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On the distributions of the MLE of the first-order autoregressive parameter and certain related estimators

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Abstract

An integral representation of the moments of the Yule-Walker and Burg's estimators of the parameter of an autoregressive process of order one is provided. Since the moments of the MLE cannot be explicitly determined, they are approximated by means of simulations. As it turns out, the distribution of the MLE is nearly indistinguishable from that of Burg's estimator. It is shown graphically that the MLE is always less biased than the Yule-Walker estimator, and a methodology is suggested for reducing their biases. A 'beta-polynomial' approximation of the density functions of these estimators is also being proposed.

Keywords and Phrases: AR(1) process; serial correlation; Yule-Walker estimator; maximum likelihood estimator; Burg's estimator; moments; bias; beta distribution; density approximation.

AMS Classification: Primary 62M10, 62F10; Secondary 62G07.

1 Introduction

Consider a zero-mean first-order autoregressive process, which is defined by the following difference equation:

$$X_t = \alpha X_{t-1} + \epsilon_t,\ t = 0, \pm 1, \pm 2, \ldots, \tag{1}$$

where $\{\epsilon_t\}$ is a sequence of independently and identically distributed standard normal random variables. It is assumed that $\{X_t\}$ is stationary or equivalently, that $|\alpha| < 1$.

Given a realization of length n denoted by $\mathbf{x} = (x_1, \ldots, x_n)'$, one requires a reliable estimator of the parameter α in order to test the model assumptions from the residuals.

Several estimates of α have been proposed in the literature. Commonly used are those of the form Q_1/S_{ab}, where $Q_1 = \sum_{t=2}^{n} x_t x_{t-1}$ and $S_{ab} = a x_1^2 + \sum_{t=2}^{n-1} x_t^2 + b x_n^2$. For instance, the ordinary least-squares estimate is Q_1/S_{10}, the Yule-Walker estimate, α_{YW}, is Q_1/S_{11} and Burg's estimate, α_B, is $Q_1/S_{\frac{1}{2}\frac{1}{2}}$.

The maximum likelihood estimator, α_{MLE}, can be determined by solving the following equation in the interval $(-1, 1)$:

$$\alpha_{MLE}^3 S_{00}(n-1)/n - \alpha_{MLE}^2 Q_1(n-2)/n - \alpha_{MLE}(S_{00} + S_{11}/n) + Q_1 = 0. \quad (2)$$

Hasza (1980) (see also Minozzo and Azzalini, (1990)) showed that Equation 2 has a unique root in $(-1, 1)$. The solution is given by

$$\alpha_{MLE} = \frac{(n-2)Q_1}{3(n-1)S_{00}} - 2\sqrt{\frac{p}{3}} \cos\left[\frac{\pi}{3} + \frac{1}{3}\cos^{-1}\left(\frac{q}{2}\sqrt{\frac{27}{p^3}}\right)\right] \quad (3)$$

where

$$p = \frac{(n-2)^2 Q_1^2 + 3(n-1)S_{00}(nS_{00} + S_{11})}{3(n-1)^2 S_{00}^2} \quad (4)$$

and

$$q = \frac{Q_1(2(n-2)^3 Q_1^2 - 9(n-1))S_{00}(2n^2 S_{00} + 2S_1 - n(S_{00} + S_{11}))}{27(n-1)^3 S_{00}^3}. \quad (5)$$

The first four moments of α_{YW} have been derived by Anderson (1990) and its exact distribution function has been given in closed form by Provost and Rudiuk (1995) whose technique can also be employed to obtain the distribution of α_B; both these estimators are defined for instance in Brockwell and Davis (2002). As for α_{MLE}, its distribution has not been much investigated. The distributional properties of the ordinary least-squares estimator of α, whose support is not confined to the interval $(-1, 1)$, will not be investigated any further in this paper.

A representation of the exact moments of α_B and α_{YW} is given in Section 2 where it is shown graphically that α_{MLE} is less biased than α_{YW} and that α_B and α_{MLE} are very similarly distributed. In addition, a methodology is proposed for reducing the bias of these estimators. A beta-polynomial density approximation which is based on the moments of the estimators is presented in Section 3.

2 The Moments and the Bias of the Estimators

The moments of α_{YW} can be evaluated from the identity,

$$\begin{aligned} E(Q_1^h S_{11}^{-h}) &= E(Q_1^h \frac{1}{\Gamma(h)} \int_0^\infty y^{h-1} e^{-yS_{11}} dy) \\ &= \frac{1}{\Gamma(h)} \int_0^\infty y^{h-1} E(Q_1^h e^{-yS_{11}}) dy\ , \end{aligned} \quad (6)$$

wherein $E(Q_1^h e^{-yS_{11}})$ is obtained as follows from the joint moment generating function of (Q_1, S_{11}), denoted by $\varphi_{Q_1,S_{11}}(s,t)$:

$$E(Q_1^h e^{-S_{11}y}) = \frac{\mathrm{d}^h}{\mathrm{d}s^h}\varphi_{Q_1,S_{11}}(s,-y)|_{s=0} \tag{7}$$

where $\varphi_{Q_1,S_{11}}(s,-y) = |\mathrm{I} - 2sA\Sigma + 2y\Sigma|^{-1/2}$, $\Sigma = (\sigma_{ij})$ with $\sigma_{ij} = (\alpha^{|i-j|}/(1-\alpha^2))$ is the covariance matrix associated with $\mathbf{X} = (X_1, \ldots, X_n)'$, the X_i's being specified by Equation 1, I is the identity matrix, and $A = (a_{ij})$ with $a_{ij} = 1/2$ if $|i-j| = 1$ and 0 elsewhere, A being the matrix of the quadratic form Q_1. Note that $\varphi_{Q_1,S_{11}}(s,-y) = |(\Sigma^{-1} - 2sA)/2y + \mathrm{I}|^{-1/2}|\Sigma|^{-1/2}(2\mathrm{y})^{-n/2}$ where $\Sigma^{-1} = (\sigma^{ij})$ with σ^{ij} equal to 1 if $i = j = 1$ or n, $1+\alpha^2$ if $i = j = 2, \ldots, n-1$, $-\alpha$ if $|i-j| = 1$, and 0 elsewhere, and $|\Sigma^{-1}| = 1-\alpha^2$. In this form, $\varphi_{Q_1,S_{11}}(s,-y)$ is the determinant of a tridiagonal matrix which can be easily evaluated. An alternative method for computing the moments of α_{YW} can be found in Shenton and Johnson (1965). The moments of α_B are obtained by replacing S_{11} with $S_{\frac{1}{2}\frac{1}{2}}$ in Equation 6 and in Equation 7; then $\varphi_{Q_1,S_{\frac{1}{2}\frac{1}{2}}}(s,-y) = |\mathrm{I} - 2sA\Sigma + 2yB\Sigma|^{-1/2}$ where $B = (b_{ij})$ is a diagonal matrix such that $b_{ii} = \frac{1}{2}$ when $i = 1$ or n, and $b_{ii} = 1$ for $i = 2, \ldots, n-1$. Additional results on the moments of ratios of quadratic forms are included in Chapter 3 of Mathai and Provost (1992).

The first six moments of α_{YW} evaluated from Equation 6 are tabulated for $\alpha = 0.05, 0.5$ and 0.95 in Table 1 and Table 2 for series of length 10 and 30, respectively. The corresponding moments are given in Tables 3 and 4 for Burg's estimator. It is seen that in each case, the exact moments closely agree with the simulated moments obtained on the basis of 100,000 replications. In light of the symmetry of the distributions, when $\alpha < 0$, the odd moments of α_{YW} and α_B are the negative of their counterparts for $\alpha > 0$, while the even moments remain unchanged.

Table 1: Exact and Simulated Moments of α_{YW} for $n = 10$

α	0.05		0.5		0.95	
Moments	Exact	Simulated	Exact	Simulated	Exact	Simulated
1	0.037501	0.037777	0.375817	0.375421	0.750780	0.750641
2	0.076325	0.076032	0.207814	0.207713	0.596178	0.596094
3	0.007355	0.007486	0.118014	0.117859	0.485507	0.485472
4	0.015078	0.015075	0.074808	0.074720	0.402667	0.402696
5	0.002118	0.002198	0.048497	0.048447	0.338231	0.338298
6	0.004366	0.004414	0.033238	0.033208	0.286972	0.287067

For comparison purposes, the expected values of α_{YW}, α_B and α_{MLE} were plotted in Figures 1, 2, and 3 for $\alpha = 0, 0.01, \ldots, 0.99$, and $n = 10$, 20, 30, respectively. The moments of α_{MLE} were determined by means of simulations since they cannot be

Table 2: Exact and Simulated Moments of α_{YW} for $n = 30$

α	0.05		0.5		0.95	
Moments	Exact	Simulated	Exact	Simulated	Exact	Simulated
1	0.045312	0.045736	0.453187	0.453235	0.867347	0.867005
2	0.032206	0.031995	0.230043	0.229289	0.760851	0.760092
3	0.003958	0.004667	0.124672	0.124162	0.673402	0.672267
4	0.002908	0.003024	0.071018	0.070870	0.600368	0.598925
5	0.000544	0.000757	0.042061	0.042197	0.538548	0.536865
6	0.000411	0.000472	0.025726	0.026029	0.485640	0.483778

Table 3: Exact and Simulated Moments of α_B for $n = 10$

$n = 10$	0.05		0.5		0.95	
Moments	Exact	Simulated	Exact	Simulated	Exact	Simulated
1	0.041381	0.041759	0.416128	0.415847	0.843484	0.844485
2	0.093637	0.093187	0.252564	0.252606	0.745829	0.747354
3	0.009715	0.009801	0.155619	0.155635	0.672332	0.674033
4	0.022144	0.022013	0.106763	0.106846	0.614698	0.616508
5	0.003289	0.003278	0.074329	0.074409	0.567353	0.569167
6	0.007539	0.007512	0.054641	0.054730	0.527602	0.529396

Table 4: Exact and Simulated Moments of α_B for $n = 30$

$n = 30$	0.05		0.5		0.95	
Moments	Exact	Simulated	Exact	Simulated	Exact	Simulated
1	0.046829	0.046396	0.468629	0.470251	0.901899	0.901886
2	0.034427	0.034343	0.245431	0.245744	0.820982	0.820649
3	0.004356	0.004968	0.136989	0.137099	0.752716	0.751985
4	0.003309	0.003424	0.080251	0.080455	0.694159	0.693037
5	0.000635	0.000831	0.048817	0.049169	0.643277	0.641797
6	0.000496	0.000553	0.030634	0.031089	0.598596	0.596802

expressed in closed form. The graphs clearly show that α_{MLE} is less biased than the Yule-Walker estimator and that Burg's estimator and the MLE are nearly identically biased. As expected, the longer the series, the smaller the bias. Table 5 suggests that α_B and α_{MLE} also have similar variances for any given values of α and n. As can be seen from Figure 4 and Figure 5, the distributional similarities between α_B and α_{MLE} hold even for short series.

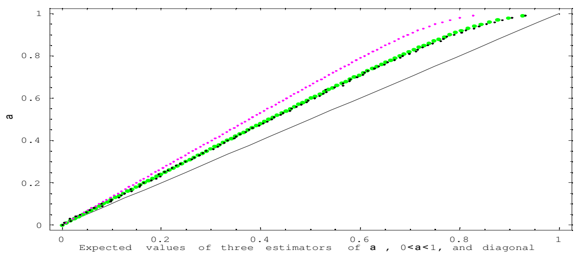

Figure 1. Graphs of α vs $E(\alpha_{YW})$ [..............], $E(\alpha_B)$ [• • • • • •] and $E(\alpha_{MLE})$ [..............], for n=10, [————] denoting the diagonal

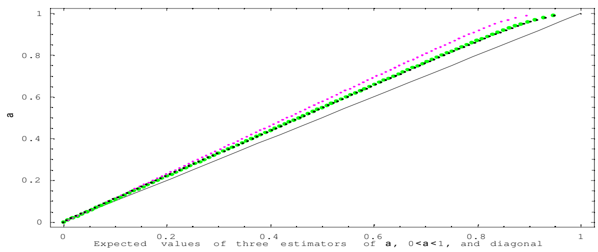

Figure 2. Graphs of α vs $E(\alpha_{YW})$ [..............], $E(\alpha_B)$ [• • • • • •] and $E(\alpha_{MLE})$ [..............], for n=20, [————] denoting the diagonal

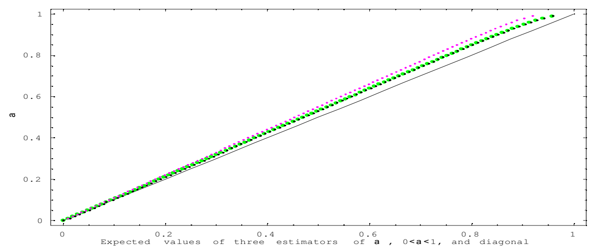

Figure 3. Graphs of α vs $E(\alpha_{YW})$ [..............], $E(\alpha_B)$ [• • • • • •] and $E(\alpha_{MLE})$ [..............], for n=30, [————] denoting the diagonal

In the remainder of this section, we propose a methodology for reducing the bias of the estimators when replicated series are available. First, a mathematical representation of the bias is obtained as follows: the expected value of a given estimator is evaluated (or simulated when an exact representation is not available) for $\alpha=-0.99, -0.98, \ldots, 0, 0.01, \ldots, 0.99$, and a tenth-degree polynomial, $p_{10}(\alpha)$, is fitted to those values. Since $E(\hat{\alpha}|\alpha = \alpha^*)$ is the expected value of the estimator $\hat{\alpha}$ when $\alpha = \alpha^*$,

Table 5: Variances of α_B and α_{MLE} for selected values of n and α

	$n = 10$		$n = 30$	
α	Burg	MLE	Burg	MLE
0.05	0.091925	0.095563	0.032234	0.032399
0.50	0.079401	0.081387	0.025819	0.024803
0.95	0.034363	0.032429	0.007561	0.006880

then, given $\bar{\alpha}$, the average of r estimates obtained from AR(1) series of length n assumed to have common autoregressive parameter, α^*, the inverse polynomial, $p_{10}^{-1}(\bar{\alpha})$, should provide a markedly less biased estimate of α^*, as n becomes large. This follows from the fact that by virtue of the *Law of Large Numbers*, $\bar{\alpha}$ tends to $E(\hat{\alpha} \mid \alpha = \alpha^*)$

The following tenth-degree polynomials were used to obtain the bias-corrected MLE's in Tables 6 and 7 and the bias-corrected Yule-Walker estimates in Tables 8 and 9.

For $n = 10,\ r = 5000$:
$0.001159 + 1.202319\bar{\alpha}_{MLE} - 0.031087\bar{\alpha}_{MLE}^2 - 0.074610\bar{\alpha}_{MLE}^3 + 0.220963\bar{\alpha}_{MLE}^4 + 0.098105\bar{\alpha}_{MLE}^5 - 0.612084\bar{\alpha}_{MLE}^6 - 0.201949\bar{\alpha}_{MLE}^7 + 0.712065\bar{\alpha}_{MLE}^8 - 0.029542\bar{\alpha}_{MLE}^9 - 0.293633\bar{\alpha}_{MLE}^{10}$.

For $n = 30,\ r = 50$:
$-0.014107 + 0.876257\bar{\alpha}_{MLE} + 0.011462\bar{\alpha}_{MLE}^2 + 1.686744\bar{\alpha}_{MLE}^3 + 0.586217\bar{\alpha}_{MLE}^4 - 5.525189\bar{\alpha}_{MLE}^5 - 2.782865\bar{\alpha}_{MLE}^6 + 7.676179\bar{\alpha}_{MLE}^7 + 4.010442\bar{\alpha}_{MLE}^8 - 3.778861\bar{\alpha}_{MLE}^9 - 1.824720\bar{\alpha}_{MLE}^{10}$.

For $n = 10,\ r = 5000$:
$5.204170^{-18} + 1.334683\bar{\alpha}_{YW} - 1.221245^{-15}\bar{\alpha}_{YW}^2 - 0.046332\bar{\alpha}_{YW}^3 - 2.131628^{-14}\bar{\alpha}_{YW}^4 + 0.140101\bar{\alpha}_{YW}^5 + 2.842171^{-14}\bar{\alpha}_{YW}^6 - 0.251772\bar{\alpha}_{YW}^7 + 1.136868^{-13}\bar{\alpha}_{YW}^8 - 0.419504\bar{\alpha}_{YW}^9 + 1.421085^{-14}\bar{\alpha}_{YW}^{10}$.

For $n = 30,\ r = 50$:
$1.058181^{-16} + 1.100272\bar{\alpha}_{YW} - 2.664535^{-15}\bar{\alpha}_{YW}^2 + 0.069487\bar{\alpha}_{YW}^3 + 2.131628^{-14}\bar{\alpha}_{YW}^4 - 0.403898\bar{\alpha}_{YW}^5 - 1.350031^{-13}\bar{\alpha}_{YW}^6 + 0.842867\bar{\alpha}_{YW}^7 + 1.847411^{-13}\bar{\alpha}_{YW}^8 - 0.589123\bar{\alpha}_{YW}^9 - 1.278977^{-13}\bar{\alpha}_{YW}^{10}$.

The bias associated with the estimators of certain time series parameters is discussed in Kendall (1954), Marriott and Pope (1954), and Tjϕstheim and Paulsen (1983). Moreover certain results which rely on replicated observations in connection with the estimation of parameters in low order autoregressive time series are available from Azzalini (1981).

Table 6: Simulated (5000 replications) and bias-corrected values of α_{MLE} for $n = 10$ and given values of α

Assumed α	-0.95	-0.50	-0.25	-0.05	0.05	0.25	0.50	0.95
Sim α_{MLE}	-0.850721	-0.417331	-0.203374	-0.042403	0.041563	0.208214	0.419142	0.854860
BiasCor. α_{MLE}	-0.949173	-0.497305	-0.243715	-0.049873	0.051072	0.249880	0.499074	0.950863

Table 7: Simulated (50 replications) and bias-corrected values of α_{MLE} for $n = 30$ and given values of α

Assumed α	-0.95	-0.50	-0.25	-0.05	0.05	0.25	0.50	0.95
Sim α_{MLE}	-0.887495	-0.479947	-0.221546	-0.033974	0.076837	0.277205	0.430188	0.887960
BiasCor. α_{MLE}	-0.963656	-0.510521	-0.222159	-0.043929	0.054060	0.259825	0.443621	0.942424

Table 8: Simulated (5000 replications) and bias-corrected values of α_{YW} for $n = 10$ and given values of α

Assumed α	-0.95	-0.50	-0.25	-0.05	0.05	0.25	0.50	0.95
Sim α_{YW}	-0.749736	-0.386005	-0.190061	-0.037236	0.036131	0.180907	0.370460	0.750499
BiasCor. α_{YW}	-0.949399	-0.513328	-0.253368	-0.049696	0.048222	0.241205	0.492772	0.949998

Table 9: Simulated (50 replications) and bias-corrected values of α_{YW} for $n = 30$ and given values of α

Assumed α	-0.95	-0.50	-0.25	-0.05	0.05	0.25	0.50	0.95
Sim α_{YW}	-0.855452	-0.448885	-0.218246	-0.043929	0.045812	0.208842	0.426413	0.862341
BiasCor. α_{YW}	-0.937735	-0.495478	-0.240672	-0.048339	0.050412	0.230270	0.470749	0.944303

3 A Beta-Polynomial Density Approximation

In this section, we show that accurate beta-type approximations to the density functions of α_{YW}, α_B or α_{MLE} can be obtained from the moments of those estimators. One can readily determine a simple beta approximation to the density functions of these estimators from their first two moments. This technique is described for instance in Johnson and Kotz (1970, Chapter 24). The same approximation can be obtained by making use of the beta-polynomial methodology described below, assuming that only two moments are available.

The proposed technique is based on the first d moments of a random variable whose distribution is assumed to behave similarly to that of a beta random variable. Durbin and Watson (1951) made use of four moments in conjunction with a related approach which was based on Jacobi polynomials.

First, the support of the random variable Y, assumed to be the bounded interval (q, r), is mapped onto the interval $(0, 1)$ with the affine transformation:

$$x = (y - q)/(r - q) \Leftrightarrow y = x(r - q) + q \; ; \tag{8}$$

the jth moment of X then becomes

$$\mu_X[j] = \sum_{i=0}^{j} \binom{j}{i} \frac{\mu_Y[i]}{(r-q)^j} (-q)^{j-i} \tag{9}$$

where $\mu_Y[i]$ denotes the ith moment of Y about the origin. Then, on the basis of the first d moments of X, a density approximation of the following form is assumed for X:

$$g(x) = \phi(x) \sum_{j=0}^{d} \alpha_j x^j \tag{10}$$

where

$$\phi(x) \;=\; \frac{1}{\mathrm{B}[\mathrm{a}_1, \mathrm{b}_1]}\, x^{a_1 - 1}(1 - x)^{b_1 - 1} \tag{11}$$

is a beta density function with parameters $[a_1, b_1]$,

$$a_1 \;=\; \frac{\mu_X[1](\mu_X[1] - \mu_X[2])}{\mu_X[2] - (\mu_X[1])^2}\ , \tag{12}$$

$$b_1 \;=\; \frac{(1 - \mu_X[1])(\mu_X[1] - \mu_X[2])}{\mu_X[2] - (\mu_X[1])^2}\ , \tag{13}$$

and

$$B[a, b] = \Gamma(a)\Gamma(b)/\Gamma(a + b)\ ,$$

$\Gamma(\cdot)$ denoting the gamma function. We denote the jth moment of such a beta distribution by

$$m[j] = \frac{\Gamma(j+a_1)\ \Gamma(a_1+b_1)}{\Gamma(a_1)\ \Gamma(j+a_1+b_1)} = \frac{\prod_{k=0}^{j-1}(k+a_1)}{\prod_{k=0}^{j-1}(k+a_1+b_1)}\ . \tag{14}$$

In order to determine the polynomial coefficients, α_j, we equate the hth moment of X to the hth moment of the approximate distribution specified by $g(x)$. That is,

$$\begin{aligned}\mu_X[h] &= \int_0^1 x^h \phi(x) \sum_{j=0}^{d} \alpha_j x^j \mathrm{d}x \\ &= \sum_{j=0}^{d} \alpha_j \int_0^1 x^{h+j}\phi(x)\mathrm{d}x \\ &= \sum_{j=1}^{d} \alpha_j\ m[h+j],\ \ h=0,1,\ldots,d.\end{aligned} \tag{15}$$

This leads to a linear system of $(d+1)$ equations in $(d+1)$ unknowns whose solution is

$$\begin{bmatrix}\alpha_0\\ \alpha_1\\ \vdots\\ \alpha_d\end{bmatrix} = \begin{bmatrix} m[0] & m[1] & \cdots & m[d-1] & m[d] \\ m[1] & m[2] & \cdots & m[d] & m[d+1] \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ m[d] & m[d+1] & \cdots & m[2d-1] & m[2d]\end{bmatrix}^{-1} \begin{bmatrix}\mu_X[0]\\ \mu_X[1]\\ \vdots\\ \mu_X[d]\end{bmatrix}\ . \tag{16}$$

The resulting representation of the density function of X will be referred to as a beta-polynomial density approximant. On applying the inverse transformation specified by Equation 8, the beta-polynomial density approximant of Y is given by

$$\frac{1}{(r-q)}\ g\left(\frac{y-q}{r-q}\right). \tag{17}$$

For the estimators at hand, $q=-1$ and $r=1$. The graphs in Figures 6, 7, 8, 9, 10 and 11 show the simulated and approximate CDF's of α_{YW} for certain values of α, n and d. (The CDF approximations were obtained by integrating the beta-polynomial density approximants.) While approximations based on two moments may be adequate for small values of $|\alpha|$, it is apparent that at least six moments are required when $|\alpha|$ is close to one.

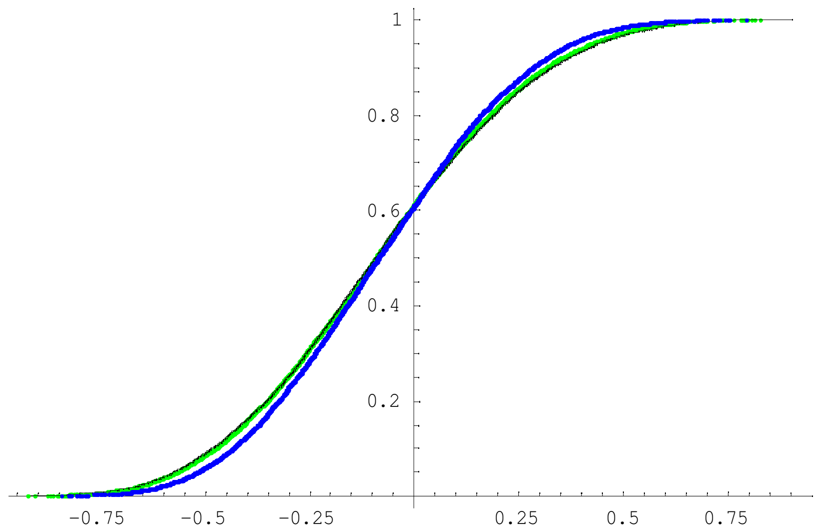

Figure 4. Simulated CDF's for $\alpha = -0.10$ and n = 10; α_{MLE} [———], α_B [———] and α_{YW} [———]

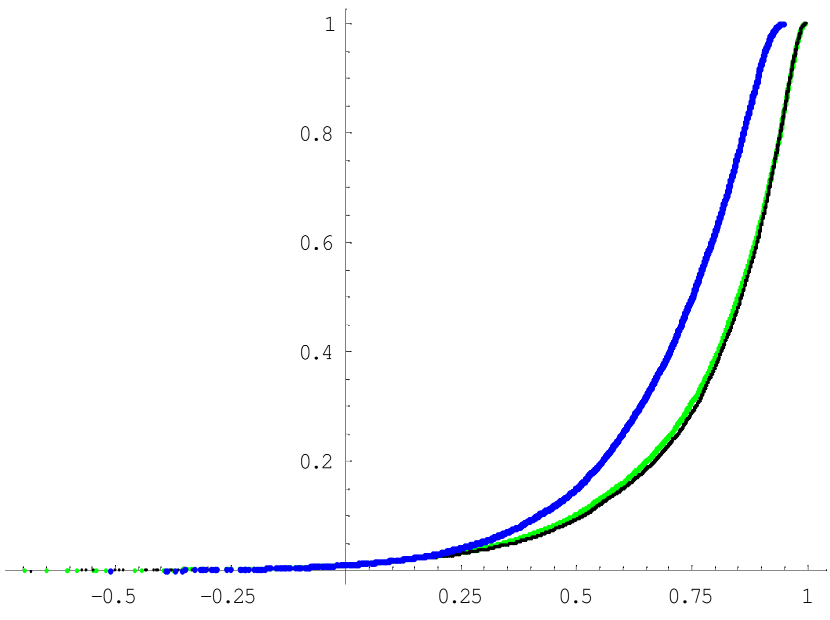

Figure 5. Simulated CDF's for $\alpha = 0.90$ and n = 10; α_{MLE} [———], α_B [———] and α_{YW} [———]

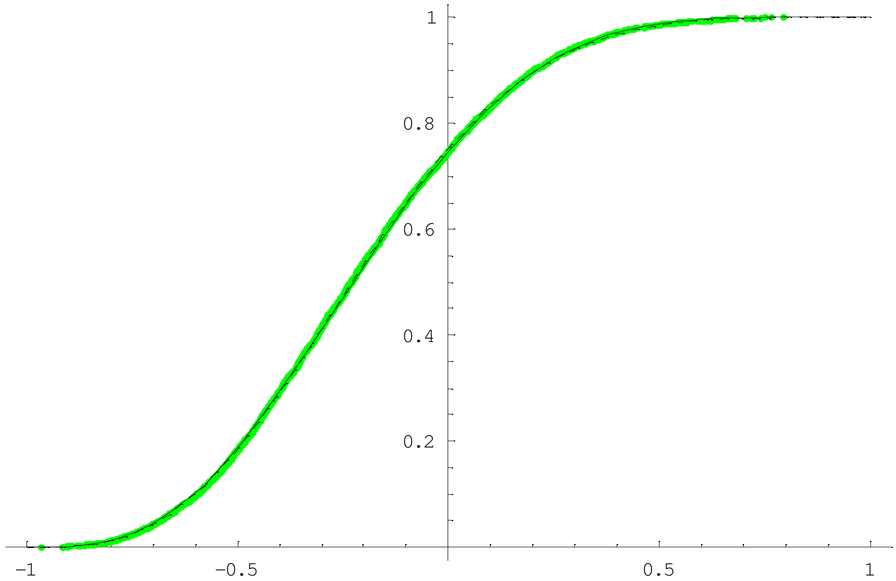

Figure 6. Four [- - - - - - -] and two [————] moment beta-polynomial CDF approximations superimposed on the simulated CDF[▬▬▬] of α_{YW} for $\alpha = -0.25$ and n = 10

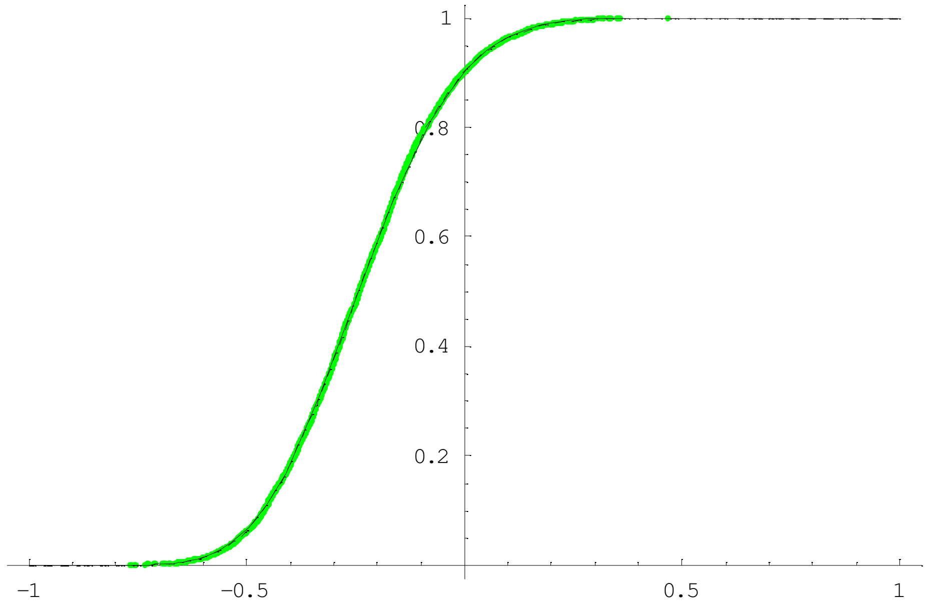

Figure 7. Four [- - - - - - -] and two [————] moment beta-polynomial CDF approximations superimposed on the simulated CDF[▬▬▬] of α_{YW} for $\alpha = -0.25$ and n =30

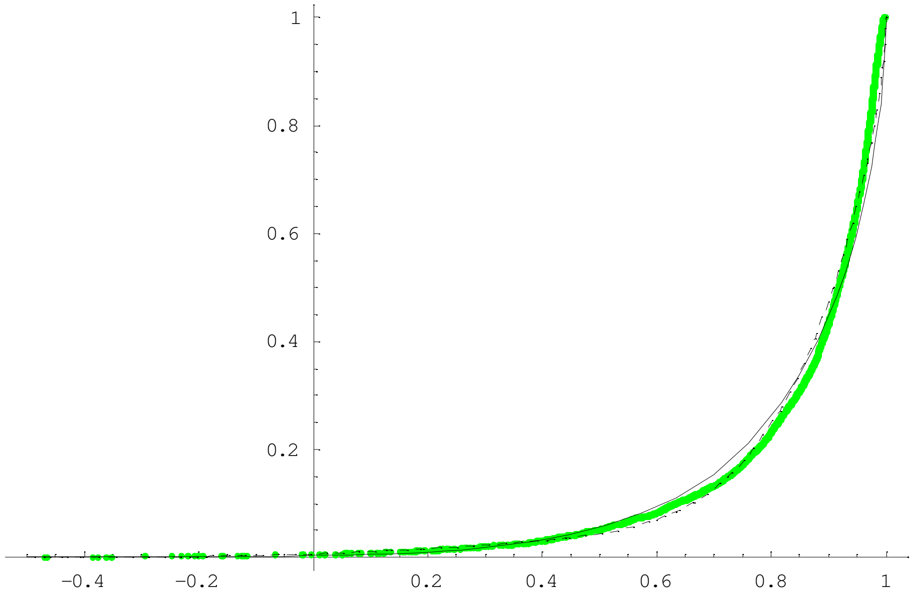

Figure 8. Four [- - - - - - -] and two [———] moment beta-polynomial CDF approximations superimposed on the simulated CDF [▬▬▬] of α_{YW} for $\alpha = 0.95$ and n=10

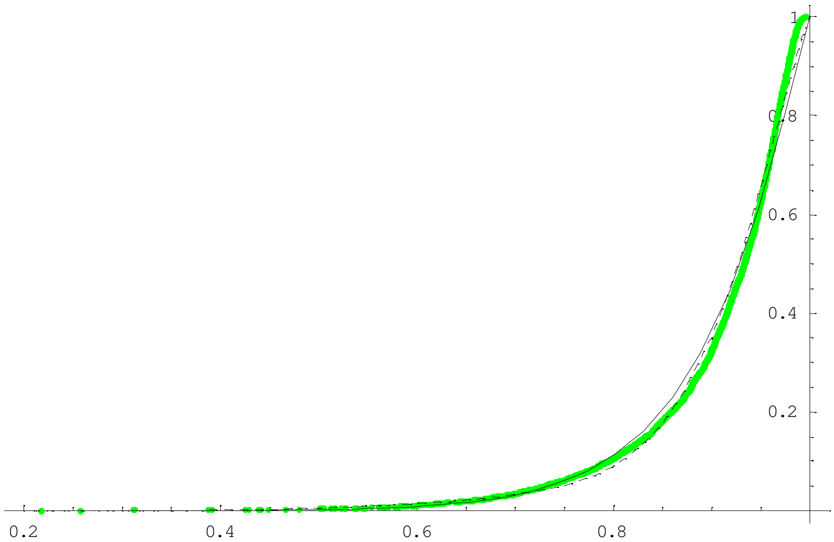

Figure 9. Four [- - - - - - -] and two [———] moment beta-polynomial CDF approximations superimposed on the simulated CDF [▬▬▬] of α_{YW} for $\alpha = 0.95$ and n=30

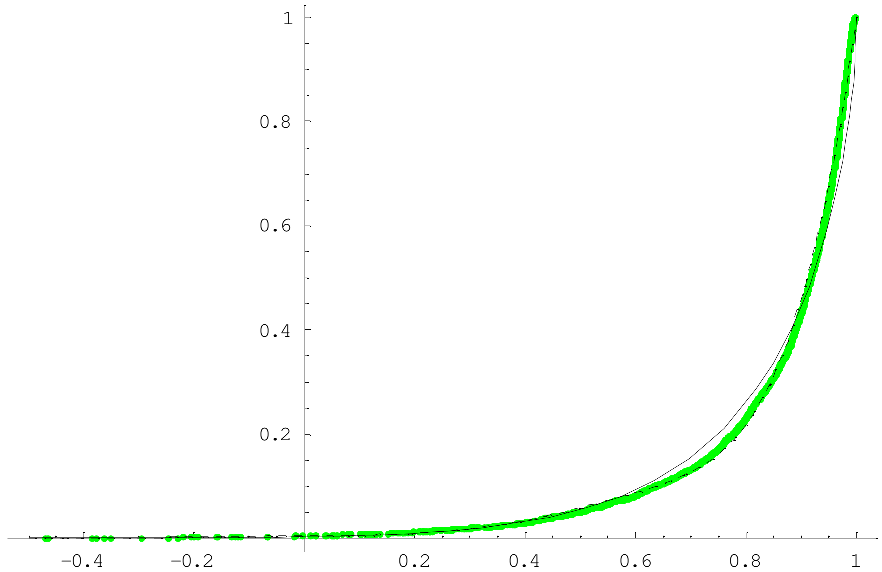

Figure 10. Six [-------] and two [――――] moment beta-polynomial CDF approximations superimposed on the simulated CDF[▬▬▬] of α_{YW} for $\alpha = 0.95$ and n=10

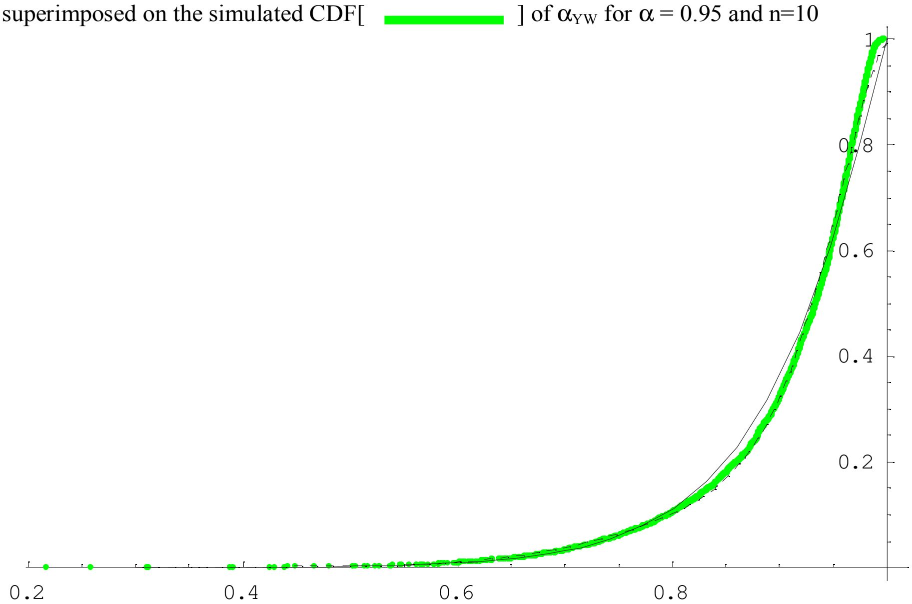

Figure 11. Six [-------] and two [――――] moment beta-polynomial CDF approximations superimposed on the simulated CDF[▬▬▬] of α_{YW} for $\alpha = 0.95$ and n=30

4 Conclusion

It was determined that Burg's estimator is less biased than Yule-Walker estimator and that its distribution is nearly identical to that of the MLE whose moments cannot be explicitly determined.

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