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Modified Inference about the Mean of the Normal Distribution Using Type II Censored Sampling

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Abstract

The Maximum Likelihood Estimator (MLE) and the Likelihood Ratio Test (LRT) can't be found in closed form in general. Mehrotra and Nanda (1974) suggested an approximation of the MLE by replacing some of the terms of the likelihood equations by their expectations to get estimators in closed form. We will compare this new estimator with the MLE of the mean of the normal distribution in case of type II censored samples through their mean square errors. Then we will use this new estimator to construct a new test in closed form which is a modification of the LRT for testing a simple hypothesis against one sided alternatives. Then the LRT and the new test will be compared through their power functions by simulation. The simulation results show that the new estimator and the new test are good competitors of the MLE and the LRT respectively.

Keywords and Phrases: Simple Random Sampling; Type II Censored Sampling, Maximum Likelihood Estimator, Modified Maximum Likelihood Estimator, Likelihood Ratio Test, Modified Likelihood Ratio Test, Mean Square Error, Power Function, Normal Distribution.

AMS Classification: $94A_{xx}$, $35B_{xx}$, $11H_{xx}$, $32A_{xx}$, $62H_{xx}$.

1 Introduction

The normal distribution is widely used in reliability and in many other areas. This distribution is a location-scale distribution and is symmetric about its mean. The random variable X has a normal distribution if its pdf is given by:

$$f\left(x,\mu\right)=\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}=\frac{1}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right),\qquad -\infty<x<\infty\qquad -\infty<\mu<\infty,\qquad \sigma>0$$

and its df is given by:

$$\mathrm{F}\left(\mathrm{x};\mu\right)=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\mathrm{x}}\mathrm{e}^{\frac{-1}{2}\left(\frac{\mathrm{u}-\mu}{\sigma}\right)^2}\mathrm{du}=\Phi\left(\frac{\mathrm{x}-\mu}{\sigma}\right),\qquad -\infty\prec x\prec\infty$$

where $\phi$and Φ are the pdf and the df of standard normal distribution, respectively. Next, we will consider the maximum likelihood estimator (MLE) of the mean (μ) for the normal distribution based on type II censored samples in case the standard deviation σ is known which will be assumed without loss of generality to be one.

Let $X_1 \leq X_2 \leq ..... \leq X_r$ be a failure-censored sample from the normal distribution. Then the likelihood function is given by:

$$L(\mu,x)=\frac{n!}{(n-r)!}\frac{e^{-\sum_{i=1}^{r}\frac{(x_i-\mu)^2}{2}}}{(2\pi)^{r/2}}\left(1-\Phi(x_r-\mu)\right)^{n-r}\tag{1}$$

and the log-likelihood function will be

$$L^*\left(\mu,x\right)=c-\sum_{i=1}^{r}\frac{\left(x_i-\mu\right)^2}{2}-rIn(2\pi)-\left(n-r\right)\left(\ln(1-\Phi(x_r-\mu))\right)\tag{2}$$

where $c=\ln\frac{n!}{(n-r)!}$. Then the first derivative of the log-likelihood function with respect to μ is given by:

$$\frac{\partial L^*\left(\mu,x\right)}{\partial\mu}=\sum_{i=1}^{r}\left(x_i-\mu\right)+\left(n-r\right)\left(\frac{\phi(x_r-\mu)}{1-\Phi(x_r-\mu)}\right)\overset{(set)}{=}0\tag{3}$$

which implies that its second derivative is given by:

$$\frac{\partial L^{*2}\left(\mu,x\right)}{\partial^2\mu}=-r+\left(n-r\right)\phi(x_r-\mu)\left(\frac{\left(x_r-\mu\right)\left(1-\Phi(x_r-\mu)\right)-\phi(x_r-\mu)}{\left(1-\Phi(x_r-\mu)\right)^2}\right)$$

It is easy to show that $\frac{\partial L^{*2}(\mu,x)}{\partial^2\mu}<0$ for any μ using the Mill's ratio (see Gordan (1941)). Note that as $\mu\to-\infty$, the LHS of (3) goes to ∞, and as $\mu\to\infty$ it goes to -r ($r\geq 1$) which is a negative number. Hence, (3) has a unique solution. Therefore the MLE of μ exists and it will be denoted by $\hat{\mu}$. However, this estimator can't be found in closed form.

Mehrotra and Nanda (1974) replaced $(n-r)\left(\frac{\phi(x_r-\mu)}{1-\Phi(x_r-\mu)}\right)$ by its expectation which equals to $\sum\limits_{i=r+1}^{n}E(V_i)$, where $V_1\leq V_2\leq\ldots..\leq V_n$ are the ordered statistics of a random sample of size n from standard normal distribution. Then they solved (3) for μ to get the estimator: $\hat{\mu}_m=\frac{\sum\limits_{i=1}^{r}X_i+\sum\limits_{i=r+1}^{n}E(V_i)}{r}$. $\hat{\mu}_m$ will be called a modified Maximum Likelihood Estimator (MMLE). Also, they used this approach to estimate the variance of the normal distribution when the mean is known. Furthermore, they considered the approximate MLE for the gamma distribution, whose pdf is given by: $f(x,\theta)=\frac{1}{\Gamma(p)\theta^p}e^{-\frac{x}{\theta}}x^{p-1},\ 0<x<\infty,\ 0<p,\ 0<\theta$

They obtained two unbiased approximate ML estimators of θ when P is known. They proved that these estimators are unbiased estimators of the corresponding parameters. Then these modified estimators have compared with the Lloyd's (1952) best linear unbiased estimators of these parameters through their variances.

Zheng and Al-Saleh (2002) considered this approach in case of ranked set sample for estimating general parameters. Their results are summarized as follows:

•For the location parameter, the MMLE is always more efficient than the MLE using SRS.

•For the scale parameter, the MMLE is at least as efficient as the MLE using SRS.

The numerical study conducted by them provided evidence showing that perfect judgment ranking of the MMLE has good efficiency relative to the MLE based on RSS. When the judgment error ranking is present, they show by simulation that the MMLE is more robust than the MLE using RSS.

Also, this approach was used by Al-Saleh & Al-Hadrami (2003,a,b) in case of moving extreme ranked set sample to estimate the location parameter of the normal distribution and the scale parameter of the exponential distribution respectively The MMLE that they obtained was compared with the exact MLE based on MERSS and with the MLE based on SRS. The MMLE showed a high efficiency relative to the MLE in both cases.

Other types of modified MLE can be found in the following references. Cordeiro, et al. (1999) they proposed a new pivotal quantity which is a function of the maximum likelihood estimate of a scalar parameter θ and whose distribution is standard normal excluding terms of order $O(n[-3/2])$ and smaller, where n is the sample size. The proposed pivot is a polynomial transformation of the standardized maximum likelihood estimate of at most third degree. They applied their main result to the one-parameter exponential family model and to a number of special distributions of this family. Some simulation results illustrate the superiority of the new estimator over the usual standardized maximum likelihood estimate with regard to third-order asymptotic theory. Yu and Wong (2005) consider a linear regression model where the response variable may be right-censored. The standard maximum likelihood estimator (MLE)-based parametric approach to estimation of regression coefficients requires that the parametric form of the error distribution be known. Given a dataset, it may not be possible to find a valid parametric form for the error distribution. In such a case the error distribution is unknown and arbitrary, and a semiparametric approach is plausible. A special modified semiparametric MLE (MSMLE) of the regression coefficients is proposed. Simulation suggests that the MSMLE is consistent is asymptotically normally distributed and may be efficient. The new procedure is applied to engineering data. Sartori (2006) studied the skew normal model which is a class of distributions that extends the Gaussian family by including a shape parameter. Despite its nice properties, this model presents some problems with the estimation of the shape parameter. In particular, for moderate sample sizes, the maximum likelihood estimator is infinite with positive probability. As a solution, they use a modified score function as an estimating equation for the shape parameter. It is proved that the resulting modified maximum likelihood estimator is always finite.

In this paper, we will compare $\hat{\mu}_m$ with $\hat{\mu}$ through their biases and mean square errors by using simulation. Then we will use the modified estimator to get a test in closed form for testing a simple hypothesis against one sided alternatives which is a modification of the LRT (MLRT) and then we will compare the LRT and the MLRT through the power functions by using simulation. Note that the biases and the mse's of both estimators don't depend on μ. Therefore, all our computations have been done for $\mu = 0$.

2 Comparisons of the MLE and the MMLE

Mehrotra and Nanda (1974) proved that $\hat{\mu}_m$ is an unbiased estimator of μ. The bias and the variance of $\hat{\mu}$ can't be found in closed form. So, we used simulation to compare between the biases and mse's of $\hat{\mu}$ and $\hat{\mu}_m$. Note that all our computations are done by using Mathematica 4.

The bias of the MLE $(\hat{\mu})$ is given in Tables 1, for n=3,... ,24 and r=2,3 ... $[\frac{n}{2}]$. Table 2 gives the efficiency of $\hat{\mu}_m$ with respect to $\hat{\mu}$ which is defined as follows

$$Eff(\hat{\mu}_m, \hat{\mu}) = \frac{mse(\hat{\mu})}{mse(\hat{\mu}_m)}$$

Table 1: $Bias(\hat{\mu})$

	r										
n	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
3	-0.0088										
4	-0.0195										
5	-0.0352	-0.0097									
6	-0.0323	-0.0202									
7	-0.0393	-0.0153									
8	-0.0388	-0.0210	-0.0148								
9	-0.0477	-0.0181	-0.0094								
10	-0.0390	-0.0268	-0.0127	-0.0064							
11	-0.0424	-0.0245	-0.0193	-0.0060							
12	-0.0502	-0.0238	-0.0161	-0.0133	-0.0074						
13	-0.0514	-0.0286	-0.0107	-0.0137	-0.0135						
14	-0.0557	-0.0384	-0.0226	-0.0119	-0.0057	-0.0095					
15	-0.0511	-0.0284	-0.0144	-0.0157	-0.0054	-0.0086					
16	-0.0562	-0.0335	-0.0208	-0.0145	-0.0067	-0.0080	-0.0064				
17	-0.0596	-0.0280	-0.0167	-0.0151	-0.0135	-0.0079	-0.0054				
18	-0.0456	-0.0265	-0.0269	-0.0115	-0.0079	-0.0131	-0.0020	-0.0035			
19	-0.0601	-0.0281	-0.0208	-0.0109	-0.0110	-0.0075	-0.0029	-0.0047			
20	-0.0560	-0.0327	-0.0220	-0.0191	-0.0078	-0.0101	-0.0050	-0.0027	-0.0008		
21	-0.0539	-0.0330	-0.0217	-0.0167	-0.0158	-0.0076	-0.0066	-0.0046	-0.0043		
22	-0.0561	-0.0318	-0.0247	-0.0124	-0.0053	-0.0092	-0.0067	-0.0032	-0.0060	-0.0071	
23	-0.0605	-0.0286	-0.0226	-0.0237	-0.0128	-0.0062	-0.0090	-0.0081	-0.0064	-0.0064	
24	-0.0539	-0.0302	-0.0192	-0.0125	-0.0168	-0.0105	-0.0045	-0.0059	-0.0054	-0.0049	-0.0045

Table 2: $Eff(\hat{\mu}_m, \hat{\mu})$

	r										
n	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
3	0.9826										
4	0.9633										
5	0.9508	0.9638									
6	0.9412	0.9480									
7	0.9302	0.9347									
8	0.9301	0.9313	0.9385								
9	0.9343	0.9149	0.9276								
10	0.9086	0.9129	0.9185	0.9339							
11	0.9087	0.9003	0.9153	0.9193							
12	0.9125	0.9072	0.9028	0.9118	0.9256						
13	0.9149	0.8896	0.8858	0.8964	0.9171						
14	0.9069	0.8840	0.8831	0.8968	0.9165	0.9118					
15	0.9068	0.8893	0.8801	0.8884	0.9055	0.9060					
16	0.9101	0.8882	0.8761	0.8817	0.8918	0.9202	0.9136				
17	0.9136	0.8758	0.8679	0.8874	0.8821	0.9072	0.9098				
18	0.8912	0.8733	0.8656	0.8830	0.8805	0.8985	0.9065	0.9082			
19	0.9049	0.8693	0.8585	0.8676	0.8707	0.8838	0.9010	0.9093			
20	0.9020	0.8664	0.8742	0.8563	0.8729	0.8929	0.8974	0.8910	0.9083		
21	0.8993	0.8719	0.8534	0.8658	0.8794	0.8706	0.8889	0.8960	0.9104		
22	0.8983	0.8536	0.8477	0.8634	0.8719	0.8655	0.8831	0.9032	0.9102	0.9186	
23	0.9001	0.8615	0.8520	0.8664	0.8586	0.8639	0.8870	0.8933	0.8927	0.9105	
24	0.8923	0.8561	0.8546	0.8535	0.8640	0.8573	0.8920	0.8859	0.8998	0.9126	0.9144

3 MLRT about the Mean of the Normal Distribution

In this section, the LRT and a modification of it (MLRT) based on the MLRT will be considered in case of normal distribution for testing

$$H_0 : \mu = \mu_0 \ vs \ H_1 : \mu > \mu_0$$

It can be assumed without loss of generality that $\mu_0 = 0$. The MMLE will be used to construct the MLRT for testing H_0 vs. H_1 by putting $\hat{\mu}_m$ instead of $\hat{\mu}$ in the statistic defining the LRT. Then the MLRT and the LRT in this case are given by:

$$\varphi^m\left(x_1, x_2, ..., x_r\right) = \left\{ \begin{array}{ll} 1, & \frac{\prod_{i=1}^{r} \phi(X_i - \hat{\mu}_1)(1-\Phi(X_r-\hat{\mu}_1))}{\prod_{i=1}^{r} \phi(x_i)(1-\Phi(x_r))} > k \\ 0, & otherwise \end{array} \right\} \tag{4}$$

and

$$\varphi\left(x_1, x_2, ..., x_r\right) = \left\{ \begin{array}{ll} 1, & \frac{\prod_{i=1}^{r} \phi(x_i - \hat{\mu})(1-\Phi(x_r-\hat{\mu}))}{\prod_{i=1}^{r} \phi(x_i)(1-\Phi(x_r))} > k \\ 0, & otherwise \end{array} \right\}$$

respectively where $\hat{\mu}_1$ is the MLE of μ under Ω=[0,∞) It is easy to show that $\hat{\mu}_1 = \max(\hat{\mu}_m, 0)$. Next, we will compare the two tests via their power functions. Since, their power functions are not in closed forms, we will use simulation to compare between them. We take α =0.05.The calculations have been done for μ = 0.1, 0.5, 1, 2, 3,4 , n=3,4& 10 and r=2,3,...,n-1. Tables 3 to 8 give the efficiency of φ^m with respect to φ, which is defined by

$$e_\mu(\phi^m, \phi) = \frac{K_{\phi^m}(\mu)}{K_\phi(\mu)}$$

where $K_\phi(\mu)$ is the power function of the test φ.

Table 3: $e_{0.1}(\phi^m, \phi)$

	r							
n	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
3	1.019356	-	-	-	-	-	-	-
4	1.036333	1.029785	-	-	-	-	-	-
5	1.034509	1.047538	1.02535	-	-	-	-	-
6	1.043756	1.026639	1.064324	1.059507	-	-	-	-
7	1.047855	1.067205	1.060215	1.102	1.040465	-	-	-
8	10.47984	1.097575	1.094236	1.054241	1.062245	1.035274	-	-
9	1.0553619	1.093222	1.08653	1.115712	1.073779	1.036489	1.029834	-
10	1.056414	1.095442	1.092708	1.098754	1.097223	1.08893	1.030478	1.040404

Table 4: $e_{0.5}(\phi^m, \phi)$

	R							
n	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
3	1.031773	-	-	-	-	-	-	-
4	1.072094	1.068025	-	-	-	-	-	-
5	1.077449	1.121504	1.085425	-	-	-	-	-
6	1.096131	1.143093	1.16956	1.083485	-	-	-	-
7	1.086400	1.201836	1.209432	1.190768	1.092366	-	-	-
8	1.076919	1.240609	1.276604	1.209154	1.178503	1.093353	-	-
9	1.100718	1.254991	1.294724	1.288058	1.234802	1.174566	1.083404	-
10	1.086708	1.244217	1.314406	1.336267	1.309016	1.25621	1.157557	1.085028

Table 5: $e_{1.0}(\phi^m, \phi)$

	R							
n	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
3	1.010227	-	-	-	-	-	-	-
4	1.01382	1.051683	-	-	-	-	-	-
5	1.000165	1.085459	1.059051	-	-	-	-	-
6	0.984683	1.089705	1.117408	1.059924	-	-	-	-
7	0.963758	1.115417	1.163836	1.142742	1.063997	-	-	-
8	0.942815	1.140454	1.199268	1.178346	1.131538	1.054871	-	-
9	0.946122	1.132572	1.225971	1.237984	1.175801	1.116907	1.049551	-
10	0.915088	1.132556	1.236788	1.261284	1.237669	1.175195	1.097328	1.043003

Table 6: $e_{2.0}(\phi^m, \phi)$

	R							
n	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
3	0.980084	-	-	-	-	-	-	-
4	0.936485	0.999179	-	-	-	-	-	-
5	0.888357	0.983766	1.004245	-	-	-	-	-
6	0.835928	0.952241	1.003751	1.003258	-	-	-	-
7	0.802889	0.935068	0.99355	1.011499	1.002807	-	-	-
8	0.751298	0.911349	0.972391	1.005359	1.008800	1.001588	-	-
9	0.71807	0.882511	0.96845	1.008918	1.009592	1.005091	1.001242	-
10	0.681656	0.857897	0.953919	1.136028	1.013097	1.010789	1.003223	1.000468

Table 7: $e_{3.0}(\phi^m, \phi)$

	R							
n	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
3	0.993572	-	-	-	-	-	-	-
4	0.972804	0.999225	-	-	-	-	-	-
5	0.939063	0.99169	0.999893	-	-	-	-	-
6	0.901789	0.975551	0.998284	1.000000	-	-	-	-
7	0.861982	0.956187	0.992142	0.999766	1.000000	-	-	-
8	0.822444	0.934709	0.984869	0.997172	0.999933	1.000000	-	-
9	0.792866	0.909553	0.971146	0.994342	0.999431	0.99997	1.000000	-
10	0.758311	0.88586	0.957212	0.988607	0.998223	0.999767	0.999967	1.000000

Table 8: $e_{4.0}(\phi^m, \phi)$

	R							
n	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
3	0.990661	-	-	-	-	-	-	-
4	0.998227	1.000000	-	-	-	-	-	-
5	0.993964	0.999667	1.000000	-	-	-	-	-
6	0.986886	0.999065	1.000000	1.000000	-	-	-	-
7	0.975878	0.996958	0.999700	1.000000	1.000000	-	-	-
8	0.962404	0.993207	0.999733	1.000000	1.000000	1.000000	-	-
9	0.956029	0.989974	0.998332	0.999900	1.000000	1.000000	1.000000	-
10	0.937783	0.983659	0.997596	0.999767	1.000000	1.000000	1.000000	1.000000

4 Conclusions

Based on Tables 1 to 8, we may conclude the following:

- The MLE is slightly biased and the bias decreases as r increases to $\frac{n}{2}$ for fixed n while it increases as n increases for fixed r.

- Table 2 indicates that the MMLE is a good competitor for the exact MLE when $r \leq \left[\frac{n}{2}\right]$ for fixed n and $n \leq 24$.

- From the above 2 comments, we expect that $\hat{\mu}_m$ will be a good competitor of $\hat{\mu}$ for large n as long as $r \leq \left[\frac{n}{2}\right]$.

- For both tests, power increases as r and/or the mean increases for fixed n. For fixed r, the power for both tests increases as n and/or the mean increases.

- The efficiency of the MLRT with respect to the exact LRT is nearly one. . This means that the MLRT is a good competitor for the exact LRT.

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