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Truncated Moments of the Reciprocal of a Noncentral F Variable

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Abstract

The bias and risk of the positive part Stein type estimators involve the evaluation of truncated moments of the reciprocal of noncentral F-variables (Saleh 2004). Bock et al (1984) obtained simple form for the inverse moments of non-central χ^2 and F random variables, which depend on the Dawson's integral (Abramowitz and Stegun 1964). However, they have not indicated about the truncated moments for the reciprocal of noncentral F variables. In this note, we find the truncated moments of the reciprocal of a noncentral F variable in terms of incomplete beta function.

Keywords and Phrases: Truncated moment; Non-central F; Incomplete Beta.

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Inverse moments of a truncated noncentral F random variable

Suppose

$$Z_1 \sim \chi^2_{n_1,\Delta} \quad \text{and} \quad Z_2 \sim \chi^2_{n_2},$$

where Z_1 and Z_2 are mutually independent. Then

$$X = \frac{Z_1/n_1}{Z_2/n_2}$$

is distributed as noncentral F-distribution with n_1, n_2 degrees of freedom and noncentrality parameter Δ. The *pdf* of X can be written as a function of central F and a power series

$$f(x;\Delta) = f(x)e^{-\Delta/2}\sum_{j=0}^{\infty}\left(\frac{\frac{1}{2}\Delta n_1 x}{n_2+n_1x}\right)^j h(n_j), \tag{1}$$

where $f(x)$ is the central F distribution with (n_1, n_2) degrees of freedom and

$$h(n_j) = \frac{(n_1+n_2)(n_1+n_2+2)....(n_1+n_2+2.\overline{j-1}))}{j!n_1(n_1+2)...(n_1+2.\overline{j-1})}.$$

The *cdf* of noncentral F, which can also be expressed as incomplete noncentral beta function ratio as

$$\begin{aligned} P(x_{n_1,n_2}(\Delta) \le f) &= I_{\left(\frac{n_1 f}{n_2+n_1 f}\right)}\left(\frac{n_1}{2}, \frac{n_2}{2}; \Delta\right) \\ &= \sum_{j=0}^{\infty}\left(\frac{(\frac{1}{2}\Delta)^j e^{-\Delta/2}}{j!}\right) I_{\left(\frac{n_1 f}{n_2+n_1 f}\right)}\left(\frac{n_1}{2}+j, \frac{n_2}{2}\right), \end{aligned} \tag{2}$$

where $I_p(\alpha, \beta)$ is the incomplete beta function ratio given by

$$I_p(\alpha,\beta) = \frac{1}{B(\alpha,\beta)}\int_0^p t^{\alpha-1}(1-t)^{\beta-1}dt.$$

Our interest is to find the truncated r^{th} moment of the reciprocal of a noncentral F variable and this will be done by induction method.

Case 1: For, $r = 1$, to find $E(x^{-1}I_{x\le d_1})$, note that

$$\begin{aligned} E(x^{-1}I_{x\le d_1}) &= \int_0^{d_1} x^{-1}\frac{(\frac{n_1}{n_2})^{n_1/2}}{B(\frac{n_1}{2},\frac{n_2}{2})}\frac{x^{\frac{n_1}{2}-1}}{(1+\frac{n_1}{n_2}x)^{\frac{n_1+n_2}{2}}}e^{-\Delta/2}\sum_{j=0}^{\infty}\left(\frac{\frac{1}{2}\Delta n_1x}{n_2+n_1x}\right)^j h(n_j)dx \\ &= \int_0^{d_1}\frac{(\frac{n_1}{n_2})^{n_1/2}}{B(\frac{n_1}{2},\frac{n_2}{2})}\frac{x^{\frac{n_1-2}{2}-1}}{(1+\frac{n_1}{n_2}x)^{\frac{n_1+n_2}{2}}}e^{-\Delta/2}\sum_{j=0}^{\infty}\left(\frac{\frac{1}{2}\Delta n_1x}{n_2+n_1x}\right)^j h(n_j)dx. \end{aligned}$$

Let us make the following transformation

$$\frac{n_1}{n_2}x = \frac{n_1-2}{n_2+2}y \Rightarrow x = \frac{n_2(n_1-2)}{n_1(n_2+2)}y \Rightarrow dx = \frac{n_2(n_1-2)}{n_1(n_2+2)}dy$$

If $x = 0$, then $y = 0$, if $x = d_1$, $y = d_1^* = \frac{n_1(n_2+2)}{n_2(n_1-2)}$. We note after simple calculation that,

$$\frac{\frac{1}{2}\Delta n_1 x}{n_2+n_1x} = \frac{\frac{1}{2}\Delta(n_1-2)y}{(n_2+2)+(n_1-2)y}.$$

We have,

$$\begin{aligned}
E(x^{-1}I_{x\leq d_1}) &= \int_0^{d_1^*} \frac{(\frac{n_1}{n_2})^{n_1/2}}{B(\frac{n_1}{2},\frac{n_2}{2})} \frac{\left(\frac{n_2(n_1-2)}{n_1(n_2+2)}y\right)^{\frac{n_1-2}{2}-1}}{(1+\frac{n_1-2}{n_2+2}y)^{\frac{n_1+n_2}{2}}} \frac{n_2(n_1-2)}{n_1(n_2+2)} \\
&\times \; e^{-\Delta/2}\sum_{j=0}^{\infty}\left(\frac{\frac{1}{2}\Delta(n_1-2)y}{(n_2+2)+(n_1-2)y}\right)^j h(n_j)dy \\
&= \int_0^{d_1^*} \frac{(\frac{n_1}{n_2})\left(\frac{(n_1-2)}{(n_2+2)}\right)^{\frac{n_1-2}{2}}}{B(\frac{n_1}{2},\frac{n_2}{2})} \frac{(y)^{\frac{n_1-2}{2}-1}}{(1+\frac{n_1-2}{n_2+2}y)^{\frac{n_1+n_2}{2}}} e^{-\Delta/2} \\
&\times \; \sum_{j=0}^{\infty}\left(\frac{\frac{1}{2}\Delta(n_1-2)y}{(n_2+2)+(n_1-2)y}\right)^j h(n_j)dy.
\end{aligned} \tag{3}$$

Since

$$\frac{(\frac{n_1}{n_2})}{B(\frac{n_1}{2},\frac{n_2}{2})} = \frac{(\frac{n_1}{n_2-2})}{B(\frac{n_1-2}{2},\frac{n_2+2}{2})},$$

the equation (3) can be written as

$$\begin{aligned}
E(x^{-1}I_{x\leq d_1}) &= \int_0^{d_1^*} \frac{(\frac{n_1}{n_1-2})\left(\frac{(n_1-2)}{(n_2+2)}\right)^{\frac{n_1-2}{2}}}{B(\frac{n_1-2}{2},\frac{n_2+2}{2})} \frac{(y)^{\frac{n_1-2}{2}-1}}{(1+\frac{n_1-2}{n_2+2}y)^{\frac{n_1+n_2}{2}}} \\
&\quad \times e^{-\Delta/2}\sum_{j=0}^{\infty}\left(\frac{\frac{1}{2}\Delta(n_1-2)y}{(n_2+2)+(n_1-2)y}\right)^j h(n_j)dy \\
&= n_1\int_0^{d_1^*} \frac{\left(\frac{(n_1-2)}{(n_2+2)}\right)^{\frac{n_1-2}{2}}}{B(\frac{n_1-2}{2},\frac{n_2+2}{2})} \frac{(y)^{\frac{n_1-2}{2}-1}}{(1+\frac{n_1-2}{n_2+2}y)^{\frac{n_1+n_2}{2}}} e^{-\Delta/2} \\
&\quad \times \sum_{j=0}^{\infty}\left(\frac{\frac{1}{2}\Delta(n_1-2)y}{(n_2+2)+(n_1-2)y}\right)^j h_1(n_j)dy \\
&= n_1\int_0^{d_1^*} f(y,n_1-2,n_2+2)e^{-\Delta/2} \\
&\quad \times \sum_{j=0}^{\infty}\left(\frac{\frac{1}{2}\Delta(n_1-2)y}{(n_2+2)+(n_1-2)y}\right)^j h_1(n_j)dy \\
&= n_1 P(Y_{n_1-2,n_2+2}(\Delta)\leq d_1^*),
\end{aligned} \tag{4}$$

where $f(y, n_1-2, n_2+2)$ is the *pdf* of central F distribution with (n_1-2, n_2+2) degrees of freedom, $Y_{n_1-2,n_2+2}(\Delta)$ has a noncentral F distribution with (n_1-2, n_2+2) degrees of freedom and noncentrality parameter Δ, and

$$h_1(n_j) = \frac{(n_1+n_2)(n_1+n_2+2)....(n_1+n_2+2.\overline{j-1})}{j!(n_1-2)n_1(n_1+2)...(n_1-2+2.\overline{j-1})}.$$

Now comparing equations (2) and (4), we have

$$E(x^{-1}I_{x\le d_1}) = n_1 \sum_{j=0}^{\infty} \frac{(\frac{\Delta}{2})^j e^{-\Delta/2}}{j!} I_{\left(\frac{(n_1-2)d_1^*}{n_2+2+(n_1-1)d_1^*}\right)} \left(\frac{n_1-2}{2}+j, \frac{n_2+2}{2}\right).$$

Since

$$\frac{(n_1-2)d_1^*}{n_2+2+(n_1-1)d_1^*} = \frac{n_1 d_1}{n_2+n_1 d_1},$$

we get

$$\begin{aligned} E(x^{-1}I_{x\le d_1}) &= n_1 \sum_{j=o}^{\infty} \frac{(\frac{\Delta}{2})^j e^{-\Delta/2}}{j!} I_{\left(\frac{n_1 d_1}{n_2+n_1 d_1}\right)} \left(\frac{n_1-2}{2}+j, \frac{n_2+2}{2}\right) \\ &= n_1 \times I_{\left(\frac{n_1 d_1}{n_2+n_1 d_1}\right)} \left(\frac{n_1-2}{2}, \frac{n_2+2}{2}; \Delta\right). \qquad (5) \end{aligned}$$

Case 2: For, $r = 2$, to find $E(x^{-2}I_{x\le d_2})$, note that

$$\begin{aligned} E(x^{-2}I_{x\le d_2}) &= \int_0^{d_2} x^{-2} \frac{(\frac{n_1}{n_2})^{n_1/2}}{B(\frac{n_1}{2},\frac{n_2}{2})} \frac{x^{\frac{n_1}{2}-1}}{(1+\frac{n_1}{n_2}x)^{\frac{n_1+n_2}{2}}} e^{-\Delta/2} \sum_{j=0}^{\infty} \left(\frac{\frac{1}{2}\Delta n_1 x}{n_2+n_1 x}\right)^j h(n_j)dx \\ &= \int_0^{d_2} \frac{(\frac{n_1}{n_2})^{n_1/2}}{B(\frac{n_1}{2},\frac{n_2}{2})} \frac{x^{\frac{n_1-4}{2}-1}}{(1+\frac{n_1}{n_2}x)^{\frac{n_1+n_2}{2}}} e^{-\Delta/2} \sum_{j=0}^{\infty} \left(\frac{\frac{1}{2}\Delta n_1 x}{n_2+n_1 x}\right)^j h(n_j)dx \end{aligned}$$

Let us make the following transformation

$$\frac{n_1}{n_2}x = \frac{n_1-4}{n_2+4}y \Rightarrow x = \frac{n_2(n_1-4)}{n_1(n_2+4)}y$$

If $x = 0$, then $y = 0$, if $x = d_2$, $y = d_2^* = \frac{n_1(n_2+4)}{n_2(n_1-4)}$. Also,

$$\frac{\frac{1}{2}\Delta n_1 x}{n_2+n_1 x} = \frac{\frac{1}{2}\Delta(n_1-4)y}{(n_2+4)+(n_1-4)y}.$$

Thus

$$E(x^{-2}I_{x\le d_2}) = \int_0^{d_2^*} \frac{(\frac{n_1}{n_2})^{n_1/2}}{B(\frac{n_1}{2},\frac{n_2}{2})} \frac{\left(\frac{n_2(n_1-4)}{n_1(n_2+4)}y\right)^{\frac{n_1-4}{2}-1}}{(1+\frac{n_1-4}{n_2+4}y)^{\frac{n_1+n_2}{2}}} \frac{n_2(n_1-4)}{n_1(n_2+4)}$$

$$\times \quad e^{-\Delta/2} \sum_{j=0}^{\infty} \left(\frac{\frac{1}{2}\Delta(n_1-4)y}{(n_2+4)+(n_1-4)y} \right)^j h(n_j)dy. \qquad (6)$$

Since

$$\frac{(\frac{n_1}{n_2})^2}{B(\frac{n_1}{2}, \frac{n_2}{2})} = \frac{\frac{n_1^2(n_2+2)}{n_2(n_1-2)(n_1-4)}}{B(\frac{n_1-4}{2}, \frac{n_2+4}{2})},$$

equation (6) can be written as

$$\begin{aligned} E(x^{-2}I_{x\le d_2}) &= \frac{n_1^2(n_2+2)}{n_2} \int_0^{d_2^*} \frac{\left(\frac{(n_1-4)}{(n_2+4)}\right)^{\frac{n_1-4}{2}}}{B(\frac{n_1-4}{2}, \frac{n_2+4}{2})} \frac{(y)^{\frac{n_1-4}{2}-1}}{(1+\frac{n_1-4}{n_2+4}y)^{\frac{n_1+n_2}{2}}} \\ &\times \quad e^{-\Delta/2} \sum_{j=0}^{\infty} \left(\frac{\frac{1}{2}\Delta(n_1-4)y}{(n_2+4)+(n_1-4)y} \right)^j h(n_j)dy \\ &= \frac{n_1^2(n_2+2)}{n_2} \int_0^{d_2^*} f(y, n_1-4, n_2+4)e^{-\Delta/2} \\ &\quad \times \sum_{j=0}^{\infty} \left(\frac{\frac{1}{2}\Delta(n_1-4)y}{(n_2+4)+(n_1-4)y} \right)^j h_2(n_j)dy \\ &= \frac{n_1^2(n_2+2)}{n_2} P(Y_{n_1-4,n_2+4}(\Delta) \le d_2^*), \end{aligned} \qquad (7)$$

where $f(y, n_1-4, n_2+4)$ is the *pdf* of central F distribution with (n_1-4, n_2+4) degrees of freedom, $Y_{n_1-4,n_2+4}(\Delta)$ has a noncentral F distribution with (n_1-4, n_2+4) degrees of freedom, and noncentrality parameter Δ and

$$h_2(n_j) = \frac{(n_1+n_2)(n_1+n_2+2)....(n_1+n_2+2(j-1))}{j!(n_1-4)(n_1-2)n_1(n_1+2)...(n_1-4+2(j-1))}.$$

Now comparing equations (2) and (7), we have

$$E(x^{-2}I_{x\le d_2}) = \frac{n_1^2(n_2+2)}{n_2} \sum_{j=0}^{\infty} \frac{(\frac{\Delta}{2})^j e^{-\Delta/2}}{j!} I_{\left(\frac{(n_1-4)d_2^*}{n_2+4+(n_1-4)d_2^*}\right)} \left(\frac{n_1-4}{2}+j, \frac{n_2+4}{2} \right).$$

Since

$$\frac{(n_1-4)d_2^*}{n_2+4+(n_1-4)d_2^*} = \frac{n_1 d_2}{n_2+n_1 d_2},$$

we have

$$E(x^{-2}I_{x\le d_2}) \quad = \quad \frac{n_1^2(n_2+2)}{n_2} \sum_{j=0}^{\infty} \frac{(\frac{\Delta}{2})^j e^{-\Delta/2}}{j!} I_{\left(\frac{n_1 d_2}{n_2+n_1 d_2}\right)} \left(\frac{n_1-4}{2}+j, \frac{n_2+4}{2} \right)$$

$$= \frac{n_1^2(n_2+2)}{n_2} \times I_{\left(\frac{n_1 d_2}{n_2+n_1 d_2}\right)} \left(\frac{n_1-4}{2}, \frac{n_2+4}{2}; \Delta\right). \tag{8}$$

Case 3: For, $r = 3$, to find $E(x^{-3} I_{x \leq d_3})$, following all steps in cases (1) and (2), we have

$$\begin{aligned} E(x^{-3} I_{x \leq d_3}) &= \frac{n_1^3(n_2+2)(n_2+4)}{n_2^2} \sum_{j=0}^{\infty} \frac{(\frac{\Delta}{2})^j e^{-\Delta/2}}{j!} I_{\left(\frac{n_1 d_3}{n_2+n_1 d_3}\right)} \left(\frac{n_1-6}{2}+j, \frac{n_2+6}{2}\right) \\ &= \frac{n_1^3(n_2+2)(n_2+4)}{n_2^2} \times I_{\left(\frac{n_1 d_3}{n_2+n_1 d_3}\right)} \left(\frac{n_1-6}{2}, \frac{n_2+6}{2}; \Delta\right). \end{aligned} \tag{9}$$

Finally by induction, the r^{th} truncated moment of the reciprocal of a noncentral F variable in terms of incomplete beta function ratio is given by

$$\begin{aligned} E(x^{-r} I_{x \leq d_r}) &= \frac{n_1^r(n_2+2)(n_2+4)\cdots(n_2+2.\overline{r-1})}{n_2^{r-1}} \\ &\quad \times \sum_{j=0}^{\infty} \frac{(\frac{\Delta}{2})^j e^{-\Delta/2}}{j!} I_{\left(\frac{n_1 d_r}{n_2+n_1 d_r}\right)} \left(\frac{n_1-2r}{2}+j, \frac{n_2+2r}{2}\right) \\ &= \left(\frac{n_1}{n_2}\right)^r \prod_{i=1}^{r} [n_2+2(i-1)] I_{\left(\frac{n_1 d_r}{n_2+n_1 d_r}\right)} \left(\frac{n_1-2r}{2}, \frac{n_2+2r}{2}; \Delta\right). \end{aligned} \tag{10}$$

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