Characterization of Mixtures from Exponential and Pareto Distributions Using Left Truncated Moments

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Abstract

In this work, the concept of left truncated moments are used to charaterize a mixtures from two exponential distribution as well as a mixture from two Pareto Pareto distribution, conditional variance are obtained. We obtained the results of Talwalker (1977) for identifying the exponential and power distributions as a special cases. The recurrence relation of the conditional moments are obtained.

Keywords and Phrases: Characterization, left truncated moments, mixture of two exponential distributions, mixture of two Pareto distributions, failure rate.

AMS Classification: 62G30

1 Introduction

Identifying distributions is an important topic in statistics, reliability studies. There Are various tolls used to characterize the probability distributions. Galambos and Kots (1978) and Ahmed (1991) have used the concept of left truncated moments to characterize some probability distributions like exponential, Pareto, Power, Gamma, negative binomial, beta, binomial and Piosson. Talwalker (1977) used the concept of right truncated moments to identify different distributions like exponential and power Distributions. Characterizations by right truncated moments are very important in practice. For example, in reliability studies some measuring devices may be unable

to record values greater than time t. the main object of this work is to identify
a mixture from two exponential as well as a mixture from the Pareto distributions
using left truncated moments. Nassar and Mohmoud (1985) characterized a mixture
from two exponential distributions using the left truncated moments. A mixture from
two generalized gamma distributions has been characterized by Yehia (1992) using
left truncated moments. Fakhry (1993) used the concept of conditional variance to
identify a mixture from two Weibull distributions. In sections 2 and 3 we extended the
results of talwalker (1977) concerning power and exponential distributions to mixtures
of exponential as well as a mixture from two Pareto distributions.

2   Mixture of Exponential Distribution

The probability distribution function of the mixtures of two exponential distributions
given by

$$f(x) = \lambda_1 \theta_1 \, e^{-\theta_1 x} \, + \, \lambda_2 \, \theta_2 \, e^{-\theta_1 x} \tag{1}$$

The cumulative distribution function is

$$f(x) = 1 - \lambda_1 e^{-\theta_1 x} \, - \, \lambda_2 \, e^{-\theta_1 x} \tag{2}$$

The reliability function is given by

$$R(x) = \lambda_1 \, e^{-\theta_1 \, x} \, + \, \lambda_2 \, e^{-\theta_2 \, x} \tag{3}$$

where $\lambda_1 + \lambda_2 = 1$, θ_1 and θ_2 are parameters.

Theorem (2.1):

Let X be a continuous random variable with finite rth moment, then x is a mixtures
from two exponential distributions with positive parameters θ_1 and θ_2 iff

$$E(x/x > y) = y \, + \, \frac{\theta_1 \, + \, \theta_2}{\theta_1 \, \theta_2} \, - \, \frac{r(y)}{\theta_1 \, \theta_2} \tag{4}$$

Multiplying (3) by $-\theta_2$ then add to (1) we get

$$\lambda_1 \; = \; \frac{\theta_2 \, R(y) - \, f(y)}{(\theta_2 \, - \, \theta_1) \, e^{-\theta_1 y}} \tag{5}$$

Multiplying (3) by $-\theta_1$ then add to (1) we get

$$\lambda_2 = \frac{f(y) - \theta_1\, R(y)}{(\theta_2 - \theta_1)e^{-\theta_2\, y}} \tag{6}$$

Now

$$E(x/x > y) = \frac{\int\limits_y^\infty x\, f(x)dx}{R(x)}$$

$$\int\limits_y^\infty x f(x)dx = \lambda_1\,\theta_1 \int\limits_y^\infty x e^{-\theta_1\, x}dx + \lambda_2\theta_2 \int\limits_y^\infty x\, e^{-\theta_2\, x}dx$$

Integrating by parts and substituting for λ_1 and λ_2 from equations (5) and (6), we have

$$\int\limits_y^\infty x f(x)dx = yR(y) - \frac{f(y)}{\theta_1\,\theta_2} + \frac{(\theta_1 + \theta_2)\,R(y)}{\theta_1\,\theta_2}$$

Differentiating both sides w.r.t. y and using the fact that $R'(y) = -f(y)$, we get

$$R''(y) + (\theta_1 + \theta_2)\,R'(y) + \theta_1\,\theta_2\,R(y) = 0$$

which is a second order differential equation has a solution of

$$R(y) = A\,e^{-\theta_1\, y} + B\,e^{-\theta_2\, y}$$

for some constants A and B. Since $R(0) = 1$, one gets $A + B = 1$ and thus X is a mixtures from two exponential distribution, and

$$E(x/x > y) = y + \frac{\theta_1 + \theta_2}{\theta_1\theta_2} - \frac{r(y)}{\theta_1\theta_2}$$

$$E(x^2/x > y) = \frac{\int\limits_y^\infty x^2 f(x)dx}{R(y)}$$

$$\int\limits_{y}^{\infty} x^2 f(x)dx = \lambda_1 \theta_1 \int\limits_{y}^{\infty} x^2 e^{-\theta_1 x}dx + \lambda_2 \theta_2 \int\limits_{y}^{\infty} x^2 e^{-\theta_2 x}dx$$

Integrating by parts twice and substituting for λ_1 and λ_2 from equations (5) and (6), we have

$$E(x^2/x > y) = y^2 + 2y \left(\frac{\theta_1 + \theta_2 - r(y)}{\theta_1 \theta_2}\right) + 2\left(\frac{\theta_1^2 + \theta_1 \theta_2 + \theta_2^2 - (\theta_1 + \theta_2)r(y)}{\theta_1^2 \theta_2^2}\right)$$

(7)

The conditional variance can be obtained as

$$Var(x/x > y) = E(x^2/x > y) - [E(x/x > y)]^2$$

$$Var(x/x > y) = \frac{\theta_1^2 + \theta_2^2 - r^2(x)}{\theta_1^2 \theta_2^2}$$

where

$$r(y) = \frac{f(y)}{R(y)}$$

is the failure rate function.

$\lambda_1 = 1$, $\lambda_2 = 0$, $\theta_1 = \theta$ and $\theta_2 = 0$, then

$$\int\limits_{y}^{\infty} xf(x)dx = \left(y + \frac{1}{\theta}\right) e^{-\theta y}$$

i.e.

$$E(x/x > y) = y + \frac{1}{\theta}$$

This result is a special case obtained by Talwalker (1977) for characterizing the exponential distribution.

3 Mixtures of two Pareto Distribution

The probability distribution function of the mixtures of two Pareto distribution is given by

$$f(x) = \lambda_1 a_1 (1/x)^{(a_1+1)} + \lambda_2 a_2 (1/x)^{(a_2+1)} \tag{8}$$

The reliability function is given by

$$R(x) = \lambda_1 (1/x)^{a_1} + \lambda_2 (1/x)^{a_2} \tag{9}$$

where $\lambda_1 + \lambda_2 = 1$, a_1 and a_2 are parameters.

Solving (8) and (9) for λ_1 and λ_2, we get

$$\lambda_1 = \frac{f(x) - a_2 R(x)/x}{(a_1 - a_2)(1/x)^{a_1+1}} \tag{10}$$

and

$$\lambda_2 = \frac{a_1 R(x)/x - f(x)}{(a_1 - a_2)(1/x)^{a_2+1}} \tag{11}$$

Theorem (3.1):

$$E(x/x > y) = \frac{\int\limits_y^\infty x f(x) dx}{R(x)}$$

$$\int\limits_y^\infty x f(x)\, dx = \int\limits_y^\infty \lambda_1 a_1 x (1/x)^{a_1+1} dx + \int\limits_y^\infty \lambda_2 a_2 x (1/x)^{a_2+1} dx$$

integrating and eliminating λ_1 and λ_2, we have

$$\int\limits_y^\infty x f(x) dx = \frac{y^2 f(y) - a_1 a_2 y R(y)}{(1 - a_1)(1 - a_2)}$$

or

$$(1 - a_1)(1 - a_2) \int_y^\infty x f(x) dx = y^2 f(y) - a_1 a_2 y R(y)$$

Differentiating both sides w.r.t. y , we have

$$y^2 R''(y) + (a_1 + a_2 + 1) y R'(y) + a_1 a_2 R(y) = 0$$

Which is the second order homogenous differential equation and has a solution of

$$R(y) = A(1/y)^{a_1} + B(1/y)^{a_2}$$

Since $R(1) = 1$, one gets $A + B = 1$, and thus X is a mixtures of two Pareto distribution.

$$E(x/x > y) = \frac{\int_y^\infty x f(x) dx}{R(y)}$$

$$E(x/x > y) = \frac{a_1 a_2 y - y^2 r(y)}{(1 - a_1)(1 - a_2)} \tag{12}$$

$$E(x^2/x > y) = \frac{\int_y^\infty x^2 f(x) dx}{R(y)}$$

$$\int_y^\infty x^2 f(x) dx = \int_y^\infty \lambda_1 a_1 x^2 x^{-a_1 - 1} dx + \int_y^\infty \lambda_2 a_2 x^2 x^{-a_2 - 1} dx$$

$$= \frac{\lambda_1 a_1}{2 - a_1} y^{2 - a_1} + \frac{\lambda_2 a_2}{2 - a_2} y^{2 - a_2}$$

Subsitituting for λ_1 and λ_2 from equations (10) and (11) respectively we have

$$\int\limits_{y}^{\infty} x^2\, f(x)dx = \frac{2y^3 f(x) - a_1 a_2 y^2 R(y)}{(2 - a_1)(2 - a_2)}$$

and hence

$$E(x^2/x > y) = \frac{a_1 a_2 y^2 - 2y^3 r(y)}{(2 - a_1)(2 - a_2)} \qquad (13)$$

Theorem (3.2):

$$E(x^m/x > y) = \frac{a_1 a_2 y^m - my^{m+1} r(y)}{(a_1 - m)(a_2 - m)} \qquad (14)$$

Proof:

$$E(x^m/x > y) = \frac{\int\limits_{y}^{\infty} x^m f(x)dx}{R(y)}$$

$$\int\limits_{y}^{\infty} x^m f(x)dx = \int\limits_{y}^{\infty} \lambda_1\, a_1 x^m\, x^{-a_1-1}dx + \int\limits_{y}^{\infty} \lambda_2\, a_2 x^m x^{-a_2-1}dx$$

$$= \frac{\lambda_1\, a_1}{m - a_1} y^{m-a_1} + \frac{\lambda_2\, a_2}{m - a_2} y^{m-a_2}$$

Subsitituting for λ_1 and λ_2 from equations (10) and (11) respectively we have

$$E(x^m/x > y) = \frac{a_1 a_2 y^m - my^{m+1} r(y)}{(a_1 - m)(a_2 - m)}$$

from (8) if $\lambda_1 = 1$, $\lambda_2 = 0$, $a_1 = a$ and $a_2 = 0$, then

$$\int\limits_{y}^{\infty} x f(x)dx = \frac{a}{1 - a} y^{-a+1}$$

And

$$E(x/x > y) \ = \ \frac{a}{1-a}y$$

which is the result obtained by Talwalker (1977) for characterizing Pareto distribution. From equations (12) and (13) the recurrence relation of moments can be obtained as

$$\frac{1}{y}(2-a_1)(2-a_2)E(x^2/x > y) - (1-a_1)(1-a_2)E(x/x > y) \ = \ -y^2 r(y)$$

from (14)we have

$$E(x^{m+1}/x > y) \ = \ \frac{a_1 a_2 y^{m+2} - (m+1)y^{m+2}r(y)}{(m-a_1)(m-a_2)} \tag{15}$$

From (14) and (15) we have the following recurrence equation

$$\frac{1}{y}(m-a_1)(m-a_2)E(x^{m+1}/x > y) - (m-a_1)(m-a_2)E(x^m/x > y) = -y^{m+1}r(y)$$

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