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Estimating the Parameters of the Second Order Spatial Unilateral Autoregressive Model

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Abstract

Various types of model have been suggested to describe spatial process namely, the simultaneous autoregressive model, the conditional autoregressive model and the moving average model. The problem of estimating the parameters of spatial models has been taken up by many researchers. In this paper, a procedure for estimating the parameters for the non-separable second order spatial unilateral autoregressive, AR(2,1) model is presented. Using our proposed procedure, we obtain good estimates, in the sense that the estimated parameters were found to be close to the true values. The performance of this estimator is also compared with other estimators such as Yule-Walker and conditional least squares estimators via simulation studies. The significance of this study is that it enables user to use an alternative procedure to estimate the parameters of the non-separable second order spatial unilateral AR(2,1) model.

Keywords and Phrases: Parameter estimation; Spatial autoregressive model; Spatial unilateral model.

AMS Classification: Primary 62M30.

1 Introduction

Spatial processes have been analyzed in various fields such as biology, agriculture, geography and meteorology. In this paper, consideration will be given to the spatial process in two-dimensional regular grid where a random variable is defined at each intersection point. Various types of models have been suggested to describe the process namely, the simultaneous autoregressive (SAR) model [see Whittle (1954)], the conditional autoregressive (CAR) model [see Besag (1974)] and the moving average (MA) model [see Haining (1978b)]. Basu and Reinsel (1992 and 1993) examined a model called the unilateral autoregressive moving average (ARMA) model of the quadrant type and attention has been given to the first order case. Martin (1979, 1990 and 1996) studied in detail a special case of this model called linear-by-linear or separable models.

Estimation of the parameters of spatial models remains a difficult task. They are frequently estimated by maximum likelihood [see Basu and Reinsel (1993), Haining (1978a) and Ord (1975)], least squares [see Haining (1978a)] or Yule-Walker [see Basu and Reinsel (1992) and Tjøstheim (1978)] methods. Some attempts have been made to overcome the computational difficulties by considering *unilateral* models [see Basu and Reinsel (1993)] and *separable* models [see Basawa, Brockwell and Mandrekar (1991) and Martin (1979), (1990), (1996)]. Separable models have a product correlation structure and this considerably simplifies the estimation. Shitan and Brockwell (1995) provided an asymptotic test for separability for spatial autoregressive model by translating the spatial problem to a multiple time series problem. For the more general *non-separable first* order unilateral model, some results on the estimation of the parameters by maximum likelihood have been provided by Basu and Reinsel (1993).

Shitan and Brockwell (1996) discussed the problem of estimation for higher order non-separable unilateral autoregressive models. They have taken up the approach of transforming the 2-D spatial problem to a multiple time series problem, treating one of the spatial coordinates as a time index and the other coordinate as a multivariate index and then carried out the multivariate least squares estimation (unconstrained and constrained) procedures.

In this paper we look at the problem of estimation from a different perspective from Shitan and Brockwell (1996). Our approach is to use maximum likelihood (ML) to estimate the parameters of the *second* order non-separable spatial unilateral autoregressive, AR(2,1) model defined as,

$$\begin{aligned} Y_{ij} = \alpha_{10} Y_{i-1,j} + \alpha_{01} Y_{i,j-1} + \alpha_{11} Y_{i-1,j-1} + \alpha_{20} Y_{i-2,j} \\ + \alpha_{21} Y_{i-2,j-1} + \varepsilon_{ij} \end{aligned} \tag{1}$$

where $\{Y_{ij}\}$ is a sequence of two dimensional random variable with zero mean and the errors ε_{ij} are assumed to be normal and independent with mean 0 and variance σ^2. This procedure is an adaptation of the estimation method for the *one* parameter case using maximum likelihood as discussed in Ord (1975). Here, we make an extension to the case of *five* parameters. We will also show how the weight matrices are constructed in order to apply the maximum likelihood method. It can be shown that with certain transformation for the weight matrices, this procedure is analogues of the modified least squares estimation as given in Cliff and Ord (1981).

In Section 2, the construction of the weight matrices is presented. The derivation of this procedure of estimation is discussed in Section 3. In Section 4, we discuss other type of estimation methods for spatial unilateral AR model namely, the Yule-Walker method and the conditional least squared methods. Some simulation studies are presented in Section 5 to compare the performance of these estimators. In Section 6, the conclusions are presented.

2 Construction of the Weight Matrices for the Second Order Spatial Autoregressive Model

We consider a second order spatial unilateral autoregressive, AR(2,1) model defined as,

$$\begin{aligned} Y_{ij} = \alpha_{10}Y_{i-1,j} + \alpha_{01}Y_{i,j-1} + \alpha_{11}Y_{i-1,j-1} + \alpha_{20}Y_{i-2,j} \\ + \alpha_{21}Y_{i-2,j-1} + \varepsilon_{ij}, \\ i = 1,\ 2,\ ...,\ m \text{ and } j = 1,\ 2,\ ...,\ n, \end{aligned} \tag{2}$$

where $\{Y_{ij}\}$ is a sequence of two dimensional random variable with zero mean and the errors ε_{ij} are assumed to be normally distributed with mean 0 and common variance σ^2.

By assuming that the unobserved values to be zeroes, and letting the observation vector, $\mathbf{Y} = (Y_{11},\ Y_{12},\ ...,\ Y_{1n},\ Y_{21},\ Y_{22},\ ...,\ Y_{2n},\ ...,\ Y_{m1},\ Y_{m2},\ ...,\ Y_{mn})'$ $= (\mathbf{Y_1},\ \mathbf{Y}_2,\ ...,\ \mathbf{Y_m})'$, where $\mathbf{Y}_i = (Y_{i1},\ Y_{i2},\ ...,\ Y_{in})'$, $i = 1,\ 2,\ ...,\ m$ and the error vector, $\varepsilon = (\varepsilon_{11},\ \varepsilon_{12},\ ...,\ \varepsilon_{1n},\ \varepsilon_{21},\ \varepsilon_{22},\ ...,\ \varepsilon_{2n},\ ...,\ \varepsilon_{m1},\ \varepsilon_{m2},\ ...,\ \varepsilon_{mn})' = (\varepsilon_{\mathbf{1}},\ \varepsilon_{\mathbf{2}},\ ...,\ \varepsilon_{\mathbf{m}})'$, where $\varepsilon_{\mathbf{i}} = (\varepsilon_{i1},\ \varepsilon_{i2},\ ...,\ \varepsilon_{in})'$, $i = 1,\ 2,\ ...,\ m$, we can rewrite equation (2) in the matrix form as,

$$\begin{bmatrix} \mathbf{Y_1} \\ \mathbf{Y_2} \\ \mathbf{Y_3} \\ \mathbf{Y_4} \\ \vdots \\ \mathbf{Y_m} \end{bmatrix} = \begin{bmatrix} \mathbf{A_1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{A_2} & \mathbf{A_1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{A_3} & \mathbf{A_2} & \mathbf{A_1} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A_3} & \mathbf{A_2} & \mathbf{A_1} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{A_3} & \mathbf{A_2} & \mathbf{A_1} \end{bmatrix} \begin{bmatrix} \mathbf{Y_1} \\ \mathbf{Y_2} \\ \mathbf{Y_3} \\ \mathbf{Y_4} \\ \vdots \\ \mathbf{Y_m} \end{bmatrix} + \begin{bmatrix} \varepsilon_{\mathbf{1}} \\ \varepsilon_{\mathbf{2}} \\ \varepsilon_{\mathbf{3}} \\ \varepsilon_{\mathbf{4}} \\ \vdots \\ \varepsilon_{\mathbf{m}} \end{bmatrix}, \tag{3}$$

where $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$ are $n \times n$ matrices given as,

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ \alpha_{01} & 0 & 0 & \cdots & 0 & 0 \\ 0 & \alpha_{01} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \alpha_{01} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_{01} & 0 \end{bmatrix},$$

$$\mathbf{A}_2 = \begin{bmatrix} \alpha_{10} & 0 & 0 & \cdots & 0 & 0 \\ \alpha_{11} & \alpha_{10} & 0 & \cdots & 0 & 0 \\ 0 & \alpha_{11} & \alpha_{10} & \cdots & 0 & 0 \\ 0 & 0 & \alpha_{11} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_{11} & \alpha_{10} \end{bmatrix}$$

and

$$\mathbf{A}_3 = \begin{bmatrix} \alpha_{20} & 0 & 0 & \cdots & 0 & 0 \\ \alpha_{21} & \alpha_{20} & 0 & \cdots & 0 & 0 \\ 0 & \alpha_{21} & \alpha_{20} & \cdots & 0 & 0 \\ 0 & 0 & \alpha_{21} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_{21} & \alpha_{20} \end{bmatrix}.$$

Equation (3) can be written more compactly as,

$$\mathbf{Y} = \mathbf{AY} + \varepsilon, \tag{4}$$

where $\mathbf{A}$ is $N \times N$ matrix, $N = mn$. It is clear that $\mathbf{A}$ is a lower triangular matrix with zeros on the main diagonal. Then, if we decompose $\mathbf{A}$ into five matrices such that it isolates different parameters, we obtain

$$\mathbf{Y} = (\alpha_{10}\mathbf{W}_1 + \alpha_{01}\mathbf{W}_2 + \alpha_{11}\mathbf{W}_3 + \alpha_{20}\mathbf{W}_4 + \alpha_{21}\mathbf{W}_5)\mathbf{Y} + \varepsilon, \tag{5}$$

where, $\mathbf{A} = \alpha_{10}\mathbf{W}_1 + \alpha_{01}\mathbf{W}_2 + \alpha_{11}\mathbf{W}_3 + \alpha_{20}\mathbf{W}_4 + \alpha_{21}\mathbf{W}_5$ and $\mathbf{W}_k$, k =1, 2, ..., 5, are the $N \times N$ weight matrices with elements ones and zeros, given as

$$\mathbf{W}_1 = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{B_1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B_1} & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & \ddots & \ldots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{B_1} & \mathbf{0} \end{bmatrix},$$

$$\mathbf{W}_2 = \begin{bmatrix} \mathbf{B_2} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B_2} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{B_2} & \ldots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & \ddots & \ldots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{B_2} \end{bmatrix},$$

$$\mathbf{W}_3 = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_2 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_2 & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & \ddots & \ldots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{B}_2 & \mathbf{0} \end{bmatrix},$$

$$\mathbf{W}_4 = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{B_1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B_1} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{B_1} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

and

$$\mathbf{W}_5 = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_2 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_2 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{B}_2 & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

as well as, $\mathbf{B}_1$ and $\mathbf{B}_2$ are $n \times n$ matrices defined as

$$\mathbf{B}_1 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}$$

and

$$\mathbf{B}_2 = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix}.$$

3 Parameter Estimation for the Second Order Spatial Autoregressive Model Using Maximum Likelihood (ML)

Equation (5) can then be written as,

$$\mathbf{Y} = [\,\mathbf{I} - (\,\alpha_{10}\mathbf{W}_1 + \alpha_{01}\mathbf{W}_2 + \alpha_{11}\mathbf{W}_3 + \alpha_{20}\mathbf{W}_4 + \alpha_{21}\mathbf{W}_5)]^{-1}\,\varepsilon \tag{6}$$

or

$$\mathbf{Y} = (\,\mathbf{I} - \mathbf{A}\,)^{-1}\,\varepsilon \tag{7}$$

where $\mathbf{I}$ is an $N \times N$ identity matrix.

Therefore, the covariance matrix of $\mathbf{Y}$, $\mathbf{V}$ is given as,

$$\mathbf{V} = \sigma^2\,(\,\mathbf{I} - \mathbf{A}\,)^{-1}\left[(\,\mathbf{I} - \mathbf{A}\,)^{-1}\right]'. \tag{8}$$

The square root of the determinant of $\mathbf{V}$ is given as,

$$|\mathbf{V}|^{1/2} = \left(\sigma^2\right)^{N/2}\left|\,(\,\mathbf{I} - \mathbf{A}\,)^{-1}\right| . \tag{9}$$

Since $(\mathbf{I}-\mathbf{A})$ is the lower triangular matrix with diagonal elements 1, $\left|\,(\,\mathbf{I} - \mathbf{A}\,)^{-1}\right| = 1$. This leads to

$$|\mathbf{V}|^{1/2} = \left(\sigma^2\right)^{N/2}. \tag{10}$$

Therefore, the likelihood function l is given as,

$$\begin{aligned} l &= \tfrac{1}{(2\pi)^{N/2}|\mathbf{V}|^{1/2}}.\exp\left\{-\tfrac{1}{2}\mathbf{Y}'\mathbf{V}^{-1}\mathbf{Y}\right\} \\ &= (2\pi)^{-N/2}(\sigma^2)^{-N/2}\exp\left\{-\tfrac{1}{2\sigma^2}\mathbf{Y}'\left[(\mathbf{I}-\mathbf{A})^{-1}(\mathbf{I}-\mathbf{A}')^{-1}\right]^{-1}\mathbf{Y}\right\} \\ &= (2\pi)^{-N/2}(\sigma^2)^{-N/2}\exp\left\{-\tfrac{1}{2\sigma^2}\mathbf{Y}'(\mathbf{I}-\mathbf{A}')(\mathbf{I}-\mathbf{A})\mathbf{Y}\right\}\ . \end{aligned}$$

Thus we obtain the log likelihood, L as

$$L = -\frac{N}{2}\ln(2\pi) - \frac{N}{2}\ln(\sigma^2) - \frac{1}{2\sigma^2}\mathbf{Y}'(\mathbf{I}-\mathbf{A}')(\mathbf{I}-\mathbf{A})\mathbf{Y}. \tag{11}$$

Denoting $\alpha' = (\alpha_{10},\ \alpha_{01},\ \alpha_{11},\ \alpha_{20},\ \alpha_{21}) = (\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ \alpha_5)$, for each $i = 1,\ 2,\ ...,\ 5$,

$$\frac{\partial L}{\partial \alpha_i} = -\frac{1}{\sigma^2}\left[-\mathbf{Y}'\,\mathbf{W}'_i\,\mathbf{Y} + \alpha_i\mathbf{Y}'\,\mathbf{W}'_i\,\mathbf{W}_i\mathbf{Y} + \sum_{\forall j \neq i}\alpha_j\mathbf{Y}'\,\mathbf{W}'_j\,\mathbf{W}_i\mathbf{Y}\right] \tag{12}$$

for $j = 1, 2, \ldots, 5$.

Equating (12) to zero leads to

$$\left[\alpha_i \mathbf{Y}' \mathbf{W}'_i \mathbf{W}_i \mathbf{Y} \;+\; \sum_{\forall j \neq i} \alpha_j \mathbf{Y}' \mathbf{W}'_j \mathbf{W}_i \mathbf{Y} \right] \;=\; \mathbf{Y}' \mathbf{W}'_i \mathbf{Y}. \tag{13}$$

Therefore, denoting $\mathbf{Z}_i = \mathbf{W}_i \mathbf{Y}$, the maximum likelihood for $\alpha' = (\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ \alpha_5)$ can be obtained by solving the equation

$$\begin{bmatrix} \mathbf{Z}'_1 \mathbf{Z}_1 & \mathbf{Z}'_2 \mathbf{Z}_1 & \mathbf{Z}'_3 \mathbf{Z}_1 & \mathbf{Z}'_4 \mathbf{Z}_1 & \mathbf{Z}'_5 \mathbf{Z}_1 \\ \mathbf{Z}'_1 \mathbf{Z}_2 & \mathbf{Z}'_2 \mathbf{Z}_2 & \mathbf{Z}'_3 \mathbf{Z}_2 & \mathbf{Z}'_4 \mathbf{Z}_2 & \mathbf{Z}'_5 \mathbf{Z}_2 \\ \mathbf{Z}'_1 \mathbf{Z}_3 & \mathbf{Z}'_2 \mathbf{Z}_3 & \mathbf{Z}'_3 \mathbf{Z}_3 & \mathbf{Z}'_4 \mathbf{Z}_3 & \mathbf{Z}'_5 \mathbf{Z}_3 \\ \mathbf{Z}'_1 \mathbf{Z}_4 & \mathbf{Z}'_2 \mathbf{Z}_4 & \mathbf{Z}'_3 \mathbf{Z}_4 & \mathbf{Z}'_4 \mathbf{Z}_4 & \mathbf{Z}'_5 \mathbf{Z}_4 \\ \mathbf{Z}'_1 \mathbf{Z}_5 & \mathbf{Z}'_2 \mathbf{Z}_5 & \mathbf{Z}'_3 \mathbf{Z}_5 & \mathbf{Z}'_4 \mathbf{Z}_5 & \mathbf{Z}'_5 \mathbf{Z}_5 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{bmatrix} = \begin{bmatrix} \mathbf{Y}' \mathbf{Z}_1 \\ \mathbf{Y}' \mathbf{Z}_2 \\ \mathbf{Y}' \mathbf{Z}_3 \\ \mathbf{Y}' \mathbf{Z}_4 \\ \mathbf{Y}' \mathbf{Z}_5 \end{bmatrix},$$

or

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{bmatrix} = \begin{bmatrix} \mathbf{Z}'_1 \mathbf{Z}_1 & \mathbf{Z}'_2 \mathbf{Z}_1 & \mathbf{Z}'_3 \mathbf{Z}_1 & \mathbf{Z}'_4 \mathbf{Z}_1 & \mathbf{Z}'_5 \mathbf{Z}_1 \\ \mathbf{Z}'_1 \mathbf{Z}_2 & \mathbf{Z}'_2 \mathbf{Z}_2 & \mathbf{Z}'_3 \mathbf{Z}_2 & \mathbf{Z}'_4 \mathbf{Z}_2 & \mathbf{Z}'_5 \mathbf{Z}_2 \\ \mathbf{Z}'_1 \mathbf{Z}_3 & \mathbf{Z}'_2 \mathbf{Z}_3 & \mathbf{Z}'_3 \mathbf{Z}_3 & \mathbf{Z}'_4 \mathbf{Z}_3 & \mathbf{Z}'_5 \mathbf{Z}_3 \\ \mathbf{Z}'_1 \mathbf{Z}_4 & \mathbf{Z}'_2 \mathbf{Z}_4 & \mathbf{Z}'_3 \mathbf{Z}_4 & \mathbf{Z}'_4 \mathbf{Z}_4 & \mathbf{Z}'_5 \mathbf{Z}_4 \\ \mathbf{Z}'_1 \mathbf{Z}_5 & \mathbf{Z}'_2 \mathbf{Z}_5 & \mathbf{Z}'_3 \mathbf{Z}_5 & \mathbf{Z}'_4 \mathbf{Z}_5 & \mathbf{Z}'_5 \mathbf{Z}_5 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{Y}' \mathbf{Z}_1 \\ \mathbf{Y}' \mathbf{Z}_2 \\ \mathbf{Y}' \mathbf{Z}_3 \\ \mathbf{Y}' \mathbf{Z}_4 \\ \mathbf{Y}' \mathbf{Z}_5 \end{bmatrix}. \tag{14}$$

4 Other Estimation Methods

In this section, we shall review other alternative methods of parameters estimation for spatial unilateral autoregressive models.

4.1 Spatial Yule-Walker Estimation

Tjøstheim (1978) considered Yule-Walker method to estimate the parameters of spatial AR models. For data given on two-dimensional regular grid, Y_{ij}, $i = 1, \ldots, m$ and $j = 1, \ldots, n$, the sample autocovariances at lag (s, t) and $(s, -t)$ for $s \geq 0$ and $t \geq 0$ are defined respectively as,

$$R(s,t) = \frac{1}{mn} \sum_{i=1}^{m-s} \sum_{j=1}^{n-t} Y_{ij} Y_{i+s,j+t} \tag{15}$$

and

$$R(s,-t) = \frac{1}{mn} \sum_{i=1}^{m-s} \sum_{j=1+1}^{n} Y_{ij} Y_{i+s,j-t}. \tag{16}$$

From these definitions, *R(s,t) = R(-s,-t)* and *R(-s,t) = R(s,-t)*. For a spatial unilateral AR(p_1, p_2) model, the spatial analogue of the one-dimensional Yule-Walker equations as in time-series case is given as,

$$R(s,t) = \sum_{k=0}^{p_1} \sum_{l=0}^{p_2} \alpha_{kl} R(s-k, t-l). \tag{17}$$

Then, if we define

$$\alpha = (\alpha_{10} \, ..., \; \alpha_{p_1,0}, \; \alpha_{01}, \; ..., \; \alpha_{p_1,1}, \; ..., \; \alpha_{0,p_2}, \; ..., \; \alpha_{p_1,p_2})',$$

$\mathbf{r} = (R(1,0), \; ..., \; R(p_1,0), R(0,1), \; ..., , \; R(p_1,1), \; ..., \; R(0,p_2), \; ..., \; R(p_1,p_2))'$ and

$$\mathbf{R} = \begin{bmatrix} R(0,0) & R(-1,0) & \cdots & R(1-p_1,p_2) \\ R(1,0) & R(0,0) & \cdots & R(2-p_1,-p_2) \\ \vdots & \vdots & \cdots & \vdots \\ R(p_1-1,p_2) & R(p_1-2,p_2) & \cdots & R(0,0) \end{bmatrix},$$

the spatial Yule-Walker estimates, $\hat{\alpha}$ can be obtained by solving equation,

$$\hat{\alpha} = \mathbf{R}^{-1}\mathbf{r}. \tag{18}$$

Guyon (1982), Basu and Reinsel (1993) and Ha and Newton (1993) showed that this estimate is asymptotically biased. If the divisor *mn* in the sample autocovariance is replaced by *(m-s)(n-t),* the 'unbiased' version of this estimator will be obtained [see Ha and Newton (1993)].

Since our interest is on a spatial unilateral AR(2,1) model as defined in equation (2) above, α, $\mathbf{r}$ and $\mathbf{R}$ will take the forms, $\alpha = (\alpha_{10}, \; \alpha_{20}, \; \alpha_{01}, \; \alpha_{11}, \; \alpha_{21})'$, $\mathbf{r} = (R(1,0), \; R(2,0), \; R(0,1), \; R(1,1), \; R(2,1))'$ and

$$R = \begin{bmatrix} R(0,0) & R(1,0) & R(1,-1) & R(0,1) & R(1,1) \\ R(1,0) & R(0,0) & R(2,-1) & R(1,-1) & R(0,1) \\ R(1,-1) & R(2,-1) & R(0,0) & R(1,0) & R(2,0) \\ R(0,1) & R(1,-1) & R(1,0) & R(0,0) & R(-1,0) \\ R(1,1) & R(0,1) & R(2,0) & R(1,0) & R(0,0) \end{bmatrix}.$$

Then the estimate of $\alpha = (\alpha_{10}, \; \alpha_{20}, \; \alpha_{01}, \; \alpha_{11}, \; \alpha_{21})'$ is obtained by using equation (18) above. The divisor *mn* is replaced by *(m-s)(n-t)* to obtain the 'unbiased' Yule-Walker estimate.

4.2 Spatial Conditional Least Squares Estimation

We discuss here two types of conditional least squares estimation method. For *Type 1*, as in maximum likelihood method discussed in Sections 2 and 3, we assume that the unobserved border values are all zero, that is $\mathbf{Y_b'} = (Y_{-1,0}, \; ..., \; Y_{-1,n}, \; Y_{00}, \; ..., \; Y_{0n}, \; Y_{10}, \; ...$

$, Y_{m0}) = \mathbf{0}$. The least squares estimates of $\alpha' = (\alpha_{10},\ \alpha_{01},\ \alpha_{11},\ \alpha_{20},\ \alpha_{21})$ is given as

$$\hat{\alpha}' = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}, \tag{19}$$

where

$$\mathbf{Y} = (Y_{11},\ Y_{12},\ ...,\ Y_{1n},\ Y_{21},\ Y_{22},\ ...,\ Y_{2n},\ ...,\ Y_{m1},\ Y_{m2},\ ...,\ Y_{mn})'$$

and $\mathbf{X}$ is a matrix of dimension $(mn) \times 5$ given as,

$$\mathbf{X} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & Y_{11} & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & Y_{1,n-1} & 0 & 0 & 0 \\ Y_{11} & 0 & 0 & 0 & 0 \\ Y_{12} & Y_{21} & Y_{11} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ Y_{1n} & Y_{2,n-1} & Y_{1,n-1} & 0 & 0 \\ Y_{21} & 0 & 0 & Y_{11} & 0 \\ Y_{22} & Y_{31} & Y_{21} & Y_{12} & Y_{11} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ Y_{2n} & Y_{3,n-1} & Y_{2,n-1} & Y_{1n} & Y_{1,n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ Y_{m-1,1} & 0 & 0 & Y_{m-2,1} & 0 \\ Y_{m-1,2} & Y_{m,1} & Y_{m-1,1} & Y_{m-2,2} & Y_{m-2,1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ Y_{m-1,n} & Y_{m,n-1} & Y_{m-1,n-1} & Y_{m-2,n} & Y_{m-2,n-1} \end{bmatrix}.$$

It is obvious that the estimates from equation (19) are equivalent to the maximum likelihood estimates as given in equation (14). This is due to our previous assumption that the unobserved border values are all fixed to zero, that is $\mathbf{Y_b'} = (Y_{-1,0},\ ...,\ Y_{-1,n},\ Y_{00},\ ...,\ Y_{0n},\ Y_{10},\ ...,\ Y_{m0}) = \mathbf{0}$ and also the square root of the determinant of the covariance matrix of $\mathbf{Y}$, $|\mathbf{V}|^{\ 1/2} = \left(\sigma^2\right)^{N/2}$ does not involved any function of α'. Therefore maximizing equation (11) with respect of α' is analogue of the procedure to obtain the conditional least squares estimates as in equation (19).

In *Type 2*, we obtain the conditional least squares estimates by conditioning on the given observed border, $\mathbf{Y_o}' = (Y_{11},\ ...,\ Y_{1n},\ Y_{21},\ ...,\ Y_{2n},\ Y_{31},\ ...,\ Y_{m1})$. Then, the estimator is given as,

$$\hat{\alpha}_{\mathbf{o}} = (\mathbf{X_o}'\mathbf{X_o})^{-1}\mathbf{X_o}'\mathbf{Y_{(1)}} \tag{20}$$

where

$$\mathbf{Y_{(1)}}' = (Y_{32},\ ...,\ Y_{3n},\ Y_{42},\ ...,\ Y_{4n},\ ...,\ Y_{m2},\ ...,\ Y_{mn}),$$

and $\mathbf{X}_o$ is a matrix of dimension $(m-2)(n-1) \times 5$ defined as,

$$\mathbf{X_o} = \begin{bmatrix} Y_{22} & Y_{31} & Y_{21} & Y_{12} & Y_{11} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ Y_{2n} & Y_{3,n-1} & Y_{2,n-1} & Y_{1n} & Y_{1,n-1} \\ Y_{32} & Y_{41} & Y_{31} & Y_{22} & Y_{21} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ Y_{3n} & Y_{4,n-1} & Y_{3,n-1} & Y_{2n} & Y_{2,n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ Y_{m-1,2} & Y_{m1} & Y_{m-1,1} & Y_{m-2,2} & Y_{m-2,1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ Y_{m-1,n} & Y_{m,n-1} & Y_{m-1,n-1} & Y_{m-2,n} & Y_{m-2,n-1} \end{bmatrix}.$$

5 Simulation Results

A simulation study is conducted to evaluate the performance of this estimation procedure for the second order spatial unilateral AR model as defined in equation (2). The estimation programmes were written in S-Plus. We began by generating independent standard normal variates ε_{ij} and then the border values $Y_{i,1}$, $i = 1, 2, \ldots, n$, $Y_{1,j}$ and $Y_{2,j}$, $j = 2, 3, \ldots, m$ are determined by assuming that the cells bordering the lattice have fixed values of zeros. Then, the remaining $Y_{i,j}$, $i = 3, 4, \ldots, m$; $j = 2, 3, \ldots, n$, are obtained recursively from equation (2). The simulations were done for two different sets of α-values and nine different grid sizes, i.e. (6×10), (8×8), (8×10), (6×15), (5×20), (16×20), (12×30), (15×25) and (20×20). The estimates of $\alpha' = (\alpha_{10}, \alpha_{01}, \alpha_{11}, \alpha_{20}, \alpha_{21})$ are obtained using maximum likelihood (ML), Yule-Walker, 'unbiased' Yule-Walker and least squares (LS Type 2) estimators as given in equations (14), (18) and (19), respectively. For each set of α-values, we made 500 replications for each of the grids and obtained the averages of the estimates. The root mean squared errors (RMSE) as well as absolute bias are used as criteria to compare the estimators. The RMSE is a square root of the mean squared error (MSE) which is given as,

$$\mathrm{MSE}(\hat{\alpha}) = E(\hat{\alpha} - \alpha)^2 = \mathrm{Var}(\hat{\alpha}) + [\mathrm{b}(\hat{\alpha})]^2,$$

where $\mathrm{b}(\hat{\alpha})$ is a bias vector of the estimators.

Tables 1, 3, 5 and 7 show the results of the average estimates of α' from 500 replications for α' fixed at (-0.6, 0.3, 0.5, -0.1, 0.4), that is $\alpha_{10} = -0.6$, $\alpha_{01} = 0.3$, $\alpha_{11} = 0.5$, $\alpha_{20} = -0.1$, $\alpha_{21} = 0.4$ and σ^2= 1.0000 using ML, Yule-Walker, 'unbiased' Yule-Walker and LS Type 2, respectively. Similarly Tables 2, 4, 6 and 8 show the results of the average estimates of α' from 500 replications for α' fixed at (0.2, 0.3, 0.2, 0.1, 0.1), that is $\alpha_{10} = 0.2$, $\alpha_{01} = 0.3$, $\alpha_{11} = 0.2$, $\alpha_{20} = 0.1$, $\alpha_{21} = 0.1$ and σ^2= 1.0000 using ML, Yule-Walker, 'unbiased' Yule-Walker and LS Type2, respectively.

The averages of the estimated σ^2 from 500 replications are shown in the last columns. σ^2 is estimated as the mean of the squared residuals. The RMSE for all estimated parameters and $\hat{\sigma}^2$ are shown in parentheses.

Figures 1 and 2 show the plots of absolute bias of the estimates against grid size and RMSE against grid size from 500 replications for α' fixed at (-0.6, 0.3, 0.5, -0.1, 0.4) and σ^2= 1.0000, while Figures 3 and 4 show the plots of absolute bias of the estimates against grid size and RMSE against grid size from 500 replications for α' fixed at (0.2, 0.3, 0.2, 0.1, 0.1) and σ^2= 1.0000 for all estimators discussed above.

From Figure 1, it is seen that for α' fixed at (-0.6, 0.3, 0.5, -0.1, 0.4), the magnitude of the bias was largest for Yule-Walker for all parameters and it was obvious for small grid sizes. The performance of the other three was considerably equivalent but ML is seen the best. The magnitude of the bias also decreases as the grid size increases for all estimators and parameters. Although the magnitude of the bias for Yule-Walker estimates was largest for all parameters, its bias was smallest for $\hat{\sigma}^2$.

From Figure 2, it is seen that Yule-Walker is still the worst if we compare the performance based on RMSE value for α' fixed at (-0.6, 0.3, 0.5, -0.1, 0.4), except for $\hat{\alpha}_{01}$ and $\hat{\alpha}_{20}$. LS Type 2 performed worst for $\hat{\alpha}_{01}$. ML estimator was always the best for all parameters except for $\hat{\alpha}_{20}$. However, the RMSE for $\hat{\sigma}^2$ are considerably equivalent for all estimators. The RMSE also decreases as the grid size increases for all estimators and for all parameters.

For α' fixed at (0.2, 0.3, 0.2, 0.1, 0.1), as shown in Figure 3, it is seen that Yule-Walker estimates are more biased than the other three for $\hat{\alpha}_{11}$, $\hat{\alpha}_{20}$ and $\hat{\alpha}_{21}$. The 'unbiased' Yule-Walker estimates performed worst for $\hat{\alpha}_{10}$ and $\hat{\alpha}_{01}$. ML and LS Type 2 estimators performed equally except for $\hat{\alpha}_{01}$ where the bias for LS Type 2 estimates was larger. The bias for $\hat{\sigma}^2$ was slightly higher for ML estimate compared to the other three and the bias was smallest for Yule-Walker estimate. However, as the results for the previous set of α', the absolute biases decrease as the grid sizes increase for all parameters and for all estimators.

Lastly, from Figure 4, the RMSE values for Yule-Walker estimate were always smallest for all parameters compared to the other three. The RMSE of LS Type 2 estimates was always largest except for $\hat{\alpha}_{21}$ which is highest for 'unbiased' Yule-Walker estimates. As in previous set, all estimators performed equally for $\hat{\sigma}^2$. The RMSE values also decrease as the grid sizes increase and the performance of all estimators was considerably equivalent for moderate grid size.

From this simulation studies, it is seen that ML estimator is the best in overall performance compared to the other three for these two set of α' values. It performed the best in the sense that the magnitude of the biases and RMSE values for all parameters were generally smallest with a few exceptions like for $\hat{\sigma}^2$. Furthermore, as the grid size increases, the estimates approach the true values and absolute bias and the RMSE values decrease for all parameters.

Table 1: Average estimated value of parameters and RMSE (in parentheses) from 500 replications of ML estimators for α' fixed at (-0.6, 0.3, 0.5, -0.1, 0.4) and σ^2= 1.0000.

Grid size	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\alpha}_3$	$\hat{\alpha}_4$	$\hat{\alpha}_5$	$\hat{\sigma}^2$
6 × 10	-0.5952 (0.1356)	0.2870 (0.1221)	0.4892 (0.1533)	-0.1171 (0.1453)	0.3790 (0.1712)	0.9201 (0.1921)
8 × 8	-0.5748 (0.1342)	0.2862 (0.1200)	0.4718 (0.1487)	-0.1027 (0.1435)	0.3683 (0.1533)	0.9198 (0.1896)
8 × 10	-0.5953 (0.1114)	0.2903 (0.1030)	0.4849 (0.1315)	-0.1063 (0.1145)	0.3818 (0.1296)	0.9370 (0.1642)
6 × 15	-0.5909 (0.1082)	0.2884 (0.0954)	0.4767 (0.1261)	-0.1028 (0.1253)	0.3786 (0.1330)	0.9431 (0.1545)
5 × 20	-0.5865 (0.1077)	0.2914 (0.0877)	0.4753 (0.1175)	-0.0977 (0.1288)	0.3849 (0.1311)	0.9507 (0.1469)
16 × 20	-0.5953 (0.0548)	0.2969 (0.0500)	0.4922 (0.0574)	-0.1003 (0.0548)	0.3915 (0.0548)	0.9877 (0.0803)
12 × 30	-0.5974 (0.0510)	0.2966 (0.0458)	0.4967 (0.0557)	-0.1037 (0.0557)	0.3928 (0.0577)	0.9889 (0.0770)
15 × 25	-0.5972 (0.0500)	0.2970 (0.0447)	0.4981 (0.0548)	-0.1006 (0.0520)	0.3949 (0.0548)	0.9895 (0.0748)
20 × 20	-0.5962 (0.0480)	0.2984 (0.0458)	0.4940 (0.0500)	-0.1003 (0.0490)	0.3932 (0.0490)	0.9900 (0.0723)

Table 2: Average estimated value of parameters and RMSE (in parentheses) from 500 replications of ML estimators for α' fixed at (0.2, 0.3, 0.2, 0.1, 0.1) and σ^2= 1.0000.

Grid size	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\alpha}_3$	$\hat{\alpha}_4$	$\hat{\alpha}_5$	$\hat{\sigma}^2$
6 × 10	0.1853 (0.1411)	0.2850 (0.1253)	0.2035 (0.1490)	0.0846 (0.1487)	0.0921 (0.1667)	0.9183 (0.1923)
8 × 8	0.2037 (0.1349)	0.2849 (0.1208)	0.1901 (0.1456)	0.0762 (0.1367)	0.0875 (0.1552)	0.9198 (0.1905)
8 × 10	0.1882 (0.1170)	0.2877 (0.1063)	0.1970 (0.1281)	0.0915 (0.1241)	0.0944 (0.1349)	0.9345 (0.1643)
6 × 15	0.1943 (0.1140)	0.2880 (0.0970)	0.1904 (0.1122)	0.0875 (0.1281)	0.1016 (0.1327)	0.9431 (0.1549)
5 × 20	0.2046 (0.1095)	0.2905 (0.0889)	0.1834 (0.1170)	0.0901 (0.1281)	0.0982 (0.1315)	0.9501 (0.1471)
16 × 20	0.2007 (0.0583)	0.2957 (0.0520)	0.1942 (0.0574)	0.0925 (0.0557)	0.0956 (0.0600)	0.9873 (0.0812)
12 × 30	0.1981 (0.0539)	0.2955 (0.0480)	0.2009 (0.0566)	0.0944 (0.0557)	0.0961 (0.0566)	0.9888 (0.0769)
15 × 25	0.1970 (0.0539)	0.2963 (0.0458)	0.2020 (0.0548)	0.0944 (0.0539)	0.0986 (0.0566)	0.9894 (0.0747)
20 × 20	0.2009 (0.0510)	0.2972 (0.0480)	0.1952 (0.0500)	0.0942 (0.0490)	0.0967 (0.0520)	0.9898 (0.0728)

Table 3: Average estimated value of parameters and RMSE (in parentheses) from 500 replications of YW estimators for α' fixed at (-0.6, 0.3, 0.5, -0.1, 0.4) and σ^2= 1.0000.

Grid size	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\alpha}_3$	$\hat{\alpha}_4$	$\hat{\alpha}_5$	$\hat{\sigma}^2$
6 × 10	-0.4565 (0.1817)	0.2403 (0.1281)	0.3258 (0.2054)	-0.0725 (0.0927)	0.2030 (0.2156)	0.9680 (0.1879)
8 × 8	-0.4714 (0.1729)	0.2379 (0.1241)	0.3300 (0.2037)	-0.0649 (0.1086)	0.2203 (0.2015)	0.9543 (0.1821)
8 × 10	-0.4875 (0.1517)	0.2483 (0.1082)	0.3462 (0.1833)	-0.0705 (0.0889)	0.2338 (0.1857)	0.9699 (0.1609)
6 × 15	-0.4599 (0.1679)	0.2553 (0.1039)	0.3326 (0.1931)	-0.0653 (0.0849)	0.2144 (0.2000)	0.9826 (0.1522)
5 × 20	-0.4322 (0.1881)	0.2624 (0.0970)	0.3195 (0.1997)	-0.0591 (0.0794)	0.1960 (0.2140)	0.9976 (0.1456)
16 × 20	-0.5403 (0.0794)	0.2759 (0.0548)	0.4191 (0.0959)	-0.0792 (0.0529)	0.3148 (0.0970)	0.9971 (0.0804)
12 × 30	-0.5277 (0.0860)	0.2790 (0.0510)	0.4163 (0.0980)	-0.0795 (0.0500)	0.3030 (0.1063)	1.0013 (0.0776)
15 × 25	-0.5393 (0.0775)	0.2987 (0.0500)	0.4252 (0.0900)	-0.0793 (0.0490)	0.3164 (0.0954)	0.9992 (0.0750)
20 × 20	-0.5503 (0.0678)	0.2782 (0.0500)	0.4295 (0.0831)	-0.0813 (0.0480)	0.3267 (0.0843)	0.9972 (0.0725)

Table 4: Average estimated value of parameters and RMSE (in parentheses) from 500 replications of YW estimators for α' fixed at (0.2, 0.3, 0.2, 0.1, 0.1) and σ^2= 1.0000.

Grid size	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\alpha}_3$	$\hat{\alpha}_4$	$\hat{\alpha}_5$	$\hat{\sigma}^2$
6 × 10	0.1758 (0.1183)	0.2814 (0.1153)	0.1349 (0.1221)	0.0459 (0.1086)	0.0298 (0.1170)	0.9453 (0.1879)
8 × 8	0.2032 (0.1225)	0.2686 (0.1105)	0.1305 (0.1292)	0.0521 (0.1063)	0.0363 (0.1153)	0.9411 (0.1858)
8 × 10	0.1881 (0.1030)	0.2803 (0.1000)	0.1396 (0.1131)	0.0645 (0.0970)	0.0406 (0.1049)	0.9536 (0.1599)
6 × 15	0.1765 (0.0980)	0.2972 (0.0985)	0.1336 (0.1054)	0.0470 (0.0959)	0.0405 (0.0985)	0.9644 (0.1514)
5 × 20	0.1731 (0.0917)	0.3071 (0.0911)	0.1250 (0.1109)	0.0387 (0.0927)	0.0323 (0.0954)	0.9740 (0.1440)
16 × 20	0.2041 (0.0574)	0.2961 (0.0510)	0.1644 (0.0624)	0.0805 (0.0529)	0.0661 (0.0600)	0.9922 (0.0812)
12 × 30	0.1948 (0.0500)	0.3062 (0.0490)	0.1694 (0.0583)	0.0754 (0.0529)	0.0634 (0.0583)	0.9949 (0.0768)
15 × 25	0.2000 (0.0520)	0.3010 (0.0469)	0.1723 (0.0566)	0.0806 (0.0510)	0.0682 (0.0566)	0.9943 (0.0747)
20 × 20	0.2073 (0.0529)	0.2952 (0.0469)	0.1678 (0.0566)	0.0858 (0.0469)	0.0699 (0.0548)	0.9937 (0.0727)

Table 5: Average estimated value of parameters and RMSE (in parentheses) from 500 replications of 'unbiased' YW estimators for α' fixed at (-0.6, 0.3, 0.5, -0.1, 0.4) and σ^2= 1.0000.

Grid size	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\alpha}_3$	$\hat{\alpha}_4$	$\hat{\alpha}_5$	$\hat{\sigma}^2$
6 × 10	-0.6512 (0.1841)	0.3128 (0.1323)	0.5508 (0.2114)	-0.1585 (0.1667)	0.4044 (0.1887)	0.9496 (0.1892)
8 × 8	-0.6265 (0.1584)	0.3013 (0.1175)	0.5218 (0.1828)	-0.1346 (0.1507)	0.3963 (0.1667)	0.9390 (0.1884)
8 ×10	-0.6374 (0.1432)	0.3077 (0.1054)	0.5290 (0.1643)	-0.1376 (0.1319)	0.3995 (0.1483)	0.9518 (0.1601)
6 ×15	-0.6304 (0.1404)	0.3076 (0.1039)	0.5287 (0.1609)	-0.1316 (0.1334)	0.3959 (0.1435)	0.9592 (0.1511)
5 × 20	-0.6196 (0.1375)	0.3162 (0.0949)	0.5321 (0.1533)	-0.1229 (0.1330)	0.3915 (0.1400)	0.9678 (0.1449)
16 × 20	-0.6153 (0.0600)	0.3032 (0.0490)	0.5142 (0.0656)	-0.1135 (0.0592)	0.4037 (0.0592)	0.9897 (0.0801)
12 × 30	-0.6166 (0.0592)	0.3062 (0.0458)	0.5234 (0.0671)	-0.1165 (0.0600)	0.4035 (0.0592)	0.9912 (0.0768)
15 × 25	-0.6151 (0.0574)	0.3037 (0.0447)	0.5193 (0.0648)	-0.1122 (0.0548)	0.4048 (0.0583)	0.9912 (0.0747)
20 × 20	-0.6140 (0.0520)	0.3030 (0.0447)	0.5122 (0.0539)	-0.1118 (0.0510)	0.4036 (0.0510)	0.9914 (0.0722)

Table 6: Average estimated value of parameters and RMSE (in parentheses) from 500 replications of 'unbiased' YW estimators for α' fixed at (0.2, 0.3, 0.2, 0.1, 0.1) and σ^2= 1.0000.

Grid size	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\alpha}_3$	$\hat{\alpha}_4$	$\hat{\alpha}_5$	$\hat{\sigma}^2$
6 × 10	0.1404 (0.1732)	0.2524 (0.1456)	0.2093 (0.1609)	0.0635 (0.1581)	0.1096 (0.1947)	0.9431 (0.2110)
8 × 8	0.1533 (0.1616)	0.2552 (0.1356)	0.2058 (0.1640)	0.0588 (0.1378)	0.1147 (0.1814)	0.9423 (0.1914)
8 × 10	0.1439 (0.1463)	0.2607 (0.1204)	0.2062 (0.1389)	0.0753 (0.1311)	0.1160 (0.1616)	0.9531 (0.1685)
6 × 15	0.1585 (0.1273)	0.2624 (0.1100)	0.1910 (0.1175)	0.0708 (0.1273)	0.1093 (0.1428)	0.9549 (0.1533)
5 × 20	0.1728 (0.1170)	0.2636 (0.1020)	0.1799 (0.1212)	0.0703 (0.1245)	0.0966 (0.1300)	0.9602 (0.1452)
16 × 20	0.1788 (0.0640)	0.2809 (0.0574)	0.1999 (0.0608)	0.0860 (0.0574)	0.1070 (0.0663)	0.9900 (0.0810)
12 × 30	0.1778 (0.0600)	0.2770 (0.0548)	0.2048 (0.0600)	0.0896 (0.0574)	0.1057 (0.0616)	0.9912 (0.0769)
15 × 25	0.1777 (0.0608)	0.2792 (0.0529)	0.2069 (0.0574)	0.0892 (0.0548)	0.1090 (0.0616)	0.9917 (0.0747)
20 × 20	0.1801 (0.0574)	0.2848 (0.0510)	0.2009 (0.0539)	0.0880 (0.0500)	0.1078 (0.0600)	0.9919 (0.0726)

Table 7: Average estimated value of parameters and RMSE (in parentheses) from 500 replications of LS (Type 2) estimators for α' fixed at (-0.6, 0.3, 0.5, -0.1, 0.4) and σ^2= 1.0000.

Grid size	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\alpha}_3$	$\hat{\alpha}_4$	$\hat{\alpha}_5$	$\hat{\sigma}^2$
6 × 10	-0.5941 (0.1581)	0.2754 (0.1549)	0.4787 (0.1794)	-0.1197 (0.1622)	0.3792 (0.1808)	0.9432 (0.1907)
8 × 8	-0.5682 (0.1565)	0.2766 (0.1425)	0.4635 (0.1694)	-0.1011 (0.1591)	0.3680 (0.1584)	0.9365 (0.1870)
8 × 10	-0.5952 (0.1249)	0.2841 (0.1204)	0.4777 (0.1449)	-0.1089 (0.1273)	0.3807 (0.1356)	0.9489 (0.1633)
6 × 15	-0.5877 (0.1273)	0.2815 (0.1221)	0.4705 (0.1473)	-0.1013 (0.1315)	0.3797 (0.1375)	0.9557 (0.1530)
5 × 20	-0.5806 (0.1273)	0.2863 (0.1153)	0.4726 (0.1421)	-0.0917 (0.1375)	0.3862 (0.1360)	0.9648 (0.1456)
16 × 20	-0.5943 (0.0574)	0.2962 (0.0539)	0.4924 (0.0608)	-0.0996 (0.0557)	0.3916 (0.0548)	0.9888 (0.0803)
12 × 30	-0.5975 (0.0548)	0.2951 (0.0510)	0.4947 (0.0592)	-0.1032 (0.0557)	0.3922 (0.0566)	0.9900 (0.0770)
15 × 25	-0.5974 (0.0539)	0.2958 (0.0480)	0.4980 (0.0566)	-0.1017 (0.0529)	0.3952 (0.0548)	0.9904 (0.0748)
20 × 20	-0.5954 (0.0490)	0.2979 (0.0490)	0.4942 (0.0520)	-0.0994 (0.0500)	0.3931 (0.0490)	0.9907 (0.0723)

Table 8: Average estimated value of parameters and RMSE (in parentheses) from 500 replications of LS (Type 2) estimators for α' fixed at (0.2, 0.3, 0.2, 0.1, 0.1) and σ^2= 1.0000.

Grid size	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\alpha}_3$	$\hat{\alpha}_4$	$\hat{\alpha}_5$	$\hat{\sigma}^2$
6 × 10	0.1763 (0.1667)	0.2729 (0.1591)	0.2035 (0.1761)	0.0829 (0.1664)	0.1017 (0.1732)	0.9418 (0.1902)
8 × 8	0.2025 (0.1578)	0.2756 (0.1449)	0.1919 (0.1606)	0.0732 (0.1470)	0.0927 (0.1591)	0.9369 (0.1877)
8 × 10	0.1825 (0.1319)	0.2804 (0.1257)	0.1963 (0.1414)	0.0905 (0.1334)	0.0999 (0.1393)	0.9468 (0.1630)
6 × 15	0.1930 (0.1345)	0.2812 (0.1257)	0.1910 (0.1292)	0.0896 (0.1315)	0.1062 (0.1375)	0.9563 (0.1532)
5 × 20	0.2082 (0.1296)	0.2842 (0.1187)	0.1853 (0.1396)	0.0930 (0.1360)	0.0996 (0.1356)	0.9650 (0.1457)
16 × 20	0.2014 (0.0608)	0.2947 (0.0557)	0.1950 (0.0592)	0.0925 (0.0566)	0.0953 (0.0600)	0.9885 (0.0812)
12 × 30	0.1975 (0.0583)	0.2939 (0.0529)	0.2004 (0.0592)	0.0951 (0.0574)	0.0970 (0.0574)	0.9899 (0.0768)
15 × 25	0.1963 (0.0574)	0.2948 (0.0500)	0.2032 (0.0566)	0.0937 (0.0548)	0.0991 (0.0574)	0.9903 (0.0747)
20 × 20	0.2015 (0.0529)	0.2965 (0.0500)	0.1959 (0.0520)	0.0945 (0.0500)	0.0962 (0.0529)	0.9906 (0.0728)

Figure 1: Absolute bias of point estimates, $\hat{\alpha}_i (i = 1, 2, 3, 4, 5)$ and $\hat{\sigma}^2$ vs. grid size from 500 replications for α' fixed at $(-0.6, 0.3, 0.5, -0.1, 0.4) - (ML(+), YW(\diamond)$, 'Unbiased' $YW(\triangle)$, LS Type $2(\times))$.

Figure 2: RMSE of point estimates, $\hat{\alpha}_i (i = 1, 2, 3, 4, 5)$ and $\hat{\sigma}^2$ vs. grid size from 500 replications for α' fixed at $(-0.6, 0.3, 0.5, -0.1, 0.4)$ – $(ML(+)$, $YW(\diamond)$, 'Unbiased' $YW(\triangle)$, LS Type $2(\times))$.

Figure 3: Absolute bias of point estimates, $\hat{\alpha}_i (i = 1, 2, 3, 4, 5)$ and $\hat{\sigma}^2$ vs. grid size from 500 replications for α' fixed at $(0.2, 0.3, 0.2, 0.1, 0.1)$ – ($ML(+)$, $YW(\diamond)$, 'Unbiased' $YW(\triangle)$, LS Type $2(\times)$).

Figure 4: RMSE of point estimates, $\hat{\alpha}_i (i = 1, 2, 3, 4, 5)$ and $\hat{\sigma}^2$ vs. grid size from 500 replications for α' fixed at $(0.2, 0.3, 0.2, 0.1, 0.1)$ – $(ML(+)$, $YW(\diamond)$, 'Unbiased' $YW(\triangle)$, LS Type $2(\times))$.

6 Conclusion

In this paper we have provided a procedure for estimating the parameters of the spatial unilateral AR(2,1) model using the maximum likelihood method. This estimation procedure was an adaptation of the method discussed in Ord (1975) which provided the maximum likelihood estimate for the *one* parameter case. Here, we have extended the procedure to the case of *five* parameters. We have shown how the matrices are constructed and the form of these matrices have a lower triangular which in turn simplifies the likelihood function.

From the simulation results, there is some evidence that this procedure provides good estimates for the parameters in sense of closeness to true values. Furthermore, as the grid size increases, the estimates approach the true values and the RMSE and absolute bias values decrease for all parameters.

In general then, by proper construction of the weight matrices it is shown that the unilateral model as defined in equation (2) can be translated to adopt the general setting in Ord (1975) and Cliff and Ord (1981) and consequently the maximum likelihood estimators can be obtained.

The method proposed herein can be extended in an analogous way for the unilateral AR(p_1,1) model where the number of parameters would be $2p_1 + 1$.

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