International Journal of Statistical Sciences *ISSN 1683–5603*
Vol. 6 (Special Issue), 2007, pp 7-18

The Family of Curvi-Triangular Distributions

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[Received September 29, 2005; Revised July 3, 2007; Accepted September 5, 2007]

Abstract

A family of distributions has been proposed which includes the symmetric triangular distribution as special case. The new family of distributions, which we refer to as generalized curvi-triangular (GCT) distributions, allows the shape of the density to be non-triangular and asymmetric, in general. The properties of GCT distributions along with the inferential procedures regarding the parameters are discussed.

Keywords and Phrases: Skewness; Skew-Normal distribution; Triangular distribution.

AMS Classification: 60E$_{xx}$.

1 Introduction

A regular three-parameter triangular distribution is given by the density

$$f(x;\alpha,\beta,\xi) = \begin{cases} \frac{2(x-\alpha)}{(\beta-\alpha)(\xi-\alpha)}, & \alpha \le x \le \xi \\ \frac{2(\beta-x)}{(\beta-\alpha)(\beta-\xi)}, & \xi \le x \le \beta \end{cases}. \tag{1}$$

This distribution has limited application, mostly because of its triangle-shape density, which makes it less suitable for real-life data. In practice, frequency curves seldom take the shape of a triangle. This kept the use of triangular density clinical. We will mention several recent works with triangular distribution here. An extension of the three-parameter triangular distribution utilized in risk analysis has been discussed by René van Dorp and Kotz (2002). They showed that the extended four-parameter family includes the triangular distribution, the power function distribution and the uniform distribution. Both Johnson (1997) and Johnson and Kotz (1999) explored the possibility of using triangular distribution as a proxy for the beta distribution in risk analysis. Johnson (1997) used triangular distributions as a proxy to the beta distribution, specifically in problems of assessment of risk and uncertainty, such as the project evaluation and review technique (PERT). In risk analysis, the parameters of a triangular distribution have a one-to-one correspondence with an optimistic estimate, most likely estimate and pessimistic estimate of a quantity under consideration, providing to the triangular distribution its intuitive appeal (René van Dorp and Kotz, 2002).

Nonetheless, the statistics community has been somewhat quiet about the application of this distribution. We will deal with a simpler version of the triangular distribution and will show that when extended in a manner similar to the extension of normal distribution to skew-normal distribution, by Azzalini (1985), the distribution becomes more exciting. We will investigate the properties of the newly defined distribution. Some inferential aspects of the parameters of the new family will also be investigated.

The article is organized as follows. In Section 2, we generalize a standard symmetric triangular distribution by introducing a shape parameter. We refer to the resulting family of distributions as generalized curvi-triangular distribution and investigate important characteristics of this distribution. Section 3 is devoted to the discussion of some inferential procedures such as random number generation, estimation and test of hypothesis concerning the shape parameter. In section 4 we present the results from a simulation study. We conclude in Section 5 with a discussion.

2 General form of Curvi-triangular Distribution

Let us consider the case where $\xi = 0$, $\alpha = -\theta$ and $\beta = \theta$. The density function of such a simple one-parameter triangular distribution (symmetric about zero) is given by

$$f(x;\theta) = \frac{\theta - |x|}{\theta^2} I_{[-\theta,\theta]}(x), \tag{2}$$

where $I_A(x) = 1, x \in A, 0$, otherwise is the characteristic function for the set A. Corresponding distribution function is defined as

$$F(x;\theta) = \frac{1}{2}\left\{1 + \frac{x\,(2\theta - |x|)}{\theta^2}\right\} I_{[-\theta,\theta]}(x) + I_{(\theta,\infty)}(x). \tag{3}$$

Although only the symmetry property (not the symmetry about zero) is required for the derivations in this paper, for simplicity we adopted this form throughout, with the understanding that in cases where the random variable Y is symmetric about $\mu, \mu \neq 0$, a transformation $X = Y - \mu$ would lead to a distribution which is symmetric about zero. One common example given in undergraduate classes for introducing this distribution is the sum of two uniform random variables. If X_1 and X_2 are two independent $U(-\theta/2, \theta/2)$ random variables, then the distribution of $X_1 + X_2$ follows a triangular distribution given by (2). The distribution has mean zero, standard deviation $\theta/\sqrt{6}$ and has a kurtosis of $\beta_2 = 2.4$. Following Azzalini (1985), we define the following distribution with one additional parameter $\lambda (\lambda \in \mathcal{R})$:

$$g(x;\theta,\lambda) = \begin{cases} \left(\frac{\theta-|x|}{\theta^2}\right)\left\{1 + \frac{\lambda x(2\theta-|\lambda x|)}{\theta^2}\right\}, & |x| \leq \theta, |\lambda x| \leq \theta, \\ \frac{2(\theta-|x|)}{\theta^2}, & |x| \leq \theta, \lambda x > \theta, \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

It is easy to verify that this distribution meets all the criteria for being a density function. Although distributions derived similar way, for example, from normal, t, logistic or Cauchy are referred to as skew-normal (Azzalini, 1985), skew-t (Branco and Day, 2001), skew-logistic (Wahed and Ali, 2001) or skew-Cauchy (Arnold and Beaver, 2000) respectively, since triangular distribution can itself be skewed, we refer to this new family as generalized curvi-triangular distribution and will denote by $GCT[\theta, \lambda]$. It can be seen from (4) that the symmetric triangular distribution defined in (2) is a special case of the generalized curvi-triangular distribution when $\lambda = 0$. The distribution $g(\cdot;\theta,\lambda)$, defined as a generalization of $f(\cdot;\theta)$, no longer is triangular-shaped, neither is it symmetric (for non-zero λ). Figure 1 shows the shapes of this distribution for different positive and negative λ values. It is seen from the figures that the density for a given value of λ is a mirror image of the corresponding density for $-\lambda$. Figure 1 also shows the corresponding cumulative distribution functions for selected λ-values. The distribution is skewed to the right for approximately $\lambda \in I = (-1.52, 0) \cup [1.52, \infty)$ and to the left for $\lambda \in \mathcal{R} - I$, as is seen in Figure 1. The degree of skewness does not depend on θ and as expected, is governed by the magnitude of λ. When $\lambda = 0$ there is no skewness (symmetric triangular) and the largest skewness is achieved when $|\lambda| \to \infty$, in which case the density becomes a three-parameter triangular distribution with bases $\alpha = -\theta$ $(\lambda > 0)$ or $\alpha = \theta$ $(\lambda < 0), \xi = 0$, and $\beta = 0$. Besides when $\lambda = 0$, the distribution is also approximately symmetric for $\lambda = \pm 1.52$, approximately.

The transformation of standard symmetric triangular distribution to the $GCT[\theta, \lambda]$ distribution does not change the flatness or peakness of the distribution much. The coefficient of kurtosis β_2, as shown in Figure 1, lies between 2.4 and 2.6 with the lowest achieved at $\lambda = 0$, and highest at around $\lambda = \pm 0.78$.

Although both mean and variance of a $GCT[\theta, \lambda]$ distribution depend on both θ and λ, the coefficient of variation (CV) is free of θ. Since $g(x;\theta,\lambda) = g(-x;\theta,-\lambda)$, the mean is an odd function increasing in λ and the variance is an even function of λ, decreasing in $|\lambda|$. The larger the λ, the larger the absolute value of the mean. The CV function has a vertical asymptote at $\lambda = 0$ and is a decreasing odd function of λ.

2.1 A Simpler Subclass

When λ is restricted in the interval $[-1,1]$, $GCT[\theta,\lambda]$ distribution given by (4) simplifies to

$$g(x;\theta,\lambda)=\left(\frac{\theta-|x|}{\theta^2}\right)\left\{1+\frac{\lambda x\,(2\theta-|\lambda x|)}{\theta^2}\right\},\ |x|\le\theta. \tag{5}$$

We will refer to this distribution as $CT[\theta,\lambda]$. Certain properties considered beforehand simplifies for this class. The distribution function for this special case is given by

$$\begin{aligned}G(x;\theta,\lambda) &= \theta^{-4}\left[\lambda^2x^4/4-\lambda(\lambda+2)\theta\,|x|^3/3+(2\lambda-|x|/x)\theta^2x^2/2\right.\\ &\quad\left.+\theta^3x\right]I(x\neq 0)+\left(\frac{1}{2}-\frac{\lambda}{3}+\frac{\lambda^2}{12}\right).\end{aligned} \tag{6}$$

As a consequence of this property and (3), we have, $G(x;\theta,1)=F^2(x;\theta)$, proving the fact that the square of a continuous distribution function is also a distribution function. The rth raw moment of the distribution about origin is given by

$$\mu_r'(\theta,\lambda)=E[Y^r]=\begin{cases}\frac{2\theta^r}{(r+1)(r+2)} & r \text{ even}\\ \frac{2\lambda\theta^r}{r+3}\left(\frac{2}{r+2}-\frac{|\lambda|}{r+4}\right) & r \text{ odd}\end{cases}. \tag{7}$$

Thus the mean and variance of a $CT[\theta,\lambda]$ distribution are respectively $\mu(\theta,\lambda)=\theta\lambda(1/3-|\lambda|/10)$, and $\sigma^2(\theta,\lambda)=\theta^2/6-\mu^2$. The coefficient of skewness, which can be obtained using the moment expressions (7), is an odd function of λ (Figure 1).

If X follows a standard symmetric triangular distribution with upper boundary θ and Y is distributed as $CT[\theta,\lambda]$, then both $|X|$ and $|Y|$ has the same triangular distribution given by (1) with $\alpha=0,\xi=0$, and $\beta=\theta$. This property is very interesting in the sense that the absolute value of a curvi-triangular random variable with parameter λ does not depend on λ. As a consequence of this property, the distribution of any function of the absolute random variable $|Y|$, will be independent of λ. In particular, Y^2 has the density,

$$f_{Y^2}(y;\theta,\lambda)=f_{Y^2}(y;\theta)=\frac{1}{\theta}\left(\frac{1}{\sqrt{y}}-\frac{1}{\theta}\right),\ 0\le y\le\theta^2. \tag{8}$$

3 Inference

3.1 Random Number Generation

Generating random numbers from the $GCT(\theta,\lambda)$ distribution can be done through the standard triangular distribution. The following proposition, similar to the one proved for skew-normal random variables in Azzalini and Dalla Valle (1996), facilitates the generation of $GCT(\theta,\lambda)$ random variables.

Proposition 3.1. *Let $U_j, j = 1, 2, 3, 4$ are independent Uniform $U(-\theta/2, \theta/2)$ random variables. Define, $V_1 = U_1 + U_2$ and $V_2 = U_3 + U_4$, and set $X = V_1$ conditional on $\lambda V_1 > V_2$. Then $X \sim GCT(\theta, \lambda)$.*

The proof of the above proposition follows directly from the assumptions that V_1 and V_2 are independent standard triangular random variables whose pdf and cdf are respectively given by Equations (2) and (3). We give a brief outline here. From the definition of X, the pdf of X can be written as

$$f_X(x) = \frac{f_{V_1}(x)P(\lambda x > V_2)}{P(\lambda V_1 > V_2)}. \tag{9}$$

Since both V_1 and V_2 are symmetric about zero, so is $\lambda V_1 - V_2$, and therefore, $P(\lambda V_1 > V_2) = .5$. Substituting this result, the pdf of V_1 and cdf of V_2 in (9), we get, $f_X(x) = 2f_{V_1}(x)F_{V_2}(\lambda x),\ |x| \leq \theta$. Azzalini and Dalla Valle (1996) suggested that, in order to avoid rejection sampling, one could use a slightly different definition of X in the above proposition, namely, set $X = V_1$ when $\lambda V_1 > V_2$, and $X = -V_1$ when $\lambda V_1 \leq V_2$.

3.2 Estimation

We consider two methods, namely, the method of moments and the method of maximum likelihood. The following derivation assumes that $(x_1, x_2, \ldots, x_n)$ is a realization of a random sample $(X_1, X_2, \ldots, X_n)$ from a $GCT(\theta, \lambda)$ distribution. Method of moments (MOM) estimators would be the easiest to apply from the technical point of view. Of all the characteristics based on moments such as variance, coefficient of variation, measure of skewness and kurtosis, the coefficient of variation is a one-to-one function of λ and is independent of θ. Thus, the method of moments estimator for λ can be obtained by calculating the sample coefficient of variation, equating it to the population coefficient of variation and solving for λ.

Although, like many other distributions defined on the interval $[-\theta, \theta]$, the MOM estimator for θ will seldom be used, nevertheless, once the estimator for λ is obtained, the moment estimator for θ can be obtained as the solution to the first moment equation. The maximum likelihood estimates can be obtained by numerically maximizing the likelihood. We will investigate the properties of these two estimators through simulation.

3.3 Test for $\lambda = 0$

Testing the hypothesis $H_0 : \lambda = 0$ would be important from the practical point of view. For a given dataset, this will tell us whether one should fit a standard symmetric triangular distribution or the generalized curvi-triangular distribution. In most practical applications, the value of θ will be known. The test that will be most straightforward to implement for this purpose is the likelihood ratio test. The likelihood ratio LR simply takes the ratio of the maximum of the likelihood of $GCT(\theta, \lambda)$ distribution to that of the standard symmetric triangular distribution with parameter θ. For n sufficiently large, under H_0, the distribution of $-2\ln LR$ is well approximated by χ_1^2. Hence the critical values from χ_1^2-distribution can be used.

A Wald test, which is based on the $\hat{\lambda}^{MLE}$ can also be used. One difficulty in using Wald test is in estimating the standard error of $\hat{\lambda}^{MLE}$. We suggest using parametric or non-parametric bootstrap method for estimating this standard error. Thus, if $\hat{\lambda}_1^{MLE}, \hat{\lambda}_2^{MLE}, \ldots, \hat{\lambda}_B^{MLE}$ are B MLE's of λ from B bootstrapped samples for the given data, then the standard error of $\hat{\lambda}^{MLE}$ is obtained as

$$SE(\hat{\lambda}^{MLE}) = \sqrt{B^{-1}\sum_{k=1}^{B}\left(\hat{\lambda}_k^{MLE} - \bar{\hat{\lambda}}\right)^2}, \tag{10}$$

where $\bar{\hat{\lambda}} = \sum_{k=1}^{B}\hat{\lambda}_k^{MLE}/B$ is the bootstrap mean of B MLE's.

The other difficulty that arises in using the Wald test is that unless the sample size is very large, the distribution of the maximum likelihood estimator of λ is far from Normal [see Section 4]. This suggests that comparing the Wald-statistic $W = \hat{\lambda}^{MLE}/S.E.(\hat{\lambda}^{MLE})$ to standard normal critical values, as is usually done, would lead to biased inference. To overcome this difficulty, a complete bootstrap test can be used details of which can be found in Boos (2003) and Efron and Tibshirani (1993).

4 Simulation Study

To evaluate the performance of the method of moments and maximum likelihood estimators and to see how the distribution fits to a given dataset, we considered several simulation scenarios. Table 1 gives a sample of size 50 generated from a $GCT[5, 1.5]$ distribution. Corresponding histogram, actual distribution and the fitted distributions appear in Figure 2. We assume that the parameter θ is known. The method of moment estimator of λ for this particular dataset is 1.44 and the MLE is 1.75. It looks like the distribution fits quite well to the data and the MLE and MOM estimator are quite close to the true value of λ.

To see how the MLE is distributed, We generated 100 samples of different sizes from $GCT[5, 1.5]$ distribution. Corresponding histograms are presented in Figure 4. It is observed that the distribution of MLE of λ is quite skewed, even for moderately large samples. Only for samples of sizes 500 or larger, the distribution becomes closer to normal. The estimator seems to be unbiased, but realizes some extreme values when sample size is small. These characteristics imply that one should take caution in using the maximum likelihood estimator as a base for large-sample inference when the sample in fact is not very large.

To investigate the power of the Chi-square approximation of the likelihood ratio test for testing $H_0 : \lambda = 0$, we considered simulations based on 200 samples for different λ values in the range $(-2.5, 2.5)$. It is observed that the test is powerful for detecting a departure from the standard symmetric triangular distribution. Figure 4 shows the power function of the test at 5% level of significance, calculated based on 200 Monte-Carlo samples of size 50 each.

5 Discussion

We have presented a family of distribution (generalized curvi-triangular distribution) which includes the standard symmetric triangular distribution as a special case. The generalized curvi-triangular family is more flexible than the triangular family in the sense that it allows for densities which are smooth curves (as opposed to straight lines). This makes the new family more suitable for practical applications. Although some complexity arises due to its irregular shape (range depends on both parameters) in estimation and testing procedures, in an era of computing revolution, this will not hold back the application of the proposed family of distributions.

Acknowledgement

I am grateful to Prof. Mir Masoom Ali for introducing me to this area of my research while I was conducting my M.A. thesis under his supervision. My warmest congratulation to him on his 70th birthday.

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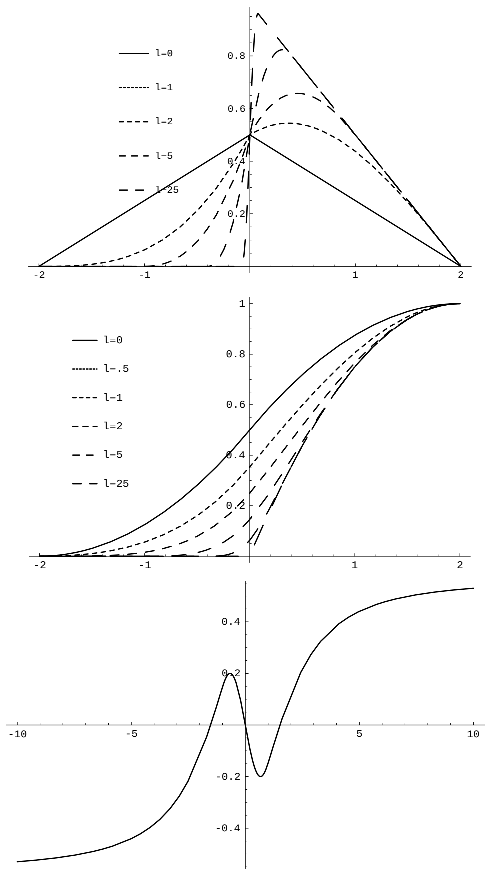

Figure 1: Some properties of $GCT[\theta = 2, l]$ distribution. Top panel: densities, and middle panel: distribution functions, and bottom panel: skewness ($\gamma_1 = \mu_3/\sqrt{\mu_2^3}$, left) and kurtosis ($\beta_2 = \mu_4/\mu_2^2$, right) plotted as a function of λ.

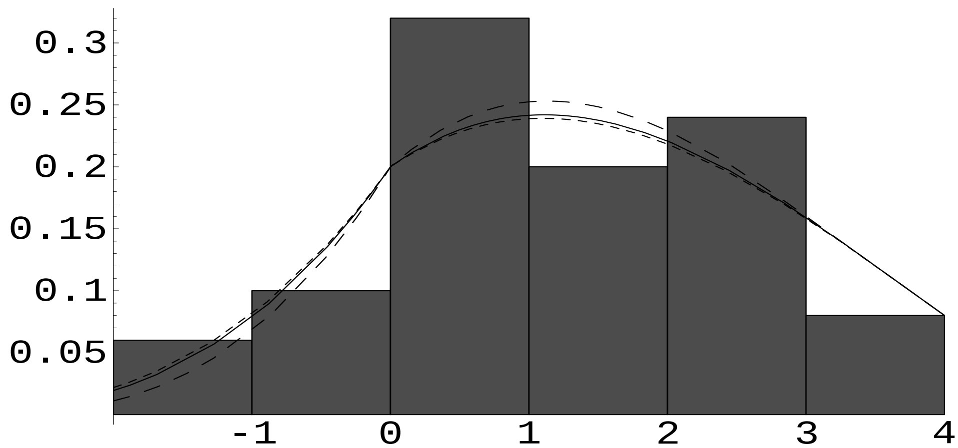

Figure 2: Histogram for the dataset presented in Table 1 and the true distribution along with the fitted curves. Solid line: true distribution, dotted line: method of moments fit, and the dashed line is the fit by maximum likelihood.

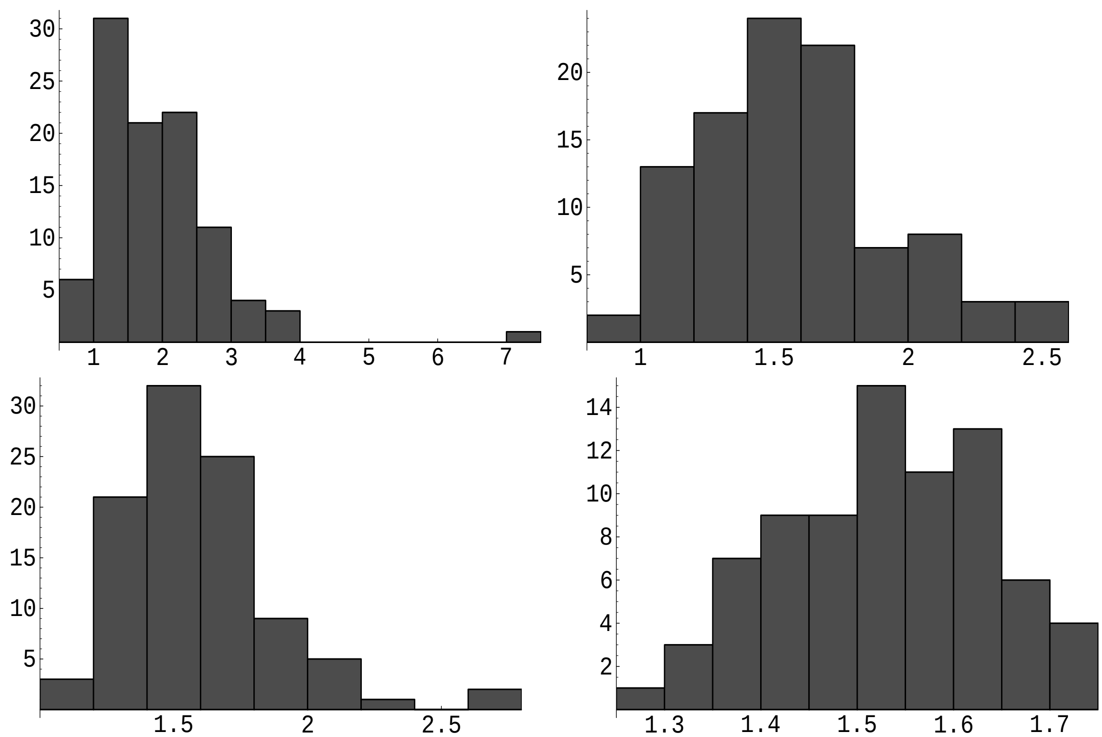

Figure 3: Distribution of maximum likelihood estimator in action for 100 samples of size n from $GCT[5, 1.5]$ distribution. Top left: $n = 20$, Top right: $n = 50$, Bottom left: $n = 100$, Bottom right: $n = 500$;

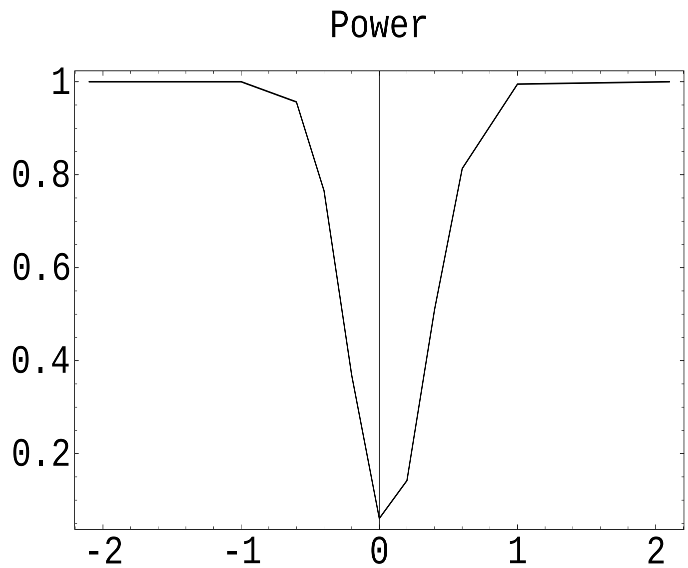

Figure 4: Power of likelihood ratio test for $H_0 : \lambda = 0$

Table 1: Sorted sample of size 50 from a $GCT[5, 1.5]$ distribution.

−1.67376	−1.10408	−1.01684	−0.786067	−0.686443
−0.118734	−0.101215	−0.098667	0.0557369	0.128512
0.160828	0.202319	0.22918	0.317187	0.330396
0.423432	0.495496	0.524236	0.594164	0.637784
0.722675	0.767406	0.851608	0.887698	1.00336
1.01973	1.21522	1.40519	1.44812	1.51873
1.62831	1.71586	1.94216	1.96565	2.12891
2.2568	2.2808	2.31929	2.55219	2.5539
2.56224	2.61798	2.6521	2.91498	2.94552
2.97861	3.44497	3.60133	3.80916	3.88195