International Journal of Statistical Sciences *ISSN 1683–5603*
Vol. 6 (Special Issue), 2007, pp 109-126

Estimation of The Edgeworth Expansion Terms in Hilbert Space and one F. Götze's Conjecture

S. V. Nagaev
Sobolev Institute of Mathematics
Koptjug prospekt 4, 630090
Novosibirsk, Russia
E-mail: nagaev@math.nsc.ru

V. I. Chebotarev
Computing Centre of the Far-Eastern Branch
of the Russian Academy of Science
Kim Yu Chen str. 153, 680042
Khabarovsk, Russia
E-mail: cheb_8@mail.ru, chebotarev@as.khb.ru

[Received December 30, 2005; Revised July 9, 2007; Accepted September 8, 2007]

Abstract

Estimation of the second term of the Edgeworth expansion for the probability of hitting to balls in Hilbert space of the sum of independent identically distributed elements is produced. Basing on the results obtained, a conjecture on a proper form of the error in the so-called brief Edgeworth expansion is proposed. We suppose that information on not less than six nonzero eigenvalues of the initial covariance operator is necessary to get the error bound of the order $O(1/n)$ in contrast to the well-known F. Götze conjecture, in which five eigenvalues play the related role.

Keywords and Phrases: Gaussian approximation, covariance operator.

AMS Classification: 60B12.

Let $\mathbf{H}$ be a separable real Hilbert space with the norm $|\cdot|$ and the inner product $(\cdot,\cdot)$. Let $X, X_1, X_2, \ldots$ be $\mathbf{H}$-valued i.i.d. random variables with $\mathbf{E}X = 0$ and a covariance operator T. Let Y be a centered Gaussian random variable with the same covariance operator.

Denote

$$S_n = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} X_j.$$

Let $a \in \mathbf{H}$, $r \in \mathbb{R}$. The expansion of the probability $F_n(r) \equiv \mathbf{P}\big(|S_n - a|^2 < r\big)$ in powers of $n^{-1/2}$ is said to be the formal Edgeworth expansion of $F_n(r)$. We write

$$\mathbf{P}\big(|S_n - a|^2 < r\big) = \mathbf{P}\big(|Y - a|^2 < r\big) + \sum_{\nu=1}^{\infty} \frac{Q_\nu(r;a)}{n^{\nu/2}}. \tag{1}$$

Define

$$\Delta_n(a) = \sup_r \Big|\mathbf{P}\big(|S_n - a|^2 < r\big) - \mathbf{P}\big(|Y - a|^2 < r\big) - \frac{Q_1(r;a)}{n^{1/2}}\Big|.$$

We shall call any estimate of $\Delta_n(a)$ by the estimate of the remainder term in the brief Edgeworth expansion of $F_n(r)$.

As C.-G. Esseen noted in his famous work in 1945, this problem is closely connected with a problem of number theory, namely with the problem of optimal bound of the difference between the number of integer points in a multivariate ellipsoid and its volume. The probability problem was being decided in [4; 5; 8; 14 - 16]. Some results in the corresponding number theory problem were also obtained in [2; 3; 6].

In [4; 5; 8; 14 - 16] a proper dependence on the number of summands $O(1/n)$, and on the moments β_4 and β_3^2 is found for the error.

In connection with this we remind the following fact which is well-known to the specialists in Gaussian approximation for multivariate distributions. The error of Gaussian approximation on balls in a many-dimensional space is inversely proportional to a positive power of a product of eigenvalues of the initial covariance operator. If the space is infinite-dimensional, the sequence of the eigenvalues tends to zero. Then an estimate of the error is so much the worse, the number of eigenvalues, incoming to the estimate, is more. Consequently, to find the true dependence of the error from the covariance operator is an importance problem of probability theory (see, for instance, the works by S. Nagaev, V. Chebotarev [12], and V. Senatov [18]).

It was in [14-16] shown also that the dependence of the error on the operator T is expressed, in particular, with the help of the factor $\Lambda_{13}^{-4/13}$, and in [8] with the help of the factor $\Lambda_{12}^{-1/2}$. The following question remaines open until now: what dependence of $\Delta_n(a)$ on moments, the covariance operator and the center of the ball a, is regular in fact?

Since the first "rejected" term in the brief expansion is the expression $Q_2(r;a)/n$, by analogy with the Taylor expansion a regular bound of $Q_2(r;a)$ might approximate us to the answer the question formulated.

In what follows we use the notations: $\sigma_1^2 \geq \sigma_2^2 \geq \ldots$ are the eigenvalues of the operator T, $e_1, e_2, \ldots$ are the following eigenvectors,

$$\Lambda_l = \prod_{j=1}^{l} \sigma_j^2, \quad \sigma^2 = \mathbf{E}|X|^2, \quad \beta_\mu = \mathbf{E}|X|^\mu, \quad \beta_\mu(a) = \mathbf{E}|(a, X)|^\mu.$$

The symbols c, c_i $(i = 1, 2, \ldots)$ will denote absolute constants.

In the paper we prove

Theorem 6. *The following bounds of $Q_2(r; a)$ hold,*

$$\sup_r |Q_2(r; a)| \le c\Big(\frac{\beta_4}{\Lambda_5^{2/5}} + \frac{\beta_3^2}{\Lambda_7^{3/7}} + \frac{\beta_4(|a|^4 + |a|^2\sigma^2)}{\Lambda_9^{4/9}} + \frac{\beta_3^2(|a|^6 + |a|^2\sigma^4)}{\Lambda_{13}^{6/13}}\Big) \tag{2}$$

$$\le c\beta_4\sigma^2\big[1/\Lambda_7^{3/7} + (|a|^6 + |a|^2\sigma^4)/\Lambda_{13}^{6/13}\big]. \tag{3}$$

Moreover,

$$\sup_r |Q_2(r; a)| \le c_1 \frac{\sigma^8}{\Lambda_8^{1/2}}\Big[\frac{\beta_4(a)}{\sigma^8} + \frac{\beta_4}{\sigma^4}\Big] + c_2 \frac{\sigma^{12}}{\Lambda_{12}^{1/2}}\Big[\frac{\beta_3^2(a)}{\sigma^{12}} + \frac{\beta_3^2}{\sigma^6}\Big]. \tag{4}$$

The analysis of the estimates (2) – (4), and their proofs lead us to the following conjecture.

Conjecture 1. *If* $\mathbf{H}$ *is the space of the dimension $d \ge 12$ (the case $d = \infty$ is considered too), and $\sigma_{12} \ne 0$, then*

$$\Delta_n(a) \le c\, n^{-1}\big[\delta(0) + \delta(a)\big], \tag{5}$$

where

$$\delta(0) = \beta_4\Lambda_4^{-1/2} + \beta_3^2\Lambda_6^{-1/2}, \quad \delta(a) = \gamma_1(a)\Lambda_8^{-1/2} + \gamma_2(a)\Lambda_{12}^{-1/2}, \tag{6}$$

$$0 < \gamma_j(a) \to 0 \ \textit{as}\ |a| \to 0,\ j = 1, 2, \tag{7}$$

$$\gamma_1(a) \le \beta_4|a|^4, \quad \gamma_2(a) \le \beta_3^2|a|^6, \quad \textit{when}\ |a| \ge \sigma. \tag{8}$$

In particular, this means that in the case $a = 0$ the error $\Delta_n(0)$ depends on *six* (or not less then six) the first eigenvalues of the covariance operator T, and this contradicts to the well-known F. Götze's conjecture [4,5], according to which $\Delta_n(0)$ must depend only on *five* eigenvalues of the operator T.

Note 1. Remark some properties of the bound (5). The right-hand side of (5) is the sum of two quantities, where the first one doesn't depend on a, and in the second one the dependence on a is expressed by the properties (7) and (8). The quantities $\delta(0)$ and $\delta(a)$ depend on the covariance operator T differently. One can say that $\delta(0)$ depends on T weaker, than $\delta(a)$: the summands, which $\delta(0)$ consists of, contain the factors $\Lambda_4^{-1/2}$ and $\Lambda_6^{-1/2}$ while $\delta(a)$ contains as the factors, $\Lambda_8^{-1/2}$ and $\Lambda_{12}^{-1/2}$. If $a = 0$, then the bound (5) becomes the more simple inequality,

$$\Delta_n(0) \le c\,n^{-1}\delta(0) \equiv c\,n^{-1}\big(\beta_4\Lambda_4^{-1/2} + \beta_3^2\Lambda_6^{-1/2}\big). \tag{9}$$

Comparing the summands in (6), we note that in the summands, containing the factor β_4, the dependence on the covariance operator is weaker, than in the summands, containing β_3. *A balance* arises: for each summand in (6) an amplification of the dependence on the moments is accompanied with the weakening of the dependence on the covariance operator.

As to $\gamma_j(a)$, one can assume, they have the following form: $\gamma_1(a) = \beta_4(a)$, $\gamma_2(a) = \beta_3^2(a)$.

Note 2. As an example of a bound, containing two parts in a sense of Note 1, we can cite the estimate of S. V. Nagaev [11],

$$\sup_r \Big|\mathbf{P}\big(|S_n - a|^2 < r\big) - \mathbf{P}\big(|Y - a|^2 < r\big)\Big| \le \frac{c\beta_3}{\sqrt{n}}\Big(\frac{\sigma}{\Lambda_4^{1/2}} + \frac{|a|^3 + |a|^{3/2}\sigma^{3/2}}{\Lambda_6^{1/2}}\Big).$$

improving the following known result [9;19],

$$\sup_r \Big|\mathbf{P}\big(|S_n - a|^2 < r\big) - \mathbf{P}\big(|Y - a|^2 < r\big)\Big| \le \frac{c\beta_3}{\sqrt{n}}\,\frac{|a|^3 + \sigma^3}{\Lambda_6^{1/2}}.$$

In what follows the symbol $\square$ will denote the end of the proof.

2. A comparison of the known results with Conjecture 1

Denote

$$\Gamma_{\mu,l} = \beta_\mu\sigma^\mu/\Lambda_l^{\mu/l}, \quad \Gamma_{\mu,l}(a) = \beta_\mu(a)/\Lambda_l^{\mu/l}, \quad L_l = \max_{1\le j\le l}\frac{\mathbf{E}|(X, e_j)|^3}{\sigma_j^3}.$$

The following bound of $\Delta_n(a)$ obtained in [1;15;16], using [10;14].

Theorem 7. *Let* $\mathbf{H}$ *be a Hilbert space of the dimension* $d \ge 13$, *and* $\sigma_{13} \ne 0$. *There exists an absolute constant* c *such that for every* $a \in \mathbf{H}$,

$$\Delta_n(a) \le \frac{c}{n}\Big(\Gamma_{4,13} + \Gamma_{3,13}^2 + L_9^2\left(\sigma^2/\Lambda_9^{1/9}\right)^2 + \Gamma_{4,9}(a) + \Gamma_{3,13}^2(a)\Big). \tag{10}$$

This bound of $\Delta_n(a)$ yields the more precise dependence on the covariance operator T and the center a, than the following result by V. Bentkus and F. Götze [5]: if $\mathbf{H}$ is a Hilbert space of the dimension $d \geq 13$, and $\sigma_{13} \neq 0$, then

$$\Delta_n(a) \leq \exp\Big\{\frac{c\sigma^2}{\sigma_{13}^2}\Big\}\Big\{\frac{\mathbf{E}\left[|X|^4 I(|X| \leq \sigma\sqrt{n})\right]}{\sigma^4 n} + \frac{\mathbf{E}\left[|X|^3 I(|X| > \sigma\sqrt{n})\right]}{\sigma^3\sqrt{n}}\Big\}\Big(1+\frac{|a|^6}{\sigma^6}\Big).$$

Formulate also the bound proved by F. Götze and V. Ulyanov in the preprint [8],

$$\Delta_n(a) \leq \frac{c}{n}\Big[\beta_4\Big(\frac{\sigma^8}{\Lambda_{12}^{1/2}} + \frac{\sigma^4\sigma_1^4}{\sigma_9^8\Lambda_9^{2/9}} + \frac{\sigma^{9/2}}{\sigma_9^4\Lambda_9^{1/4}}\Big) + \frac{\beta_4(a)}{\Lambda_{12}^{1/2}}\Big(\sigma^4 + \beta_2(a)\Big)\Big]. \tag{11}$$

Note 3. It is obvious that in the case, when σ_{13}^2 is small with respect to the previous eigenvalues $\sigma_1^2, \dots, \sigma_{12}^2$, the bound (11) is better, than (10). But one can find conditions on T, under which the relation between these bounds is inverse. Indeed, let $\sigma^2 = \sum_1^k \sigma_j^2 = 1$, $\sigma_1^2 = \sigma_2^2 = 1/4$, $\sigma_3^2 = 1/\sqrt{k}$, $\sigma_j^2 = (1/2 - 1/\sqrt{k})/(k-3)$ for $4 \leq j \leq k$. Let $k \to \infty$. Then, as immediate calculations show, firstly, in the bound (11) the quantity $\frac{\sigma_1^4}{\sigma_9^8\Lambda_9^{2/9}}$ majorizes the quantity $\frac{1}{\Lambda_{12}^{1/2}}$. Secondly, in the bound (10) $\frac{L_9^2}{\Lambda_9^{2/9}}$ majorizes $\Gamma_{3,13}^2$ (we may consider that $\mathbf{E}|(X, e_j)|^3 \geq c\beta_3$, $1 \leq j \leq 9$, for some $c > 0$). And, moreover, $\frac{\beta_4\sigma_1^4}{\sigma_9^8\Lambda_9^{2/9}}$ majorizes $\frac{L_9^2}{\Lambda_9^{2/9}}$. This means, that for the distribution of X under consideration, the bound (11) is less precise than (10).

It is shown in [8; Lemma 2.6] that in the Euclidean space $\mathbb{R}^{13}$ there exist a distribution of X and balls with centers a: $|a| > 1$, such that for given values $\sigma_1^2, \dots, \sigma_{12}^2 > 0$ of the eigenvalues of the operator T,

$$\liminf_{n\to\infty} n\Delta_n(a) \geq c\Lambda_{12}^{-1/2}|a|^6\beta_4.$$

This result implies the following

Claim 1. *Any explicit bound of $\Delta_n(a)$ has to depend on the first 12 eigenvalues of T.*

Note 4. Conjecture 1 is in accordance with Claim 1. In (11) the part, depending on a, is regular from the point of view both of Claim 1 and Conjecture 1. From this point of view, the part, depending on a, in (10), may be considered as *almost* regular. But the inequality (10) have other advantages with respect to (11). In contrast to (11), it reflects the property (8), and the *balance* property as well (see note 1, p. 112). On the other hand, it should be noted, that the parts of the bounds (10) and (11), which don't depend on a, are far from the optimal in the sense of Conjecture 1.

3. Basis of Conjecture 1

The following expansion corresponds to (1),

$$\mathbf{E}\exp\{it|S_n - a|^2\} = g(t;a) + \sum_{\nu=1}^{\infty}\frac{\widehat{Q}_\nu(t;a)}{n^{\nu/2}}, \tag{12}$$

where $g(t;a) = \mathbf{E}\exp\{it|Y-a|^2\}$, $\widehat{Q_\nu}(t;a) = \int_0^\infty e^{itr}\, dQ_\nu(r;a)$. Note that there are different algorithms of calculation of $\widehat{Q_\nu}(t;a)$, and the proof of relative identities is a particular problem (see [13; Subsection 1.4])

Now we shall find $\widehat{Q_2}(t;a)$.

Let $\mathbf{H} = \mathbb{R}$. It is well known [17] that in the formal Edgeworth expansion

$$\mathbf{E}e^{itS_n} = e^{(it)^2\sigma^2/2}\Big(1 + \sum_{\nu=1}^{\infty} \frac{p_\nu(it;X)}{n^{\nu/2}}\Big)$$

the functions $p_\nu(it;X)$ have the form

$$p_\nu(it;X) = \sum_{\{q\,\mu_q\}_1^\nu} \prod_{q=1}^{\nu} \big[(it)^{q+2}\,\varkappa_{q+2}(X)/(q+2)!\big]^{\mu_q}/\mu_q!, \tag{13}$$

where $\varkappa_{q+2}(X)$ is the cumulant of the order $q+2$, $\sum\limits_{\{q\,\mu_q\}_1^\nu}$ is the sum over all nonnegative integers $\mu_1,\ldots,\mu_\nu$, such that $\sum_{q=1}^{\nu} q\mu_q = \nu$.

Express the cumulants $\varkappa_{q+2}(X)$ via moments of X, and represent $p_\nu(it;X)$ as the following sum,

$$p_\nu(it;X) = \sum\nolimits^{\sharp(\nu)} a_\nu(j_1,\ldots,j_m)(it)^M \prod_{q=1}^{m} \mathbf{E}X_q^{j_q}, \tag{14}$$

where $M = \sum_{q=1}^{m} j_q$, and $\sum^{\sharp(\nu)}$ denotes the summation over all nonnegative integers $j_1,\ldots,j_m$ such that

$$2 \le j_q \le \nu + 2, \quad M \le \nu + 2m, \quad m \le \nu. \tag{15}$$

Let α_j be independent standard Gaussian variables, which doesn't depend on Y and X_j, $j = 1,\ldots,n$, too. Denote

$$\alpha = (\alpha_1, \alpha_2, \ldots), \qquad (x,\alpha) = \sum_{j=1}^{\infty} \alpha_j (x, e_j),$$

$$g_j(t) = (1 - 2it\sigma_j^2)^{-1/2}, \qquad A_t x = \sum_{j=1}^{\infty} g_j(t)(x,e_j)e_j, \qquad s = (2it)^{1/2}.$$

Let $\{j_q\}_{q=1}^m$ be a fixed sequence of nonnegative integers. Define $\sum^{\flat(m,\{j_q\})}$ as the summation over all matrices $\{\nu_{pq}\}_{p,q=1}^m$ and sequences $\{t_q\}_{q=1}^m$ of nonnegative integers, such that for every $1 \le q \le m$

$$\nu_q + \widetilde{\nu}_q + t_q = j_q, \tag{16}$$

where $\nu_q = \sum_{p=1}^m \nu_{pq}$, $\widetilde{\nu}_q = \sum_{p=1}^m \nu_{qp}$.

Lemma 1. *The following representations of coefficients* $\widehat{Q}_\nu(t;a)$ *from* (12) *hold,*

$$\widehat{Q}_\nu(t;a) = g(t;a)\mathbf{E}_\alpha p_\nu\big(s;(A_tX,\alpha - sA_ta)\big) = \mathbf{E}e^{it|Y-a|^2} p_\nu\Big(s;\big(X,\alpha+s(Y-a)\big)\Big)$$

$$= g(t;a)\sum\nolimits^{\sharp(\nu)} \cos(\pi M) a_\nu(j_1,\dots,j_m)\Big(\prod_{q=1}^m j_q!\Big)\sum\nolimits^{\flat(m,\{j_q\})} \frac{s^{2(M-r)}}{2^r}$$

$$\times\,\mathbf{E}\prod_{q=1}^m\Big[\frac{(A_tX_q,A_ta)^{t_q}}{t_q!}\prod_{p=1}^m\frac{(A_tX_p,A_tX_q)^{\nu_{pq}}}{\nu_{pq}!}\Big]\Big\} \quad (17)$$

$$= \sum\nolimits^{\sharp(\nu)} a_\nu(j_1,\dots,j_m)\Big(\prod_{q=1}^m j_q!\Big)$$

$$\times\sum\nolimits^{\flat(m,\{j_q\})} \frac{s^{2(M-r)}}{2^r}\mathbf{E}e^{it|Y-a|^2}\prod_{q=1}^m\Big[\frac{(X_q,Y-a)^{t_q}}{t_q!}\prod_{p=1}^m\frac{(X_p,X_q)^{\nu_{pq}}}{\nu_{pq}!}\Big], \quad (18)$$

where $a_\nu(j_1,\dots,j_m)$ *are the quantities from* (14), $r=\sum_{p,q=1}^m \nu_{pq}$.

Sketch of the proof. We suppose temporary that X is a bounded random variable. The following equalities hold,

$$\mathbf{E}e^{it|S_n-a|^2} \underset{[9]\text{ or }[1,\text{ p. }90]}{=} \mathbf{E}_\alpha\mathbf{E}_{S_n}e^{s(S_n-a,\alpha)} \underset{[1,\text{ p. }178]}{=} \mathbf{E}_\alpha e^{s^2(T\alpha,\alpha)-s(a,\alpha)}\Big(1+\sum_{\nu=1}^\infty\frac{p_\nu\big(s;(X,\alpha)\big)}{n^{\nu/2}}\Big)$$

$$\underset{[1,\text{ p. }179]}{=} g(t;a)\mathbf{E}_\alpha\Big(1+\sum_{\nu=1}^\infty\frac{p_\nu\big(s;(A_tX,\alpha-sA_ta)\big)}{n^{\nu/2}}\Big) \quad (19)$$

$$\underset{[1,\text{ p. }177]}{=} g(t;a)\Big(1+\sum_{\nu=1}^\infty\sum\nolimits^{\sharp(\nu)}\frac{a_\nu(j_1,\dots,j_m)}{n^{\nu/2}}s^M\mathbf{E}_\alpha\prod_{q=1}^m\mathbf{E}_{X_q}(A_tX_q,\alpha-sA_ta)^{j_q}\Big) \quad (20)$$

$$\underset{[1,\text{ p. }222]}{=} g(t;a)+\sum_{\nu=1}^\infty\sum\nolimits^{\sharp(\nu)}\frac{a_\nu(j_1,\dots,j_m)}{n^{\nu/2}}s^M\mathbf{E}e^{it|Y-a|^2}\prod_{q=1}^m\mathbf{E}_{X_q}(X_q,\alpha+s(Y-a))^{j_q}. \quad (21)$$

Moreover, by (20) and [1, p. 141],

$$\mathbf{E}e^{it|S_n-a|^2} = g(t;a)\Big\{1+\sum_{\nu=1}^\infty\frac{1}{n^{\nu/2}}\sum\nolimits^{\sharp(\nu)}\cos(\pi M)a_\nu(j_1,\dots,j_m)\Big(\prod_{q=1}^m j_q!\Big)$$

$$\times\sum\nolimits^{\flat(m,\{j_q\})}\frac{s^{2(M-r)}}{2^r}\mathbf{E}\prod_{q=1}^m\Big[\frac{(A_tX_q,A_ta)^{t_q}}{t_q!}\prod_{p=1}^m\frac{(A_tX_p,A_tX_q)^{\nu_{pq}}}{\nu_{pq}!}\Big]\Big\}. \quad (22)$$

Analogously, by virtue of (21) and [1, p. 141],

$$\mathbf{E}e^{it|S_n-a|^2} = g(t;a) + \sum_{\nu=1}^{\infty} \frac{1}{n^{\nu/2}} \sum\nolimits^{\sharp(\nu)} a_\nu(j_1,\dots,j_m)\Big(\prod_{q=1}^m j_q!\Big)$$
$$\times \sum\nolimits^{\flat(m,\{j_q\})} \frac{s^{2(M-r)}}{2^r} \mathbf{E}e^{it|Y-a|^2} \prod_{q=1}^m \Big[\frac{(X_q, Y-a)^{t_q}}{t_q!} \prod_{p=1}^m \frac{(X_p,X_q)^{\nu_{pq}}}{\nu_{pq}!}\Big]. \quad (23)$$

Lemma 1 follows from (19) – (23). □

Denote

$$\xi_q(a) = (A_t X_q, A_t a), \quad \xi_{pq} = (A_t X_p, A_t X_q), \quad \eta_q(a) = (X_q, Y-a), \quad \eta_{pq} = (X_p, X_q),$$

$$I(\mathcal{A}) = \begin{cases} 1, & \text{if a condition } \mathcal{A} \text{ is fulfilled,} \\ 0, & \text{otherwise.} \end{cases}$$

We shall say that the condition $\mathcal{A}_\nu$ is fulfilled (or $p_\nu(it;X)$ satisfies the condition $\mathcal{A}_\nu$), if for some sequence $j_1, \dots, j_m$ from (15) there exists a matrix $\{\nu_{pq}\}_{p,q=1}^m$, such that in the equalities (16) we have $t_1 = \dots = t_m = 0$, i.e.

$$\nu_q + \widetilde{\nu}_q = j_q, \quad q = 1, \dots, m.$$

Notice that $p_1(it;\xi) = \frac{(it)^3}{3!}\mathbf{E}X^3$ doesn't satisfy the condition $\mathcal{A}_1$, but

$$p_2(it;X) = \frac{(it)^6}{2(3!)^2}\prod_{j=1}^2 \mathbf{E}\,X_j^3 + \frac{(it)^4}{4!}\Big(\mathbf{E}\,X_1^4 - 3\prod_{j=1}^2 \mathbf{E}\,X_j^2\Big) \quad (24)$$

satisfies $\mathcal{A}_2$. We introduce this definition to select cases when $\widehat{Q}_\nu(t;a)$ may be splitted on two parts: a part depending on a, and the second one not depending on a.

Lemma 2. *The following inequality holds,*

$$|\widehat{Q}_\nu(t;a)| \le \frac{c(\nu)\,\beta_{\nu+2}}{\sigma^{\nu+2}}\,|g(t;a)|\bigg\{\Big((|t|\sigma^2)^{3\nu} + |t|\sigma^2\Big)$$
$$\times \Big[\Big(\frac{|a|}{\sigma}\Big)^{3\nu} + \frac{|a|}{\sigma}\Big] + I(\mathcal{A}_\nu)\Big((|t|\sigma^2)^{3\nu/2} + |t|\sigma^2\Big)\bigg\}.$$

Proof. Notice that

$$\big|\xi_q(a)^{t_q}\xi_{pq}^{\nu_{pq}}\big| \le |X_q|^{t_q+\nu_{pq}}|X_p|^{\nu_{pq}}|a|^{t_q}.$$

Since $\nu_q + \widetilde{\nu}_q + t_q = j_q$, $M = \sum_{q=1}^m j_q$,

$$\Big|\mathbf{E}\prod_{q=1}^m \xi_q(a)^{t_q}\prod_{p=1}^m \xi_{pq}^{\nu_{pq}}\Big| \le |a|^{M-2r}\prod_{q=1}^m \beta_{j_q}.$$

Now we shall use the Liapunov inequality: if $2 \le \mu \le N$ and $\mathbf{E}|\xi|^N < \infty$ then

$$\mathbf{E}|\xi|^\mu \le \Big(\mathbf{E}^{\mu-2}|\xi|^N\,\mathbf{E}^{N-\mu}|\xi|^2\Big)^{1/(N-2)}, \tag{25}$$

where $\mathbf{E}^k\zeta \equiv \big(\mathbf{E}\zeta\big)^k$. Putting in (25) $\mu = j_q$, $N = \nu+2$, we obtain

$$\mathbf{E}|X|^{j_q} \le \Big(\mathbf{E}^{j_q-2}|X|^{\nu+2}\,\mathbf{E}^{\nu+2-j_q}|X|^2\Big)^{1/\nu}.$$

It follows from here and the condition (15), that

$$\prod_{q=1}^m \beta_{j_q} \le \beta_{\nu+2}^{(M-2m)/\nu}\,\sigma^{2(\nu m+2m-M)/\nu} = \beta_{\nu+2}\frac{\sigma^{2(\nu m+2m-M)/\nu}}{\beta_{\nu+2}^{(\nu-M+2m)/\nu}} \le \beta_{\nu+2}\sigma^{M-\nu-2}.$$

Consequently, by Lemma 1,

$$|\widehat{Q}_\nu(t;a)| \le \frac{c(\nu)\,\beta_{\nu+2}}{\sigma^{\nu+2}}|g(t;a)|\sum\nolimits^{\sharp(\nu)}|t|^{M-r}\sigma^M|a|^{M-2r}. \tag{26}$$

Notice that by the condition (16), $2r+\sum_{q=1}^m t_q = M$. Write up

$$\sum\nolimits^{\sharp(\nu)}|t|^{M-r}\sigma^M|a|^{M-2r} = \sum\nolimits^{\sharp(\nu)}\Big(I(2r=M)+I(2r<M)\Big)|t|^{M-r}\sigma^M|a|^{M-2r}.$$

It is not hard to see that

$$\begin{aligned}\sum\nolimits^{\sharp(\nu)}I(2r=M)|t|^{M-r}\sigma^M|a|^{M-2r} &= \sum\nolimits^{\sharp(\nu)}I(2r=M)|t|^{\frac{M}{2}}\sigma^M\\ &\le I(\mathcal{A}_\nu)c(\nu)\Big((|t|\sigma^2)^{\frac{3\nu}{2}}+|t|\sigma^2\Big),\\ \sum\nolimits^{\sharp(\nu)}I(2r<M)|t|^{M-r}\sigma^M|a|^{M-2r} &= c(\nu)\sum\nolimits^{\sharp(\nu)}I(2r<M)(|t|\sigma^2)^{M-r}\Big((|a|/\sigma)^{3\nu}+|a|/\sigma\Big)\\ &\le c_1(\nu)\Big((|t|\sigma^2)^{3\nu}+|t|\sigma^2\Big)\Big((|a|/\sigma)^{3\nu}+|a|/\sigma\Big).\end{aligned}$$

Thus, Lemma 2 follows from (26). $\square$

Lemma 3. *The following equalities hold,*

$$\widehat{Q}_2(t;a) = g(t;a)K(t;a), \tag{27}$$

where

$$\begin{aligned}K(t;a) = \frac{1}{8}\bigg[&\frac{s^{12}}{9}\,\mathbf{E}^2\xi_1^3(a) + \frac{s^{10}}{3}\Big(3\mathbf{E}\xi_1^2(a)\xi_2^2(a)\xi_{12} + 2\mathbf{E}\xi_1(a)\xi_{11}\mathbf{E}\xi_1^3(a)\Big)\\ &+ s^8\Big(\mathbf{E}^2\xi_1(a)\xi_{11} + \mathbf{E}\xi_1(a)\xi_2(a)\xi_{12}^2 + \frac{1}{3}\mathbf{E}\xi_1^4(a) - \mathbf{E}^2\xi_1^2(a)\Big)\bigg]\\ &+ \frac{s^6}{4}\Big(\frac{1}{2}\mathbf{E}\xi_{11}\xi_{12}\xi_{22} + \frac{1}{3}\mathbf{E}\xi_{12}^3 + \mathbf{E}\xi_1^2(a)\xi_{11} - 2\mathbf{E}\xi_1(a)\xi_2(a)\xi_{12} - \mathbf{E}\xi_1^2(a)\mathbf{E}\xi_{11}\Big)\\ &+ \frac{s^4}{8}\Big(\mathbf{E}\xi_{11}^2 - \mathbf{E}^2\xi_{11} - \mathbf{E}\xi_{12}^2)\Big),\end{aligned}$$

and

$$\widehat{Q}_2(t;a) = \mathbf{E}e^{it|Y-a|^2}\Big\{\frac{1}{8}\Big[\frac{s^{12}}{9}\,\eta_1^3(a)\eta_2^3(a) + \frac{s^{10}}{3}\Big(3\eta_1^2(a)\eta_2^2(a)\eta_{12} + 2\eta_1(a)\eta_{11}\eta_2^3(a)\Big)$$
$$+ s^8\Big(\eta_1(a)\eta_{11}\eta_2(a)\eta_{22} + \eta_1(a)\eta_2(a)\eta_{12}^2 + \frac{1}{3}\eta_1^4(a) - \eta_1^2(a)\eta_2^2(a)\Big)\Big]$$
$$+ \frac{s^6}{4}\Big(\frac{1}{2}\eta_{11}\eta_{12}\eta_{22} + \frac{1}{3}\eta_{12}^3 + \eta_1^2(a)\eta_{11} - 2\eta_1(a)\eta_2(a)\eta_{12} - \eta_1^2(a)\eta_{22}\Big)$$
$$+ \frac{s^4}{8}\Big(\eta_{11}^2 - \eta_{11}\eta_{22} - \eta_{12}^2)\Big)\Big\}. \quad (28)$$

Moreover, the following inequalities hold,

$$|\widehat{Q}_2(t;a)| \le c\,|g(t;a)|\Big\{\beta_4\Big(t^4|a|^4 + |t|^3|a|^2 + t^2\Big) + \beta_3^2\Big(t^6|a|^6 + t^4|a|^2 + |t|^3\Big)\Big\} \quad (29)$$
$$\le \frac{c\,\beta_4}{\sigma^4}|g(t;a)|\Big\{\Big((t\sigma^2)^6 + (|t|\sigma^2)^3\Big)\Big[\Big(\frac{|a|}{\sigma}\Big)^6 + \Big(\frac{|a|}{\sigma}\Big)^2\Big] + (|t|\sigma^2)^3 + (t\sigma^2)^2\Big\}.$$

Proof. Using Lemma 1, we obtain from (24), that

$$\widehat{Q}_2(t;a) = g(t;a)\Big\{\frac{1}{2}\sum\nolimits^{\flat(2,(3,3))}\frac{s^{2(6-r)}}{2^r}\mathbf{E}\prod_{q=1}^{2}\Big[\frac{\xi_q^{t_q}(a)}{t_q!}\prod_{p=1}^{m}\frac{\xi_{pq}^{\nu_{pq}}}{\nu_{pq}!}\Big] + \sum\nolimits^{\flat(1,(4))}\frac{s^{2(4-r)}\mathbf{E}\xi_1^{t_1}(a)\xi_{11}^{\nu_{11}}}{2^r\,t_1!\,\nu_{11}!}$$
$$-\frac{3\cdot 2!\,2!}{4!}\sum\nolimits^{\flat(2,(2,2))}\frac{s^{2(4-r)}}{2^r}\mathbf{E}\prod_{q=1}^{2}\Big[\frac{\xi_q^{t_q}(a)}{t_q!}\prod_{p=1}^{m}\frac{\xi_{pq}^{\nu_{pq}}}{\nu_{pq}!}\Big]\Big\} \equiv g(t;a)K(t;a). \quad (30)$$

At first we consider $K(t;0)$. Note that $K(t;0)$ coincides with the sum of those products, which depend on zero-sequences $\{t_q\}$. Write up the sums $\sum^{\flat(2,(3,3))}\cdots$, $\sum^{\flat(1,(4))}\cdots$ and $\sum^{\flat(2,(2,2))}\cdots$ in the case $a=0$ in detail.

In what follows, ν_{pq} are nonnegative integers. There are exactly six matrices $\begin{pmatrix}\nu_{11} & \nu_{12}\\ \nu_{21} & \nu_{22}\end{pmatrix}$, satisfying the condition

$$\underbrace{\nu_{11}+\nu_{21}}_{\nu_1} + \underbrace{\nu_{11}+\nu_{12}}_{\widetilde{\nu}_1} = 3, \quad \underbrace{\nu_{12}+\nu_{22}}_{\nu_2} + \underbrace{\nu_{21}+\nu_{22}}_{\widetilde{\nu}_2} = 3.$$

They are $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$, $\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$, $\begin{pmatrix}0 & 1\\ 2 & 0\end{pmatrix}$, $\begin{pmatrix}0 & 2\\ 1 & 0\end{pmatrix}$, $\begin{pmatrix}0 & 0\\ 3 & 0\end{pmatrix}$, $\begin{pmatrix}0 & 3\\ 0 & 0\end{pmatrix}$. Since here $r \equiv \sum_{p,q=1}^{m}\nu_{pq} = 3$,

$$\sum\nolimits^{\flat(2,(3,3))}\cdots = \frac{s^6}{2^3}\Big(2\mathbf{E}\xi_{11}\xi_{12}\xi_{22} + 2\frac{\mathbf{E}\xi_{12}\xi_{21}^2}{2!} + 2\frac{\mathbf{E}\xi_{12}^3}{3!}\Big) \equiv K_{(3,3)}(t;0). \quad (31)$$

Analogously, there are only 3 matrices $\{\nu_{pq}\}_{p,q=1}^2$, such that

$$2\nu_{11}+\nu_{21}+\nu_{12}=2,\quad \nu_{12}+\nu_{21}+2\nu_{22}=2.$$

They are $\begin{pmatrix}1&0\\0&1\end{pmatrix}$, $\begin{pmatrix}0&0\\2&0\end{pmatrix}$, $\begin{pmatrix}0&2\\0&0\end{pmatrix}$. Since here $r=2$,

$$\sum\nolimits^{\flat(2,(2,2))}\cdots=\frac{s^4}{2^2}\Big(\mathbf{E}^2\xi_{11}+2\frac{\mathbf{E}\xi_{12}^2}{2!}\Big)\equiv K_{(2,2)}(t;0). \tag{32}$$

Moreover, it is easily seen that

$$\sum\nolimits^{\flat(1,(4))}\cdots=\frac{s^4}{2^2}\frac{\mathbf{E}\xi_{11}^2}{2!}\equiv K_{(4)}(t;0). \tag{33}$$

It follows from (30) – (33) that

$$\begin{aligned}K(t;0)&=\frac{1}{2}K_{(3,3)}(t;0)+K_{(4)}(t;0)-\frac{1}{2}K_{(2,2)}(t;0)\\&=\frac{s^6}{16}\Big[2\mathbf{E}\xi_{11}\xi_{12}\xi_{22}+\frac{4}{3}\mathbf{E}\xi_{12}^3\Big]+\frac{s^4}{8}\Big[\mathbf{E}\xi_{11}^2-\mathbf{E}^2\xi_{11}-\mathbf{E}\xi_{12}^2)\Big].\end{aligned} \tag{34}$$

Let $a\neq 0$. There are exactly 8 matrices $\{\nu_{pq}\}_{p,q=1}^2$, such that

$$2\nu_{11}+\nu_{21}+\nu_{12}+t_1=3,\quad \nu_{12}+\nu_{21}+2\nu_{22}+t_2=3$$

with some nonnegative integers t_1, t_2, $t_1+t_2>0$. Let us enumerate all such matrices, at the same time calculating corresponding values of r, t_1, t_2:

$$\begin{pmatrix}0&0\\0&0\end{pmatrix},\ \begin{matrix}r=0,\\t_1=t_2=3\end{matrix};\quad \begin{pmatrix}0&0\\1&0\end{pmatrix},\ \begin{matrix}r=1,\\t_1=t_2=2\end{matrix};\quad \begin{pmatrix}0&1\\0&0\end{pmatrix},\ \begin{matrix}r=1,\\t_1=t_2=2\end{matrix};\quad \begin{pmatrix}1&0\\0&0\end{pmatrix},\ \begin{matrix}r=1,\\t_1=1,\ t_2=3\end{matrix};$$
$$\begin{pmatrix}0&0\\0&1\end{pmatrix},\ \begin{matrix}r=1,\\t_1=3,\ t_2=1\end{matrix};\quad \begin{pmatrix}1&0\\0&1\end{pmatrix},\ \begin{matrix}r=2,\\t_1=t_2=1\end{matrix};\quad \begin{pmatrix}0&0\\2&1\end{pmatrix},\ \begin{matrix}r=2,\\t_1=t_2=1\end{matrix};\quad \begin{pmatrix}0&2\\0&0\end{pmatrix},\ \begin{matrix}r=2,\\t_1=t_2=1\end{matrix}.$$

Hence, in view of (30) and (31),

$$\begin{aligned}\sum\nolimits^{\flat(2,(3,3))}\cdots=K_{(3,3)}(t;0)&+s^{12}\frac{\mathbf{E}^2\xi_1^3(a)}{3!\,3!}+\frac{s^{10}}{2}\Big(2\frac{\mathbf{E}\xi_1^2(a)\xi_2^2(a)\xi_{12}}{2!\,2!}+2\frac{\mathbf{E}\xi_1(a)\xi_{11}\mathbf{E}\xi_1^3(a)}{3!}\Big)\\&+\frac{s^8}{2^2}\Big(\mathbf{E}^2\xi_1(a)\xi_{11}+2\frac{\mathbf{E}\xi_1(a)\xi_2(a)\xi_{12}^2}{2!}\Big)\equiv K_{(3,3)}(t;a).\end{aligned} \tag{35}$$

Next, there are exactly 5 matrices $\{\nu_{pq}\}_{p,q=1}^2$, such that

$$2\nu_{11}+\nu_{21}+\nu_{12}+t_1=2,\quad \nu_{12}+\nu_{21}+2\nu_{22}+t_2=2$$

with some nonnegative integers t_1, t_2, $t_1 + t_2 > 0$. They are

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \ \begin{matrix} r = 0, \\ t_1 = t_2 = 2 \end{matrix}; \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \ \begin{matrix} r = 1, \\ t_1 = t_2 = 1 \end{matrix};$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ \begin{matrix} r = 1, \\ t_1 = t_2 = 1 \end{matrix}; \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ \begin{matrix} r = 1, \\ t_1 = 0,\ t_2 = 2 \end{matrix}; \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \ \begin{matrix} r = 1, \\ t_1 = 2,\ t_2 = 0. \end{matrix}$$

In view of (30) and (32),

$$\sum\nolimits^{\flat(2,(2,2))} \cdots = K_{(2,2)}(t;0) + s^8 \frac{\mathbf{E}^2 \xi_1^2(a)}{2!\,2!} + \frac{s^6}{2}\Big(2\mathbf{E}\xi_1(a)\xi_2(a)\xi_{12} + 2\frac{\mathbf{E}\xi_1^2(a)\mathbf{E}\xi_{11}}{2!}\Big) \equiv K_{(2,2)}(t;a). \quad (36)$$

It follows from (30), (35), (36) and the equality

$$\sum\nolimits^{\flat(1,(4))} \ \cdots \ = K_{(4)}(t;0) + s^8 \frac{\mathbf{E}\xi_1^4(a)}{4!} + \frac{s^6}{2}\frac{\mathbf{E}\xi_1^2(a)\xi_{11}}{2!} \equiv K_{(4)}(t;a),$$

that

$$\begin{aligned} K(t;a) &= \frac{1}{2}K_{(3,3)}(t;a) + K_{(4)}(t;a) - \frac{1}{2}K_{(2,2)}(t;a) \\ &= K(t;0) + \frac{1}{8}\Big[\frac{s^{12}}{9}\mathbf{E}^2\xi_1^3(a) + \frac{s^{10}}{3}\Big(3\mathbf{E}\xi_1^2(a)\xi_2^2(a)\xi_{12} + 2\mathbf{E}\xi_1(a)\xi_{11}\mathbf{E}\xi_1^3(a)\Big) \\ &\quad + s^8\Big(\mathbf{E}^2\xi_1(a)\xi_{11} + \mathbf{E}\xi_1(a)\xi_2(a)\xi_{12}^2 + \frac{\mathbf{E}\xi_1^4(a)}{3} - \mathbf{E}^2\xi_1^2(a)\Big)\Big] \\ &\quad + \frac{s^6}{4}\Big(\mathbf{E}\xi_1^2(a)\xi_{11} - 2\mathbf{E}\xi_1(a)\xi_2(a)\xi_{12} - \mathbf{E}\xi_1^2(a)\mathbf{E}\xi_{11}\Big). \end{aligned} \quad (37)$$

The equality (27) follows from (30), (34) and (37). The formula (28) arises from (27) and the last equality in (18). The bound (29) follows from the immediate estimate of (37).

□

Notice that the bound (29) is more precise than Lemma 2 for $\nu = 2$.

Note 5. For the sake of comparison, we give the representation of $\widehat{Q}_\nu(t;a)$ by F. Götze [7],

$$\widehat{Q}_\nu(t;a) = p_\nu(D)\mathbf{E}\exp\Big\{it\Big|Y - a + \sum_{q=1}^m \lambda_q X_q\Big|^2\Big\}\Big|_{\lambda_1=\cdots=\lambda_\nu=0}, \quad (38)$$

where $p_\nu(D)$ is the differential operator, defined by the formula (14), in which the moments $\mathbf{E}\xi_q^{j_q}$ are replaced by the partial derivatives $D_{\lambda_q}^{j_q}$, and the expression it is replaced by 1.

Notice that the calculation of $\widehat{Q}_2(t;a)$ in detail, using (38), is not simpler the proof of Lemma 3.

Lemma 4. *Let $d \ge 0$ and an integer $l \ge 2d+3$. Then for every $a \in \mathbf{H}$,*

$$\int_{-\infty}^{\infty} |t|^d |g(t;a)|\, dt < 2^{-d+2} \Lambda_l^{-(d+1)/l}. \tag{39}$$

One can find the proof of (39) in [15;16].

Lemma 5. *The following bounds hold,*

$$\sup_r |Q_2(r;a)| \le c\Big(\frac{\beta_4}{\Lambda_5^{2/5}} + \frac{\beta_3^2}{\Lambda_7^{3/7}} + \frac{\beta_4(|a|^4+|a|^2\sigma^2)}{\Lambda_9^{4/9}} + \frac{\beta_3^2(|a|^6+|a|^2\sigma^4)}{\Lambda_{13}^{6/13}}\Big) \tag{40}$$
$$\le c\beta_4\sigma^2\big[1/\Lambda_7^{3/7} + (|a|^6+|a|^2\sigma^4)/\Lambda_{13}^{6/13}\big].$$

Proof. Using the inversion formula, the estimate (29) and Lemma 4, we obtain

$$\sup_r |Q_2(r;a)| \le c \int_{-\infty}^{\infty} \frac{|\widehat{Q}_2(t;a)|}{|t|} dt$$
$$\le c_1 \int_{-\infty}^{\infty} |g(t;a)|\big[|t|^5\beta_3^2|a|^6 + |t|^3|a|^2(\beta_4|a|^2+\beta_3^2) + t^2(\beta_4|a|^2+\beta_3^2) + |t|\beta_4\big]dt$$
$$\le \frac{\beta_3^2|a|^6}{\Lambda_{13}^{6/13}} + \frac{|a|^2(\beta_4|a|^2+\beta_3^2)}{\Lambda_9^{4/9}} + \frac{\beta_4|a|^2+\beta_3^2}{\Lambda_7^{3/7}} + \frac{\beta_4}{\Lambda_5^{2/5}}. \tag{41}$$

Lemma 5 follows from (41). □

Note 6. Apparently, more refined methods (see, for instance, [8, Lemma 2.2]) allow to obtain, instead of Lemma 5, the following bound,

$$\sup_r |Q_2(r;a)| \le c\Big(\frac{\beta_4}{\Lambda_4^{1/2}} + \frac{\beta_3^2}{\Lambda_6^{1/2}} + \frac{\beta_4(|a|^4+|a|^2\sigma^2)}{\Lambda_8^{1/2}} + \frac{\beta_3^2(|a|^6+|a|^2\sigma^4)}{\Lambda_{12}^{1/2}}\Big) \tag{42}$$
$$\le c\,\beta_4\sigma^2\big[1/\Lambda_6^{1/2} + (|a|^6+|a|^2\sigma^4)/\Lambda_{12}^{1/2}\big].$$

Conjecture 1 is based just on the inequalities (40) and (42).

The folowing statement is proved in [8].

Lemma 6 [8; Lemma 2.2]. *Let $\tau > 0$, $b \in \mathbb{R}$, $b \ne 0$. Let M be a positive integer, $Z = (Z_1, \dots, Z_{2M})$ be Gaussian random vector with independent coordinates, $\mathbf{E}Z_j = 0$, $\mathbf{E}Z_j^2 = \sigma_j^2$, $\sigma_1^2 \ge \sigma_2^2 \ge \dots \ge \sigma_{2M}^2 > 0$ $a \in \mathbb{R}^{2M}$. Then there exists a positive constant $c(M)$, such that*

$$\Big|\int_{-\tau}^{\tau} t^{M-1}\mathbf{E}\exp\big\{it|Z+a|^2\big\}\, e^{itb}\, dt\Big| \le \frac{c(M)}{\Lambda_{2M}^{1/2}}.$$

The next lemma is a consequence of Lemma 6.

Lemma 7. *Let M be a positive integer, $r \in \mathbb{R}$, $a \in \mathbf{H}$. If the dimension d of the space $\mathbf{H}$ satisfies the condition $d \geq 2M$ then for every $\tau > 0$,*

$$I \equiv \Big| \int_{-\tau}^{\tau} t^{M-1} e^{-itr} \mathbf{E} e^{it|Y-a|^2}\, dt \Big| \leq \frac{c(M)}{\Lambda_{2M}^{1/2}}. \tag{43}$$

Lemma 8. *For every sequence of nonnegative integers $\{t_p\}_1^k$, and elements $x_p \in \mathbf{H}$, $p = 1, \ldots, k$, we have*

$$\begin{aligned} I \equiv \Big| \int_{-\tau}^{\tau} t^{M-1} e^{-itr} \mathbf{E} \Big(e^{it|Y-a|^2} \prod_{p=1}^{k} (Y-a, x_p)^{t_p} \Big)\, dt \Big| \\ \leq \frac{c(M; \{t_p\}) \sigma^{2J}}{\Lambda_{2M}^{1/2}} \Big(\prod_{p=1}^{k} \frac{|(a, x_p)|^{t_p}}{\sigma^{2t_p}} + \sum_{1 \leq p \leq k} \frac{|x_p|^{t_p}}{\sigma^{t_p}} \prod_{\substack{1 \leq j \leq k \\ j \neq p}} \frac{|(a, x_j)|^{t_j}}{\sigma^{2t_j}} \\ + \sum_{1 \leq p_1 < p_2 \leq k} \frac{|x_{p_1}|^{t_{p_1}}}{\sigma^{t_{p_1}}} \cdot \frac{|x_{p_2}|^{t_{p_2}}}{\sigma^{t_{p_2}}} \prod_{\substack{1 \leq j \leq k \\ j \neq p_1, p_2}} \frac{|(a, x_j)|^{t_j}}{\sigma^{2t_j}} + \ldots + \prod_{p=1}^{k} \frac{|x_p|^{t_p}}{\sigma^{t_p}} \Big), \end{aligned}$$

where $J = \sum_{p=1}^{k} t_p$.

Proof. Let $Y_1, \ldots, Y_{J+1}$ be independent copies of the random variable $\dfrac{1}{\sqrt{J+1}}\, Y$, $a_q = \dfrac{1}{J+1}\, a$, $q = 1, \ldots, J+1$. We have

$$\prod_{p=1}^{k} (Y-a, x_p)^{t_p} = \sum_{\{m_{q1}\}_{q=1}^{J+1}}^{t_1} \cdots \sum_{\{m_{qk}\}_{q=1}^{J+1}}^{t_k} \Big(\prod_{p=1}^{k} S_{t_p}\Big(\{m_{qp}\}_{q=1}^{J+1} \Big) \Big) \prod_{p=1}^{k} \prod_{q=1}^{J+1} (Y_q - a_q, x_p)^{m_{qp}},$$

where $S_{t_p}\Big(\{m_{qp}\}_{q=1}^{J+1}\Big)$ are the polynomial coefficients, where for each collection of the sequences

$$(m_{11}, m_{21}, \ldots, m_{J+1\,1}), \quad (m_{11}, m_{22}, \ldots, m_{J+1\,2}), \quad \ldots, \quad (m_{1k}, m_{2k}, \ldots, m_{J+1\,k}),$$

there exists $1 \leq q_0 \leq J+1$, such that $m_{q_0 p} = 0$ for all $1 \leq p \leq k$. Let for simplicity $q_0 = 1$. Then

$$\begin{aligned} \mathbf{E} e^{it|Y-a|^2} \prod_{p=1}^{k} \prod_{q=1}^{J+1} (Y_q - a_q, x_p)^{m_{qp}} &= \mathbf{E} e^{it|Y-a|^2} \prod_{p=1}^{k} \prod_{q=2}^{J+1} (Y_q - a_q, x_p)^{m_{qp}} \\ &= \mathbf{E} \Big[\Big(\prod_{p=1}^{k} \prod_{q=2}^{J+1} (Y_q - a_q, x_p)^{m_{qp}} \Big) \mathbf{E}_{Y_1} e^{it|Y_1 + y_1|^2} \Big], \end{aligned}$$

where $y_1 = -a + \sum_{j=2}^{J+1} Y_j$. It follows from here and Lemma 7 that

$$I = \Bigg| \sum_{\{m_{q1}\}_{q=1}^{J+1}}^{t_1} \cdots \sum_{\{m_{qk}\}_{q=1}^{J+1}}^{t_k} \Big(\prod_{p=1}^{k} S_{t_p}\Big(\{m_{qp}\}_{q=1}^{J+1}\Big)\Big) \mathbf{E}\Big[\Big(\prod_{p=1}^{k} \prod_{q=1}^{J+1} (Y_q - a_q, x_p)^{m_{qp}} \Big)$$

$$\times \int_{-\tau}^{\tau} \mathbf{E}_{Y_1} e^{it|Y_1+y_1|^2} t^{M-1} e^{-itr}\, dt \Big]\Bigg| \le \frac{c_1(M; \{t_p\})}{\Lambda_{2M}^{1/2}}$$

$$\times \mathbf{E}\Big[\Big(\sum_{j=1}^{J+1} (|(Y_j, x_1)|^{t_1} + |(a, x_1)|^{t_1}) \Big) \cdots \Big(\sum_{j=1}^{J+1} (|(Y_j, x_k)|^{t_k} + |(a, x_k)|^{t_k}) \Big)\Big]. \quad (44)$$

Using the inequality $\mathbf{E}|(Y_j, x_p)|^{t_p} \le c(J)\sigma^{t_p}|x_p|^{t_p}$, and carrying out the square brackets the expression σ^{2J} in (44), we arrive at the statement of Lemma 8. □

Lemma 9. *The following estimate holds,*

$$\sup_r |Q_2(r; a)| \le c_1 \frac{\sigma^8}{\Lambda_8^{1/2}} \Big[\frac{\beta_4(a)}{\sigma^8} + \frac{\beta_4}{\sigma^4} \Big] + c_2 \frac{\sigma^{12}}{\Lambda_{12}^{1/2}} \Big[\frac{\beta_3^2(a)}{\sigma^{12}} + \frac{\beta_3^2}{\sigma^6} \Big]. \quad (45)$$

Proof. The function $Q_2(r; a)$ is defined only for $r \ge 0$. Let us extend it onto the negative semiaxis in the even way. Since $Q_2(-\infty; a) = Q_2(+\infty; a) = 0$, by the invertion formula (see [5, p. 381]), we have

$$Q_2(r; a) = \frac{i}{2\pi} \lim_{T\to\infty} \lim_{\varepsilon \downarrow 0} \int\limits_{\varepsilon < |t| \le T} e^{-itr} \frac{\widehat{Q}_2(t; a)}{t}\, dt = \frac{i}{2\pi} \lim_{T\to\infty} \int_{-T}^{T} e^{-itr} \frac{\widehat{Q}_2(t; a)}{t}\, dt. \quad (46)$$

The expression (28) for $\widehat{Q}_2(t; a)$ contains of 15 summands. As a result we need to estimate the following 15 integrals:

$$I_1 \equiv \int\limits_{-T}^{T} e^{-itr} t^5 \mathbf{E} e^{it|Y-a|^2} \eta_1^3(a) \eta_2^3(a)\, dt, \qquad I_2 \equiv \int\limits_{-T}^{T} e^{-itr} t^4 \mathbf{E} e^{it|Y-a|^2} \eta_1^2(a) \eta_2^2(a) \eta_{12}\, dt,$$

$$\ldots,\ I_{14} \equiv \int\limits_{-T}^{T} e^{-itr} t \mathbf{E} e^{it|Y-a|^2} \eta_{11}\eta_{22}\, dt, \qquad I_{15} \equiv \int\limits_{-T}^{T} e^{-itr} t \mathbf{E} e^{it|Y-a|^2} \eta_{12}^2\, dt.$$

We have

$$I_1 = \int\limits_{-T}^{T} e^{-itr} t^5 \mathbf{E} e^{it|Y-a|^2} (Y - a, X_1)^3 (Y - a, X_2)^3\, dt.$$

For the sake of brevity we introduce the following notations,

$$B_3(a) = \left(\frac{\beta_3(a)}{\sigma^6}\right)^2 + \left(\frac{\beta_3}{\sigma^3}\right)^2, \quad B_4(a) = \frac{\beta_4(a)}{\sigma^8} + \frac{\beta_4}{\sigma^4}.$$

Using Lemma 8, we find that

$$|I_1| \le \frac{c\,\sigma^{12}}{\Lambda_{12}^{1/2}}\Big[\left(\frac{\beta_3(a)}{\sigma^6}\right)^2 + \frac{\beta_3}{\sigma^3}\frac{\beta_3(a)}{\sigma^6} + \left(\frac{\beta_3}{\sigma^3}\right)^2\Big] \le \frac{c_1\,\sigma^{12}}{\Lambda_{12}^{1/2}}\,B_3(a) < \frac{c_1\,\sigma^{12}}{\Lambda_{12}^{1/2}}\,B_3(a) + \frac{\sigma^8}{\Lambda_8^{1/2}}\,B_4(a).$$

Now we prove the inequality

$$|I_2| \le \frac{c\,\sigma^{10}}{\Lambda_{10}^{1/2}}\,B_3(a) < \frac{c\,\sigma^{12}}{\Lambda_{12}^{1/2}}\,B_3(a) + \frac{\sigma^8}{\Lambda_8^{1/2}}\,B_4(a). \tag{47}$$

We have with the help of Lemma 8,

$$\begin{aligned} |I_2| &\equiv \left|\int_{-T}^{T} e^{-itr} t^4 \mathbf{E} e^{it|Y-a|^2} \eta_1^2(a)\,\eta_2^2(a)\,\eta_{12}\,dt\right| \\ &\le \frac{c\,\sigma^{10}}{\Lambda_{10}^{1/2}}\,\mathbf{E}\Big[\frac{(a,X_1)^2\,(a,X_2)^2}{\sigma^8} + \frac{|X_1|^2\,(a,X_2)^2}{\sigma^6} + \frac{|X_1|^2\,|X_2|^2}{\sigma^4}\Big]\,\frac{|X_1|\,|X_2|}{\sigma^2}. \end{aligned} \tag{48}$$

Denote $U_j(a) = \dfrac{|(a,X_j)|}{\sigma^2}$, $U_j = \dfrac{|X_j|}{\sigma}$, $j = 1, 2$. It is easily seen that

$$U_1^2(a)U_2^2(a)U_1U_2 \le \big(U_1^3(a) + U_1^3\big)\,\big(U_2^3(a) + U_2^3\big).$$

Then

$$\mathbf{E}U_1^2(a)U_2^2(a)U_1U_2 \le \left(\frac{\beta_3(a)}{\sigma^6}\right)^2 + 2\frac{\beta_3(a)}{\sigma^6}\,\frac{\beta_3}{\sigma^3} + \left(\frac{\beta_3}{\sigma^3}\right)^2 \le 3\,B_3(a). \tag{49}$$

In the same way we get

$$\mathbf{E}U_1^3U_2U_2^2(a) \le \mathbf{E}\big(U_1^3U_2^3 + U_1^3U_2^3(a)\big) \le 2\Big[\left(\frac{\beta_3(a)}{\sigma^6}\right)^2 + \left(\frac{\beta_3}{\sigma^3}\right)^2\Big]. \tag{50}$$

The bound (47) follows from (48) – (50).

Analogously,

$$|I_6| \equiv \left|\int_{-T}^{T} e^{-itr} t^3 \mathbf{E} e^{it|Y-a|^2} \eta_1^4(a)\,dt\right| \le \frac{c\,\sigma^8}{\Lambda_8^{1/2}}\,B_4(a) < \frac{c\,\sigma^8}{\Lambda_8^{1/2}}\,B_4(a) + \frac{\sigma^{12}}{\Lambda_{12}^{1/2}}\,B_3(a)$$

and for the rest I_p we have

$$I_p < \frac{c_1\,\sigma^{12}}{\Lambda_{12}^{1/2}}\,B_3(a) + \frac{c_2\,\sigma^8}{\Lambda_8^{1/2}}\,B_4(a).$$

The statement of the lemma follows from (46), (28) and the bounds, obtained for $|I_p|$, $p = 1, \dots, 15$.

The proof of Theorem 6. Theorem 6 follows from Lemmas 3 and 9.

Acknowledgments

The work is partially supported by the following grants: INTAS 03-51-5018, RFBR 06-01-00069 and 07-01-00054, and grant of the Far-Eastern Branch of the Russian Academy of Sciences 06-III-A-01-003.

References

[1] Chebotarev V. I. (2002) *Gaussian Approximation in Hilbert Space and Asymptotic Expansions. Thesis of the doctor of phisical and mathematical sciences: 01.01.05.* Khabarovsk. (Russian)

[2] Bentkus V., Götze F. (1994) *On the lattice point problem for ellipsoids.* Preprint 94–111 SFB 343/ Universität Bielefeld. Bielefeld.

[3] Bentkus V., Götze F. (1995) *On the lattice point problem for ellipsoids. Dokl. Ross. Akad. Nauk.* V. 343, no 4.

[4] Bentkus V., Götze F. (1996) *Optimal rates of convergence in CLT for quadratic forms. Ann. Probab.* V. 24. P. 406–490

[5] Bentkus V., Götze F. (1997) *Uniform rates of convergence in the CLT for quadratic forms in multidimensional spaces. Probab. Theory Relat. Fields.* V. 109. P. 367–416

[6] Bentkus V., Götze F. (1997) *On the lattice point problem for ellipsoids. Acta Arithmetica.* V. 80, no 2. P. 101–125.

[7] Götze F. (1981) *On Edgeworth expansions in Banach spaces. Ann. Probab.* V. 9, no 5.
P. 852–859.

[8] Götze F., Ulyanov V. (2000) *Uniform approximation in the CLT for balls in Euclidian spaces.* Preprint 00–034 SFB 343 /Universität Bielefeld. Bielefeld.

[9] Nagaev, S. V. (1989) *On a new approach to the study of the distribution of the norm of random element in Hilbert space.* Abstracts of commun. The Fifth Intern. Vilnius Conference in Prob. Theory and Math. Stat. Vilnius. LTSR Matematikos ir Kiberneticos Institutas. V. 4. P. 77–78. (Russian)

[10] Nagaev S. V. (1990) *Concentration functions and approximation with infinitely divisible laws in Hilbert space.* Preprint, 90–094, SFB343, Universität Bielefeld. Bielefeld

[11] Nagaev S. V. (1992) *On estimates of the rate of convergence in the CLT in a Hilbert space.* Abstracts of commun. Workshop on Limit Theorems and Nonparametric Statistics, Bielefeld, August 24 - 28, 1992. P. 1–3.

[12] Nagaev S. V., Chebotarev V. I. (1986) *A refinement of the error estimate of the normal approximation in a Hilbert space. Siberian Math. J.* V. 27, no. 3. P. 434–450.

[13] Nagaev S. V., Chebotarev V. I. (1993) *On Edgeworth expansion in Hilbert space. Siberian Advances in Math.* V. 3, no 3. P. 89–122.

[14] Nagaev S. V., Chebotarev V. I. (1999) *On the accuracy of Gaussian approximation in Hilbert space. Acta Applicandae.* V. 58. P. 189–215.

[15] Nagaev S. V., Chebotarev V. I. (2001) *On the estimate of accuracy of normal approximation in Hilbert space.* Research Report No. 01/55: The Russian Academy of Sciences, Far–East. Branch, Computing Centre. Khabarovsk.

[16] Nagaev S. V., Chebotarev V. I. (2005) *On the estimate of accuracy of normal approximation in Hilbert space. Siberian Advances in Math.* V. 16, no 1. P. 11–73.

[17] Petrov V. V. (1975) *Sums of independent random variables.* Berlin - Heidelberg - New York: Springer - Verlag.

[18] Senatov V. V. (1996) *Qualitative effects in the estimates of the convergence rate in the central limit theorem in multidimensional spaces. Proceedings of the Steklov Institute of Mathematics.* Moscow: MAIK Nauka, Birmingham. AL: Interperiodica Publishing.

[19] Zalesskii B. A., Sazonov V. V., and Ul'yanov V. V. (1989) *A regular estimate of the convergence rate in Central Limit Theorem in Hilbert space. Matematich. sbornik.* V. 180, no 12. P. 1587–1613. (Russian)