q_{\cdot j} = \sum_{i=1}^{k} q_{ij}$.

For uniform order statistics the variance of $W_{(i,j)}$ is given by

$$
\begin{aligned}
\operatorname{Var}(W_{(i,j)}) &= \frac{1}{4}[\operatorname{Var}(U_{(i)}) + \operatorname{Var}(U_{(j)}) + 2\operatorname{Cov}(U_{(i)}, U_{(j)})] \qquad (5)\\
&= \frac{1}{4}\left[\frac{i(k-i+1)}{(k+1)^2(k+2)} + \frac{j(k-j+1)}{(k+1)^2(k+2)} + 2\frac{i(k+1-j)}{(k+1)^2(k+2)}\right]\\
&= \frac{1}{4}\left[\frac{3i(k+1) - i^2 - 2ij + j(k+1) - j^2}{(k+1)^2(k+2)}\right].
\end{aligned}
$$

The resulting model will be referred to as the variance of the midpoints model. With $k = 5$ and $z = 0.01$, the corresponding lower and upper endpoints of the intervals are:

i	Lower	Upper
1	0.066	0.264
2	0.236	0.434
3	0.400	0.600
4	0.566	0.764
5	0.736	0.931

yielding the $\vec{P}$ matrix:

$$
\vec{P} = \begin{bmatrix}
0.635 & 0.323 & 0.099 & 0.017 & 0.002\\
0.259 & 0.343 & 0.245 & 0.093 & 0.019\\
0.086 & 0.225 & 0.311 & 0.226 & 0.088\\
0.019 & 0.092 & 0.245 & 0.344 & 0.263\\
0.002 & 0.017 & 0.099 & 0.321 & 0.628
\end{bmatrix}.
$$

  However, a number of difficulties arise when this approach is applied to a distribution other than the uniform. First, we do not have closed form expressions for the expected values of order statistics for many common distributions and only rarely are the covariances of adjacent order statistics available in closed form. These problems, however, can be addressed by using approximations for the moments of order statistics. Second, there is no longer a bounded support framework within which to work. In the standard uniform setting we set $U_{(0)} \equiv 0$ and $U_{(k+1)} \equiv 1$ for convenience, but it was also quite natural. For a distribution that has the real line or a half-infinite interval as its support, we must take a different approach. We considered several different methods for dealing with this endpoint problem. One such approach that we present here is to let $X_{(0)}$ and $X_{(k+1)}$ be the corresponding endpoints for the support of the underlying distribution (even if one or both are infinite).

  In general, let $W_{(i,j)} = \frac{X_{(i)} + X_{(j)}}{2}$. Then the extended variance of the midpoints model is to set

$$
\begin{aligned}
q_{ij} = P\Big(E[W_{(i-1,i)}] \;-\; & z\sqrt{\operatorname{Var}(W_{(i-1,i)})} \le X_{(j)} \le E[W_{(i,i+1)}]\\
+\; & z\sqrt{\operatorname{Var}(W_{(i,i+1)})}\Big), \qquad\qquad (6)
\end{aligned}
$$

where $X_{(0)}$ and $X_{(k+1)}$ are defined as the left and right endpoints for the support of the underlying distribution. As usual, then $p_{ij} = \frac{q_{ij}}{q_{\cdot j}}$, where $q_{\cdot j} = \sum_{i=1}^{k} q_{ij}$.

For the double exponential distribution with $z = 0.1$ and $k = 5$ the intervals are

i	Lower	Upper
1	$-\infty$	-0.995
2	-1.166	-0.227
3	-0.346	0.346
4	0.227	1.166
5	0.995	∞

and they result in the following probability matrix:

$$\vec{P} = \begin{bmatrix} 0.578 & 0.198 & 0.040 & 0.004 & 0.000 \\ 0.316 & 0.414 & 0.241 & 0.071 & 0.009 \\ 0.097 & 0.312 & 0.438 & 0.312 & 0.097 \\ 0.009 & 0.071 & 0.241 & 0.414 & 0.316 \\ 0.000 & 0.004 & 0.040 & 0.198 & 0.578 \end{bmatrix}.$$

Computing such a $\vec{P}$ matrix requires values for the expectations, variances and covariances of order statistics from the assumed distribution. When closed form expressions for the needed moments are available they should be used. Such closed form expressions exist for the moments and covariances of uniform and exponential order statistics. Johnson *et al.* (1995) give explicit formulas for the moments of double exponential order statistics.

When approximations for the moments of order statistics from a continuous population with c.d.f. F are needed the integral expression

$$E[X_{(i)}^t] = \frac{k!}{(i-1)!(k-i)!} \int_0^1 [F^{-1}(u)]^t u^{i-1}(1-u)^{k-i} \, du \tag{7}$$

can be approximated in R (see Ihaka and Gentleman, 1996) using the adapt function for approximating integrals. This package, as modified for R from Mike Meyer's S code by Thomas Lumley and obtained from StatLib at Carnegie Mellon University, calls A.C. Genz's Fortran ADAPT subroutine to do all the calculations. Accuracy for the relative error is set to be < 0.00001. When an approximation for the covariance between two order statistics is needed the integral expression

$$\begin{aligned} E[X_{(i)}X_{(j)}] \quad = \quad & \frac{k!}{(i-1)!(j-i-1)!(n-j)!} \times \\ & \iint_{0<u_i<u_j<1} F^{-1}(u_i)F^{-1}(u_j)u_i^{i-1}(u_j-u_i)^{j-i-1}(1-u_j)^{k-i} \, du_i \, du_j \end{aligned} \tag{8}$$

can be approximated in R (again see Ihaka and Gentleman, 1996) in the same way as equation (7). These approximations are used in this paper whenever moments of order statistics are needed and not available in closed form.

5.3   Variance of Differences Model

A modification of the model discussed in Section 5.2 is motivated by the idea that if the correlation between two variables is positive then they behave similarly. If one is large the other will also tend to be large and both being large will make it harder to judgment rank them. With negatively correlated variables if one is large the other is likely to be small so that they will tend to be separated, making it easier to judgment rank them. Of course, adjacent order statistics are always positively correlated, but in a relative sense the logic still applies. If two adjacent order statistics have a stronger positive correlation than two other adjacent order statistics, the pair having the greater positive correlation will be harder to judgment rank. Thus, in lieu of using the variance of $W_{(i,j)}$ in (5) to define the width of our intervals in (6), we consider instead the use of

$$\text{Var}(\text{Spacing}) = \text{Var}(X_{(j)} - X_{(i)}) = \text{Var}(X_{(i)}) + \text{Var}(X_{(j)}) - 2\text{Cov}(X_{(i)}, X_{(j)})$$

in (6). This results in q_{ij} equal to

$$P\Big(E[W_{(i-1,i)}] \quad - \quad z\sqrt{\text{Var}(X_{(i+1)} - X_{(i)})} \le X_{(j)} \le E[W_{(i,i+1)}]$$
$$+ \quad z\sqrt{\text{Var}(X_{(i+1)} - X_{(i)})}\Big),$$

where once again $X_{(0)}$ and $X_{(k+1)}$ are defined as the left and right endpoints for the support of the underlying distribution, respectively, and we take $p_{ij} = \frac{q_{ij}}{q_{\cdot j}}$, where $q_{\cdot j} = \sum_{i=1}^{k} q_{ij}$.

For the uniform distribution with $k = 5$ and $z = 0.5$, we get the following intervals from this approach:

i	Lower	Upper
1	0.013	0.320
2	0.180	0.487
3	0.346	0.654
4	0.513	0.820
5	0.680	0.987

with the resulting $\vec{P}$ matrix:

$$\vec{P} = \begin{bmatrix} 0.622 & 0.305 & 0.107 & 0.023 & 0.003 \\ 0.264 & 0.343 & 0.242 & 0.100 & 0.021 \\ 0.090 & 0.229 & 0.303 & 0.229 & 0.090 \\ 0.021 & 0.100 & 0.242 & 0.343 & 0.264 \\ 0.003 & 0.023 & 0.107 & 0.305 & 0.622 \end{bmatrix}.$$

The drawbacks of this variance of differences model are similar to those of the other midpoints models discussed in this section, namely, it is difficult to implement for

different distributions and will sometimes require approximations for various moments of order statistics from the underlying distribution. Also, it is clearly dependent on the underlying distribution.

Note that one common problem with all of the interval type models discussed here is that they are not capable of representing either perfect judgment rankings or random rankings.

6 Simulation Models

In addition to the baseline model, modified baseline models and the various interval models using variances and covariances of adjacent order statistics we also suggest consideration of simulation models to generate an appropriate $\vec{P}$ matrix for settings where closed form expressions for the p_{ij}'s do not exist. The first proposed model uses independent order statistics, while the second approach approximates the p_{ij}'s by using a Dell and Clutter type model. The third model of this type considered here is based on concomitant variables.

6.1 Independent Order Statistics Simulation

This method of generating a $\vec{P}$ matrix is based on independent order statistics from a specific parametric distribution. For a set of k independent order statistics, the first order statistic, for example, could be larger than the second order statistic or any other order statistic. The same is true for any of the independent order statistics.

First, for each round of simulation we generate k independent order statistics from the underlying distribution using the marginal distributions of the order statistics. Then p_{ij} is estimated by the proportion of iterations for which the generated i^{th} order statistic is actually the jth smallest in the set of k independent order statistics. The approximate $\vec{P}$ matrix from this approach based on $100,000$ simulations using R (Ihaka and Gentleman, 1996) from the uniform distribution (independent of location and scale) with $k = 5$ is given by

$$\vec{P} \approx \begin{bmatrix} 0.740 & 0.198 & 0.052 & 0.010 & 0.001 \\ 0.203 & 0.509 & 0.217 & 0.062 & 0.009 \\ 0.048 & 0.222 & 0.463 & 0.219 & 0.048 \\ 0.008 & 0.063 & 0.216 & 0.512 & 0.202 \\ 0.001 & 0.009 & 0.052 & 0.197 & 0.741 \end{bmatrix}.$$

The approximate $\vec{P}$ matrix obtained from this approach is always doubly stochastic and the model can be applied easily to any distribution for which the order statistics can be generated. The model does not depend on either the location or scale parameter of the underlying distribution and it extends easily to larger set sizes. The model is, however, computationally intensive, especially for larger set sizes.

6.2   Dell and Clutter Simulation

Our second method for simulating an appropriate $\vec{P}$ matrix is based on the Dell and Clutter formulation, but instead of using it to approximate the moments of the judgment order statistics it is used to approximate the p_{ij}'s. Recall that the model is $Y_i = X_i + \epsilon_i$, where ϵ_i represents the error in the judgment ranking process. To implement this model a set of k independent random X_i's and k random error variates, ϵ_i's, are generated and the Y_i's are computed. For this model $X_{[r]}$ is the X_i associated with $Y_{(r)}$. Then p_{ij} is defined to be the proportion of these iterations for which $X_{[j]}$ is actually $X_{(i)}$.

The variance component in the error term of this model allows the user to control the degree of judgment error. Also different distributions can be used to generate the order statistics and the errors.

For set size $k = 5$ the approximate $\vec{P}$ matrix from this approach based on $100,000$ simulations using R (Ihaka and Gentleman, 1996) for normal order statistics and normal judgment errors with an error variance of $\sigma^2 = 0.5$ is

$$
\vec{P} \approx \begin{bmatrix}
0.715 & 0.199 & 0.063 & 0.018 & 0.004 \\
0.197 & 0.493 & 0.212 & 0.079 & 0.019 \\
0.065 & 0.209 & 0.448 & 0.212 & 0.065 \\
0.019 & 0.079 & 0.213 & 0.492 & 0.197 \\
0.003 & 0.019 & 0.064 & 0.199 & 0.715
\end{bmatrix}.
$$

The approximate $\vec{P}$ matrix obtained from this approach is also always doubly stochastic and model simulation can be applied to any distribution for which order statistics can be generated. Once again, this $\vec{P}$ matrix does not depend on either the location or scale parameter for the X distribution or on the location parameter of the error distribution. The model can be extended easily to larger set sizes, but it also becomes computationally intensive as the set size increases.

6.3   Concomitant Simulation

For this approach we generate pairs of variables, one (X) being the variable of interest and the second (Y) is a concomitant variable. The concomitant variable is then used to judgment rank the variable of interest. Specifically, $X_{[r]}$ is the X_i corresponding to $Y_{(r)}$. Then p_{ij} is estimated as the proportion of these iterations for which $X_{[j]}$ is actually $X_{(i)}$. (Note that an alternative method of approximating these probabilities was proposed by David *et al.*, 1977.)

For the case when the variable of interest and the concomitant variable follow a bivariate normal distribution, the correlation coefficient represents the degree of judgment error that is present in the ranking process. A correlation of one corresponds to perfect judgment rankings and a correlation of zero corresponds to the case of random rankings.

For set size $k = 5$ and a bivariate normal distribution with correlation coefficient $\rho = 0.9$, the approximate $\vec{P}$ matrix based on $100,000$ simulations using R is

$$\vec{P} \approx \begin{bmatrix} 0.723 & 0.194 & 0.062 & 0.017 & 0.004 \\ 0.196 & 0.503 & 0.210 & 0.074 & 0.017 \\ 0.060 & 0.210 & 0.456 & 0.211 & 0.063 \\ 0.018 & 0.075 & 0.210 & 0.501 & 0.196 \\ 0.003 & 0.017 & 0.062 & 0.197 & 0.720 \end{bmatrix}.$$

This bivariate normal concomitant model is easily extended to larger set sizes, but again it becomes computationally intensive as the set size increases.

7   Discussion

It is difficult to look at a particular $\vec{P}$ matrix and understand how the individual p_{ij}'s will influence the efficiencies of different statistical procedures. It may be possible to do this for small values of k, but larger values of k present a problem. In this regard, some of the things to look for in a $\vec{P}$ matrix are the magnitudes and pattern of the p_{ij}'s on the main diagonal (these are the probabilities of correctly judgment ranking observations), how the probabilities decline off the diagonals and how large the probabilities are in the corners of the matrix. The baseline model for $k = 5$ has moderate values of p_{11} and p_{55} and the probabilities fall to zero fairly quickly off the main diagonal. The modified baseline model has very large probabilities on the main diagonal (by design) and the probabilities fall to zero very quickly off the main diagonal. The interval type models tend to have smaller probabilities on the main diagonal and fall more slowly, never quite reaching zero off the main diagonal. The simulation models tend to have larger probabilities on the main diagonal and drop quickly away from this diagonal, but once again not all the way to zero.

Of course, in this preliminary discussion of modeling $\vec{P}$ matrices nothing is being held constant across the different models so it is difficult to draw real conclusions about how well any of them reflect a given underlying distribution and associated judgment rankings. Further studies are needed to sharpen our understanding of the relative merits of these different approaches. The important result is that these models for imperfect judgment rankings can now be used to assess the impact that imperfect judgment rankings have on statistical procedures in a wide variety of settings.

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