International Journal of Statistical Sciences *ISSN 1683–5603*
Vol. 4, 2005, pp 1-12

Estimation of Parameters of A Two-Parameter Exponential Distribution and Its Characterization by $K - th$ Record Values

Jagdish Saran and Aparna Pandey
Department of Statistics
University of Delhi
Delhi-110007 (India)

[Received December 3, 2003; Revised June 20, 2005; Accepted August 16, 2005]

Abstract

In this paper best linear unbiased and maximum likelihood estimates of the parameters of a two-parameter exponential distribution based on $k-th$ record values have been derived.

Keywords and Phrases: Order statistics, record values, $k - th$ upper record values, single moments, product moments, recurrence relations, two-parameter exponential distribution, best linear unbiased estimates, maximum likelihood estimates, characterization.

AMS Classification: 62G30, 78M05, 33B$_{xx}$, 62F$_{xx}$.

1 Introduction

A random variable X is said to have a two- parameter exponential distribution (Exp(μ, σ)) if its probability density function (pdf) is of the form

$$f(x) = \frac{1}{\sigma} e^{-\frac{(X-\mu)}{\sigma}} \quad , \quad \mu \leq x < \infty, \quad \sigma > 0, \tag{1}$$

and the cumulative distribution function is of the form

$$F(x) = 1 - e^{-\frac{(X-\mu)}{\sigma}} \quad , \quad \mu \leq x < \infty, \quad \sigma > 0. \tag{2}$$

Note that for two- parameter exponential distribution we have

$$f(x) = \frac{1}{\sigma}(1 - F(x)) \qquad , \quad \mu \leq x < \infty, \quad \sigma > 0. \tag{3}$$

The one-parameter exponential distribution $Exp(\sigma)$ can be obtained from the two-parameter exponential distribution given in (1) by setting $\mu = 0$.

Record values are found in many situations of daily life as well as in many statistical applications. Often we are interested in observing new records and in recording them: e.g. Olympic records or world records in sports. Motivated by extreme weather conditions, Chandler (1952) introduced record values and record value times. Feller (1966) gave some examples of record values with respect to gambling problems.Theory of record values and its distributional properties have been extensively studied in the literature. For more details, see Nevzorov (1987), Arnold, Balakrishnan and Nagaraja (1998), Ahsanullah (1988,1995) and Kamps (1995).

We shall now consider the situations in which the record values (e.g. successive largest insurance claims in non-life insurance, highest water-levels or highest temperatures) themselves are viewed as 'outliers' and hence the second or third largest values are of special interest. Insurance claims in some non-life insurance can be used as an example. Observing successive k-th largest values in a sequence, Dziubdziela and Kopocióński (1976) proposed the following model of k-th record values, where k is some positive integer.

Let $X_1, X_2, \cdots\cdots$ be a sequence of independent and identically distributed random variables from two-parameter exponential distribution given in (1). Let $X_{j:n}$ denote the $j-th$ order statistic of a sample $(X_1, X_2, \cdots\cdots, X_n)$. For a fixed $k \geq 1$, we define the sequence $U_1^{(k)}, U_2^{(k)}, \cdots\cdots$ of $k-th$ upper record times of $X_1, X_2, \cdots\cdots$ as follows:

$$\begin{aligned} U_1^{(k)} &= 1 \\ U_{n+1}^{(k)} &= min\Big\{ j > U_n^{(k)} : X_{j:j+k-1} > X_{U_n^{(k)}:U_n^{(k)}+k+1} \Big\}. \end{aligned}$$

The sequence $\{Y_n^{(k)}, n \geq 1\}$, where $Y_n^{(k)} = X_{U_n^{(k)}}$ is called the sequence of $k-th$ upper record values of the sequence $\{X_n, n \geq 1\}$. Let us define $Y_0^{(k)} = 0$. Note that for $k = 1$ we have $Y_n^{(1)} = X_{U_n}$, $n \geq 1$, which are record values of $\{X_n, n \geq 1\}$ [Ahsanullah (1995)].

In this paper, we shall make use of the properties of the $k-th$ upper record values to develop inferential procedures such as point estimation. We shall derive the BLUE's and the maximum likelihood estimates of the parameters of the two- parameter exponential distribution in terms of $k-th$ upper record values. Similar work based on record values ($k = 1$) has been done for other distributions like two- parameter rectangular distribution (Ahsanullah,1986) and power function distribution (Ahsanullah,1989).

At the end we give characterization of the considered distribution using the recurrence relation for the single moments of $k - th$ upper record values.

2 Auxiliary Results on Distributions of $K - th$ Record Values

The probability density function of $k - th$ upper record values $Y_n^{(k)},\ n = 1, 2, \cdots\cdots$, as given by Dziubdziela and Kopocińnski (1976) is as follows :

$$f_{Y_n^{(k)}}(x) = \frac{k^n}{(n-1)!}[H(x)]^{n-1}[(1 - F(x))]^{k-1} f(x), \tag{4}$$

where $H(x) = -ln[1 - F(x)]$, ln is the natural logarithm.

The joint density function of $Y_m^{(k)}$ and $Y_n^{(k)}$, $(1 \leq m < n), n = 2, 3, \cdots\cdots$, as discussed by Grudzieón (1982) is given by :

$$f_{Y_m^{(k)}, Y_n^{(k)}}(x, y) = \frac{k^n}{(m-1)!(n-m-1)!}[H(y) - H(x)]^{n-m-1}[H(x)]^{m-1} \\ h(x)[(1 - F(y))]^{k-1} f(y), \quad x < y, \tag{5}$$

where $h(x) = H'(x)$.

Also the joint density of the first n $k - th$ record values is given by :

$$f_{Y_1^{(k)}, Y_2^{(k)}, \cdots\cdots, Y_n^{(k)}}(y_1, y_2, \cdots\cdots y_n) = k^n \prod_{i=1}^{n-1} \frac{f(y_i)}{1 - F(y_i)} (1 - F(y_n))^{k-1} f(y_n), \\ y_1 < y_2 < \cdots < y_n. \tag{6}$$

(cf. Kamps (1995)).

In order to derive BLUE's for the parameters of two-parameter exponential distribution, we need some recurrence relations for single and product moments of $k - th$ upper record values which have been established in the next section.

3 Recurrence Relations for Single and Product Moments

Theorem 3.1: Fix a positive integer $k \geq 1$. For $n \geq 1$ and $r = 0, 1, 2, \cdots\cdots$,

$$E\Big(Y_n^{(k)}\Big)^{(r+1)} = \frac{(r+1)\sigma}{k} E\Big(Y_n^{(k)}\Big)^r + E\Big(Y_{n-1}^{(k)}\Big)^{(r+1)}. \tag{7}$$

Proof: For $n \geq 1$ and $r = 0, 1, 2, \cdots\cdots$, on using (3) and (4), we get

$$E\Big(Y_n^{(k)}\Big)^r = \frac{k^n}{(n-1)!\sigma} \int_\mu^\infty x^r \ [H(x)]^{n-1} [1 - F(x)]^k \ dx. \tag{8}$$

Integrating (8) by parts, treating x^r as the part to be integrated and the rest of the integrand for differentiation, we get

$$E\Big(Y_n^{(k)}\Big)^r = \frac{k}{(r+1)\sigma} \Big[E(Y_n^{(k)})^{r+1} - E(Y_{n-1}^{(k)})^{(r+1)}\Big].$$

The relation in (7) can be obtained on rewriting the above expression.

Remark 3.1: Setting $\sigma = 1$ in (7), we deduce the recurrence relation for single moments of $k - th$ upper record values from standard exponential distribution Exp(1), which has been obtained by Pawlas and Szynal (1998).

Remark 3.2: Putting $k = 1$ and $\sigma = 1$ in (7), we shall deduce the recurrence relation for single moments of upper record values from standard exponential distribution Exp(1), established in Arnold, Balakrishnan and Nagaraja (1998 , pp.52) .

Theorem 3.2: For $1 \leq m \leq n-2$, $r, s = 0, 1, 2, \cdots\cdots$,

$$E\Big[\Big(Y_m^{(k)}\Big)^r \Big(Y_n^{(k)}\Big)^{s+1}\Big] = \frac{(s+1)\sigma}{k} E\Big[\Big(Y_m^{(k)}\Big)^r \Big(Y_n^{(k)}\Big)^s\Big] + E\Big[\Big(Y_m^{(k)}\Big)^r \Big(Y_{n-1}^{(k)}\Big)^{s+1}\Big], \tag{9}$$

and for $m \geq 1$, $r, s = 0, 1, 2, \cdots\cdots$,

$$E\Big[\Big(Y_m^{(k)}\Big)^r \Big(Y_{m+1}^{(k)}\Big)^{s+1}\Big] = \frac{(s+1)\sigma}{k} E\Big[\Big(Y_m^{(k)}\Big)^r \Big(Y_{m+1}^{(k)}\Big)^s\Big] + E\Big[\Big(Y_m^{(k)}\Big)\Big]^{r+s+1}. \tag{10}$$

Proof: For $1 \leq m \leq n-1$ and $r, s = 0, 1, 2, \cdots\cdots$, on using (5), we obtain

$$E\Big[\Big(Y_m^{(k)}\Big)^r \Big(Y_n^{(k)}\Big)^s\Big] = \frac{k^n}{(m-1)!(n-m-1)!} \int_\mu^\infty x^r [H(x)]^{m-1} \ h(x) I(x) dx, \tag{11}$$

where

$$I(x) = \int_x^\infty y^s \; [H(y) - H(x)]^{n-m-1} \; [1 - F(y)]^{k-1} \; f(y)dy.$$

Solving the integral in $I(x)$ by parts, treating y^s for integration and the rest of the integrand for differentiation, we get, on using (3)

$$\begin{aligned} I(x) &= -\frac{(n-m-1)}{(s+1)\sigma} \int_x^\infty y^{(s+1)} \; [H(y) - H(x)]^{n-m-2} \; [1 - F(y)]^k \; h(y)dy \\ &+ \frac{k}{(s+1)\sigma} \int_x^\infty y^{(s+1)} \; [H(y) - H(x)]^{n-m-1} \; [1 - F(y)]^{k-1} \; f(y)dy. \end{aligned}$$

Substituting the above expression for $I(x)$ in (11) and simplifying further, we obtain

$$\frac{(s+1)\sigma}{k} E\Big[\Big(Y_m^{(k)}\Big)^r \Big(Y_n^{(k)}\Big)^s\Big] = E\Big[\Big(Y_m^{(k)}\Big)^r \Big(Y_n^{(k)}\Big)^{s+1}\Big] - E\Big[\Big(Y_m^{(k)}\Big)^r \Big(Y_{n-1}^{(k)}\Big)^s\Big],$$

which on rearrangement of terms leads to (9). Proceeding in a similar manner one can easily obtain the relation (10) for the case $n = m + 1$.

Remark 3.3: Setting $\sigma = 1$ in (9) and (10), we deduce the recurrence relations for product moments of $k-th$ upper record values from standard exponential distribution Exp(1), which have been obtained by Pawlas and Szynal (1998).

Remark 3.4: Putting $k = 1$ and $\sigma = 1$ in (9) and (10), we shall deduce the recurrence relations for product moments of upper record values from standard exponential distribution Exp(1), established in Arnold, Balakrishnan and Nagaraja (1998 , pp.53).

4 Best Linear Unbiased Estimates of The Parameters μ and σ

On using (1), (2) and (4), we have

$$E\Big(Y_n^{(k)}\Big) = \frac{k^n}{(n-1)!\sigma} \int_\mu^\infty x \; \left(\frac{x-\mu}{\sigma}\right)^{n-1} \; e^{-\frac{k(x-\mu)}{\sigma}} \; dx,$$

which on simplification gives

$$E\Big(Y_n^{(k)}\Big) = \mu + \frac{n\sigma}{k}. \tag{12}$$

Similarly, it can easily be shown that

$$E\left(Y_n^{(k)}\right)^2 = \mu^2 + \frac{n(n+1)\sigma^2}{k^2} + \frac{2n\mu\sigma}{k}$$

and

$$Var\left(Y_n^{(k)}\right) = \frac{n\sigma^2}{k^2}. \tag{13}$$

Further, on using the recurrence relations given in (7) and (9) in the relation

$$Cov\left(Y_m^{(k)}\ ,\ Y_n^{(k)}\right) = E\left(Y_m^{(k)}\ ,\ Y_n^{(k)}\right) - E\left(Y_m^{(k)}\right)E\left(Y_n^{(k)}\right),$$

one can easily establish that

$$Cov\left(Y_m^{(k)}\ ,\ Y_n^{(k)}\right) = Cov\left(Y_m^{(k)}\ ,\ Y_{n-1}^{(k)}\right), \qquad n > m.$$

Proceeding recursively in the similar manner, we get

$$Cov\left(Y_m^{(k)}\ ,\ Y_n^{(k)}\right) = Var\left(Y_m^{(k)}\right), \qquad n > m. \tag{14}$$

For the estimation of the parameters μ and σ, let us consider the following transformation

$$\begin{array}{rcl} V_1^{(k)} & = & Y_1^{(k)} \\ V_2^{(k)} & = & Y_2^{(k)} - Y_1^{(k)} \\ V_3^{(k)} & = & Y_3^{(k)} - Y_2^{(k)} \\ \vdots & & \vdots \qquad \vdots \\ V_n^{(k)} & = & Y_n^{(k)} - Y_{n-1}^{(k)} \end{array}$$

Then on using (12), we obtain

$$E\left(V_1^{(k)}\right) = E\left(Y_1^{(k)}\right) = \mu + \frac{\sigma}{k} \tag{15}$$

and

$$E\Big(V_i^{(k)}\Big) = E\Big(Y_i^{(k)} - Y_{i-1}^{(k)}\Big) = \frac{\sigma}{k}, \qquad i = 2, 3, \cdots n. \tag{16}$$

Further on using (13), we get

$$Var\Big(V_i^{(k)}\Big) = \frac{\sigma^2}{k^2}, \qquad i = 2, 3, \cdots n. \tag{17}$$

$$Also \;\; Cov\Big(V_i^{(k)}\;,\; V_j^{(k)}\Big) = 0, \quad for \;\; i \neq j, \quad 1 \leq i < j \leq n. \tag{18}$$

Let $V' = \Big(V_1^{(k)}, V_2^{(k)}, \cdots\cdots, V_n^{(k)}\Big)$. Then

$$E(V) = A\theta, \tag{19}$$

where

$$\begin{pmatrix} 1 & \frac{1}{k} \\ 0 & \frac{1}{k} \\ 0 & \frac{1}{k} \\ \vdots & \vdots \\ 0 & \frac{1}{k} \end{pmatrix}, \qquad \theta = \begin{pmatrix} \mu \\ \sigma \end{pmatrix}.$$

The best linear unbiased estimates $\hat{\mu}$, $\hat{\sigma}$ of μ and σ, respectively, based on $Y_1^{(k)}, Y_2^{(k)}, \cdots, Y_n^{(k)}$ are given by

$$\hat{\theta} = \begin{bmatrix} \hat{\mu} \\ \hat{\sigma} \end{bmatrix} = (A'A)^{-1}\; A'V. \tag{20}$$

We have

$$(A'A)^{-1} = \frac{k^2}{(n-1)} \begin{pmatrix} \frac{n}{k^2} & -\frac{1}{k} \\ -\frac{1}{k} & 1 \end{pmatrix}.$$

Substituting for $(A'A)^{-1}$ in (20) and simplifying the resulting expression, we obtain

$$\hat{\theta} = \frac{k^2}{(n-1)} \begin{pmatrix} \frac{n-1}{k^2} & -\frac{1}{k^2} & \cdots & -\frac{1}{k^2} \\ 0 & \frac{1}{k} & \cdots & \frac{1}{k} \end{pmatrix} \begin{pmatrix} V_1^{(k)} \\ V_2^{(k)} \\ \vdots \\ V_n^{(k)} \end{pmatrix},$$

which on further simplification gives

$$\hat{\mu} = V_1^{(k)} - \frac{\hat{\sigma}}{k}$$

and

$$\hat{\sigma} = \frac{k^2}{(n-1)} \Big[\frac{1}{k} \Big(V_2^{(k)} + V_3^{(k)} + \cdots\cdots + V_n^{(k)} \Big) \Big].$$

Hence

$$Var(\hat{\mu}) = \frac{n}{n-1} \frac{\sigma^2}{k^2}, \tag{21}$$

$$Var(\hat{\sigma}) = \frac{\sigma^2}{n-1} \tag{22}$$

and

$$Cov(\hat{\mu}, \hat{\sigma}) = -\frac{\sigma^2}{(n-1)k}. \tag{23}$$

Remark 4.1: Putting $k = 1$ in (21), (22) and (23), we shall deduce the variances and covariance of BLUE's of μ and σ in terms of upper record values from two-parameter exponential distribution (Exp (μ, σ)), established in Arnold, Balakrishnan and Nagaraja (1998 , pp. 1).

The generalized variance $\hat{\Sigma}$ of $\hat{\mu}$ and $\sigma\Big(\hat{\Sigma} = Var(\hat{\mu})Var(\hat{\sigma}) - (Cov(\hat{\mu}, \hat{\sigma}))^2\Big)$ is $\frac{\sigma^4}{k^2(n-1)}$. On considering the two $k-th$ record values $Y_s^{(k)}$ and $Y_r^{(k)}$ $(s > r)$ it follows from (12) and (13) that the best linear unbiased estimates of μ and σ based on these two $k-th$ record values are as follows:

$$\mu^* = Y_r^{(k)} - \frac{r\sigma^*}{k},$$

$$\sigma^* = k\frac{\Big(Y_s^{(k)} - Y_r^{(k)}\Big)}{(s-r)}.$$

The variances and covariances of μ^* and σ^* are

$$Var(\mu^*) = \frac{rs\sigma^2}{k^2(s-r)},$$

$$Var(\sigma^*) = \frac{\sigma^2}{(s-r)}$$

and

$$Cov(\mu^*, \sigma^*) = -\frac{r\sigma^2}{k(s-r)}.$$

It can be shown that the generalized variance Σ^* of μ^* and $\sigma^* \Big\{ \Sigma^* = Var(\mu^*)Var(\sigma^*) - (Cov(\mu^*, \sigma^*))^2 \Big\}$ is minimum when $s = n$ and $r = 1$. Hence the best linear unbiased estimates of μ and σ based on two selected $k-th$ record values are

$$\tilde{\mu}^* = Y_1^{(k)} - \frac{\tilde{\sigma}^*}{k},$$

$$\tilde{\sigma}^* = \frac{k}{(n-1)} \left(Y_n^{(k)} - Y_1^{(k)} \right).$$

Also

$$Var(\tilde{\mu}^*) = \frac{n\sigma^2}{(n-1)},$$

$$Var(\tilde{\sigma}^*) = \frac{\sigma^2}{(n-1)}$$

and

$$Cov(\tilde{\mu}^*, \tilde{\sigma}^*) = -\frac{\sigma^2}{k(n-1)}.$$

Let $e_1 = \frac{Var(\hat{\mu})}{Var(\tilde{\mu}^*)}$, $e_2 = \frac{Var(\hat{\sigma})}{Var(\tilde{\sigma}^*)}$ and $e_{12} = \frac{Cov(\hat{\mu}, \hat{\sigma})}{Cov(\tilde{\mu}^*, \tilde{\sigma}^*)}$.

The generalized variance $\tilde{\Sigma}^*$ of $\tilde{\mu}^*$ and $\tilde{\sigma}^*$ is $\frac{\sigma^4}{k^2(n-1)}$. Further, it can be seen that $e_1 = e_2 = e_{12} = 1$. Thus the efficiencies of the best linear unbiased estimates of μ and σ based on two $k-th$ record values are same compared to the corresponding estimates based on a complete set of $k-th$ record values.

5 Maximum Likelihood Estimates of μ and σ

Let us denote the MLE's of the parameters μ and σ of Exp(μ, σ) distribution by $\tilde{\mu}$ and $\tilde{\sigma}$, respectively. Upon using (6) we have the log of the likelihood function in this case to be

$$logL = -nlog\sigma + nlogk - k\frac{(y_n - \mu)}{\sigma}, \qquad \mu \leq y_1 < y_2 < \cdots < y_n < \infty.$$

Then the MLE's of μ and σ can be obtained as

$$\tilde{\mu} = y_1 \text{ and } \tilde{\sigma} = \frac{k}{n}(y_n - y_1).$$

Further

$$E(\tilde{\mu}) = \mu + \frac{\sigma}{k},$$

and $$E(\tilde{\sigma}) = \frac{k(n-1)\sigma}{n},$$

Also

$$Var(\tilde{\mu}) = \frac{\sigma^2}{k^2},$$

$$Var(\tilde{\sigma}) = \frac{(n-1)k^2\sigma^2}{n}$$

and

$$Cov(\tilde{\mu}, \tilde{\sigma}) = 0.$$

The unbiased estimators for μ and σ are as follows:

$$\tilde{\tilde{\mu}} = \tilde{\mu} - \frac{n\tilde{\sigma}}{(n-1)k^2}$$

and

$$\tilde{\tilde{\sigma}} = \frac{n}{k(n-1)}\tilde{\sigma}.$$

6 Characterization

This section contains characterization of the two-parameter exponential distribution. We shall use the following result of Lin (1986).

Theorem 6.1 (Lin): Let n_o be any fixed non-negative integer, $-\infty \leq a < b \leq \infty$, and $g(x) \geq 0$ be an absolutely continuous function with $g'(x) \neq 0$ a.e. on (a, b). Then the sequence of functions $\{(g(x))^n \; e^{-g(x)}, \; n \geq n_o\}$ is complete in $L(a, b)$ iff $g(x)$ is strictly monotone on (a, b).

Theorem 6.2: A necessary and sufficient condition for a random variable X to be distributed with pdf given by (1) is that

$$E\Big(Y_n^{(k)}\Big)^{r+1} = \frac{(r+1)\sigma}{k}E\Big(Y_n^{(k)}\Big)^{r} + E\Big(Y_{n-1}^{(k)}\Big)^{r+1} \tag{24}$$

$n = 1, 2, \cdots\cdots$ and $r = 0, 1, 2, \cdots\cdots$, where $k \geq 1$ is any fixed positive integer.

Proof: The necessary part follows from (7). On the other hand if the recurrence relation given in (24) is satisfied, then

$$\frac{k^{n+1}}{(n-1)!}\int_{\mu}^{\infty} x^{r+1}[H(x)]^{n-1}[1-F(x)]^{k-1}f(x)dx = \frac{(r+1)\sigma k^n}{(n-1)!}\int_{\mu}^{\infty} x^{r}[H(x)]^{n-1} [1-F(x)]^{k-1}f(x)dx + \frac{k^n}{(n-2)!}\int_{\mu}^{\infty} x^{r+1}[H(x)]^{n-2}[1-F(x)]^{k-1}f(x)dx.$$

Integrating the last integral on the right hand side of the above equation by parts, we get

$$\frac{k^{n+1}}{(n-1)!}\int_{\mu}^{\infty} x^{r+1}[H(x)]^{n-1}[1-F(x)]^{k-1}f(x)dx = \frac{(r+1)\sigma k^n}{(n-1)!}\int_{\mu}^{\infty} x^{r}[H(x)]^{n-1} [1-F(x)]^{k-1}f(x)dx - \frac{(r+1)k^n}{(n-1)!}\int_{\mu}^{\infty} x^{r}[H(x)]^{n-1}[1-F(x)]^{k}f(x)dx + \frac{k^{n+1}}{(n-1)!}\int_{\mu}^{\infty} x^{r+1}[H(x)]^{n-1}[1-F(x)]^{k-1}f(x)dx,$$

which on simplification reduces to

$$\int_{\mu}^{\infty} x^{r}[H(x)]^{n-1}[1-F(x)]^{k-1}\Big\{k\ x\ f(x) - (r+1)\sigma f(x) + (r+1)(1-F(x)) - k\ x\ f(x)\Big\}dx = 0.$$

It now follows from Theorem 6.1, with $g(x) = -\ ln\ [1-F(x)] = H(x)$, that

$$f(x) = \frac{[1-F(x)]}{\sigma},$$

which proves that $f(x)$ has the form (1).

Acknowledgements: The authors wish to thank the referee for giving valuable comments that led to an improvementin the presentation of the paper.

References

[1] Ahsanullah, M. (1986). Estimation of the parameters of a rectangular distribution by record values. Comp. Stat. Quart. 2, 119-125.

[2] Ahsanullah, M. (1989). Intoduction to record statistics. Ginn Press, Needham Heights, MA, U.S.A.

[3] Ahsanullah, M. (1989). Estimation of the parameters of a power function distribution by record values. Pak. J. Statist. 5(2)A, 189-194.

[4] Ahsanullah, M.(1995). Record Statistics. Nova Science Publishers, Inc.Commack, NewYork.

[5] Arnold, B.C. and Balakrishnan, N. (1989). Relations, Bounds and Approximations for Order Statistics, Lecture Notes is Statistics, 53, Springer-Verlag, New York.

[6] Arnold, B.C., Balakrishnan, N. and Nagaraja, H.N. (1992). A First Course in Order Statistics, John Wiley and Sons, New York.

[7] Arnold, B.C., Balakrishnan, N. and Nagaraja, H.N. (1998). Records. John Wiley and Sons, New York.

[8] Chandler, K.M. (1952). The distribution and frequency of record values. J.Roy. Statist. Soc., Ser. B, 14, 220-228.

[9] Dziubdziela, W. and Kopoci*ó*nski, B. (1976). Limiting properties of the k-th record value. Appl. Math., 15, 187-190.

[10] Feller, W. (1966). An Introduction to Probability Theory and Its Applications,Vol. 2. John Wiley and Sons, New York.

[11] Grudzie*ó*n, Z. (1982). Charakteryzacja rokladw w terminach statystyk rekordowych oraz rozklady i momenty statystyk porzadkowych i rekordowych z prb o losowej liczebnosci. Praca doktorska, UMCS Lublin.

[12] Kamps,U. (1995). A concept of generalized order statistics. J. Statist. Plan. Infer., 48, 1-23.

[13] Lin, G.D. (1986). On a moment problem. Tohoku Math. Journal, 38, 595-598.

[14] Nevzorov, V.B. (1987). Records. Theo. Prob. Appl. , 32 (2), 201-228 (English Tanslation).

[15] Pawlas, P.and Szynal, D. (1998). Relations for single and product moments of k-th record values from exponential and Gumbel distributions, J. Appl. Statist. Sc., 7 (1) , 53-62.