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Record Values from Uniform Distribution

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Abstract

In this paper record values from uniform distribution are considered. Several distributional properties of record values are presented. A characterization of uniform distribution based on one of these distributional roperties is also given.

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1 Introduction

Suppose $\{\mathrm{X}_i, i = 1, 2, \cdots\}$ be a sequence of independent and identically distributed random variables(i.i.d.) with absolutely continuous (with respect to Lebesgue measure) cumulative distribution function (cdf) $F(x)$ and the corresponding probability density function (pdf) $f(x)$. Let $Y_n = \max(\min)\{X_1, X_2, \cdots, X_n\}$. We say X_j is an upper (lower) record value of $\{X_n,\ n \geq 1\}$ if $Y_j > (<) Y_{j-1},\ j > 1$. By definition X_1 is an upper as well as lower record value.The indices at which the upper record values occur are given by the record times $\{U(n)\}, n > 0$, where $U(n) = \min\{j | j > U(n-1), X_j > X_{U(n-1)}, n > 1\}$ and $U(1) = 1$. We will denote $L(n)$ as the indices where the lower record values occur. Let $X_i, i = 1, 2, \cdots$ be distributed uniformly in the interval $(0, 1)$. In this paper we will study several distributional properties of $X_{U(n)}$. A characteriation of the uniform distribution based on these distributional properties is given.

2 Main Results

We denote by $f_n(x)$ as the pdf of $X_{U(n)}$ and $f_{(n)}(x)$ be the pdf $X_{L(n)}$. It can be shown (see Ahsanullah (2004), Nevzorov (2001)) that

$$f_n(x) = \frac{1}{\Gamma(n)}[-ln\bar{F}(x)]^{n-1}f(x), \quad \bar{F}(x) = 1 - F(x) \tag{1}$$

and

$$f_{(n)}(x) = \frac{1}{\Gamma(n)}[-lnF(x)]^{n-1}f(x). \tag{2}$$

The corresponding cumulative distributional functions are respectively

$$\begin{aligned} \bar{F}_n(x) &= 1 - F_n(x) = \bar{F}(x)\sum_{j=0}^{n-1}\frac{(R(x))^j}{\Gamma(j+1)}, \\ R(x) &= -\ln\bar{F}(x) \end{aligned} \tag{3}$$

and

$$\begin{aligned} F_{(n)}(x) &= F(x)\sum_{j=0}^{n-1}\frac{(H(x))^j}{\Gamma(j+1)}, \\ H(x) &= -\ln F(x). \end{aligned} \tag{4}$$

The joint pdf of mth and nth ($n>m$) upper and lower record values respectively are as

$$\begin{aligned} f_{m,n}(x,y) &= \frac{(R(x))^{m-1}\{R(y)-R(x)\}^{n-m-1}}{\Gamma(m)\Gamma(n-m)}r(x)f(y), \quad -\infty < x < y < \infty, \\ &= 0, \text{ otherwise.} \end{aligned}$$

$$\begin{aligned} f_{(m),(n)}(x,y) &= \frac{(H(x))^{m-1}\{H(y)-H(x)\}^{n-m-1}}{\Gamma(m)\Gamma(n-m)}h(x)f(y), \quad -\infty < y < x < \infty, \\ &= 0, \text{ otherwise.} \end{aligned}$$

It is well known (see Bairamov and Ahsanullah (2000)) that $X_{U(n)}$ can be represented as

$$X_{U(n)} \stackrel{d}{=} g_F^{-1}(g_F(X_1) + g_F(X_2) + \cdots + g_F(X_n))$$

where $g_F(x) = -\ln(1-F(x))$ and $(g_F^{-1}(x) = F^{-1}(1-e^{-x})$, $x \geq 0$. Thus if X is distributed as uniform on (0,1), then for $n = 1, 2, \cdots$

$$X_{U(n)} \stackrel{d}{=} 1 - \Pi_{i=1}^{n}(1-X_i). \tag{5}$$

It is also true that

$$1 - X_{U(n)} \stackrel{d}{=} (1 - X_{U(n-1)}\ )V_1, \tag{6}$$

where V_1 is uniformly distributed as uniform on (0,1). The coresponding relations for the lower records are

$$X_{L(n)} \stackrel{d}{=} \Pi_{i=1}^{n} X_i \tag{7}$$

and

$$X_{L(n)} \stackrel{d}{=} X_{L(n-1)} V_1. \tag{8}$$

If X_i is distributed as uniform on (0,1) then $1 - X_i$ is also distributed as uniform on (0,1). Thus (5) can be written as

$$1 - X_{U(n)} \stackrel{d}{=} \Pi_{i=1}^{n} W_i, \tag{9}$$

where $W_1, W_2, \cdots, W_n$ are i.i.d. with cdf $F(x) = x$, $0 \le x \le 1$. Comparing (7) and (9) we can say $1 - X_{U(i)}$ and $X_{L(n)}$ have the same distribution.

Theorem 2.1. *For* $1 \le m < n, r, s = 1, 2, \cdots$, $E(X^r_{L(m)} X^s_{L(n)}) = (\frac{1}{r+s+1})^m (\frac{1}{s+1})^{n-m}$.

Proof. From (8), we have

$$P\left(\frac{X_{L(n)}}{X_{L(n-1)}} \le x\right) = P(V_1 \le x) = x, \ \ 0 \le x \le 1.$$

Therefore, for $1 \le m < n, r, s = 1, 2, \cdots$

$$X^r_{L(m)} X^s_{L(n)} \stackrel{d}{=} \Pi_{j=1}^{m} V_j^{r+s} \Pi_{j=m+1}^{n} V_j^s,$$

where $V_1, V_2, \cdots, V_n$ are i.i.d. uniform on (0,1). Hence

$$E(X^r_{L(m)} X^s_{L(n)}) = (\frac{1}{r+s+1})^m (\frac{1}{s+1})^{n-m}. \tag{10}$$

Using (10), we obtain

$$\begin{aligned} E(X^p_{L(n)}) &= (\frac{1}{p+1})^n \\ Var(X_{L(n)}) &= (\frac{1}{3})^n - (\frac{1}{4})^n \\ Cov(X_{L(m)} X_{L(n)}) &= (\frac{1}{3})^{n-m} Var(X_{L(n)}), \ \ 1 \le m < n. \end{aligned}$$

The following theorem gives a recurrence relation of the moments of $X_{U(n)}$.

Theorem 2.2. *For* $n \geq 2$, $r = 0, 1, 2, \cdots$ $E(X^{r+1}_{U(n)}) = \frac{r+1}{r+2} E(X^{r}_{U(n)}) + \frac{1}{r+2} E(X^{r+1}_{L(n-1)})$.

Proof.

$$\begin{aligned} E(X^r_{U(n)} - X^{r+1}_{U(n)}) &= \int_0^1 (x^r - x^{r+1}) f_n(x) dx \\ &= \int_0^1 (x^r - x^{r+1}) \frac{1}{\Gamma(n)} (-\ln(1-x))^{n-1} dx \\ &= \int_0^1 x^r (1-x) \frac{1}{\Gamma(n)} (-\ln(1-x))^{n-1} dx \\ &= \frac{1}{r+1} \{ \int_0^1 x^{r+1} \frac{1}{\Gamma(n)} (-\ln(1-x))^{n-1} dx \\ & - \int_0^1 x^{r+1} \frac{n-1}{\Gamma(n)} (-\ln(1-x))^{n-2} dx \} \\ &= \frac{1}{r+1} [E(X^{r+1}_{U(n)}) - E(X^{r+1}_{U(n-1)})] \end{aligned} \tag{11}$$

By simply rewriting the above equation we obtain the recurrence relation.

The following theorem gives the recurrence relations between product moments of upper record values.

Theorem 2.3. *For* $m \geq 1$ *and* $r, s = 0, 1, 2, \cdots$

$$E(X^r_{U(m)} X^{s+1}_{U(n)}) = \frac{s+1}{s+2} E(X^r_{U(m)} X^s_{U(n)}) + \frac{1}{s+2} E(X^r_{U(m)} X^{s+1}_{U(n-1)}).$$

Proof.

$$\begin{aligned} E(X^r_{U(m)} X^s_{U(m+1)} - X^r_{U(m)} X^{s+1}_{U(1)}) &= \int\int_{0 \leq x < y < 1} (x^r y^s - x^r y^{s+1}) f_{m.m+1}(x, y) dy dx \\ &= \frac{1}{\Gamma(m)} \int_0^1 x^r (-\ln \bar{F}(x))^{m-1} \frac{f(x)}{\bar{F}(x)} I_1(x) dx \end{aligned} \tag{12}$$

where

$$\begin{aligned} I_1(x) &= \int_x^1 y^s (1-y) \{-\ln(\bar{F}(y) + \ln(\bar{F}(x)\}^{n-m-1} f(y) dy \\ &= \int_x^1 y^s \{-\ln(\bar{F}\ (y) + \ln(\bar{F}(x)\}^{n-m-1} \bar{F}(y) \\ &= \frac{1}{s+1} \int_x^1 y^s [\{-\ln(\bar{F}(y) + \ln(\bar{F}(x)\}^{n-m-1} f(y) \\ & -(n-m-1)\{-\ln(\bar{F}(y) + \ln(\bar{F}(x)\}^{n-m-2} f(y)] dy \end{aligned} \tag{13}$$

Substituting (13) in (12) we obtain

$$E(X^r_{U(m)}X^s_{U(n)} - X^r_{U(m)}X^{s+1}_{U(n)}) = \frac{1}{s+1}E(X^r_{U(m)}X^{s+1}_{U(n)}) - \frac{1}{s+1}E(X^r_{U(m)}X^{s+1}_{U(n-1)}) \quad (14)$$

Rearranging the terms of (14) , we obtain the recurrence relations.

From (6) we have $1 - X_{U(n)} \stackrel{d}{=} (1 - X_{U(m)})V_1V_2\cdots V_{n-m}$, where $V_1V_2\cdots V_{n-m}$ are i.i.d uniform on (0,1). Is it a characteristic property of the uniform distribution? The following theorem gives a solution of the problem for $m = n - 1$.

Theorem 2.4. *Let $X_1, X_2, \cdots$ be a sequence of i.i.d absolutely continuous bounded random variables. We will assume without any loss of generality $F(0) = 0$ and $F(1) = 1$. Let V_1 be independent of the X's and is distributed as uniform on (0,1). Then the relation $1 - X_{U(n)} \stackrel{d}{=} (1 - X_{U(n-1)})V_1$ is true iff $X \in U(0,1)$.*

Proof. We will establish here that the relation $1 - X_{U(n)} \stackrel{d}{=} (1 - X_{U(n-1)})V_1$ implies that $X \in U(0,1)$.

$$P(1 - X_{U(n+1)} \le x) = P(X_{L(n+1)} \le x) = F_{(n+1)}(x)$$

$$\begin{aligned} P((1 - X_{U(n-1)})V_1 \le x) &= P(X_{L(n)}V_1 \le x) \\ &= \int_0^x dv + \int_x^1 F_{(n)}(\frac{x}{v})dv \\ &= x + x\int_x^1 F_{(n)}(t)\frac{1}{t^2}dt \end{aligned}$$

Since $1 - X_{U(n)} \stackrel{d}{=} (1 - X_{U(n-1)})V_1$, we must have

$$F_{(n+1)}(x) = x + x\int_x^1 F_{(n)}(t)\frac{1}{t^2}dt \quad (15)$$

Differentiating both sides of (15) with respect to x, we obtain

$$f_{(n+1)}(x) = 1 - \frac{1}{x}F_{(n)}(x) + \int_x^1 F_{(n)}(t)\frac{1}{t^2}dt \quad (16)$$

Multiplying both sides of (16) by x, we have

$$\begin{aligned} xf_{(n+1)}(x) &= x - F_{(n)}(x) + x\int_x^1 F_{(n)}(t)\frac{1}{t^2}dt \\ &= x - F_{(n)}(x) + F_{(n+1)}(x) - x \\ &= F_{(n+1)}(x) - F_{(n)}(x) \quad (17) \end{aligned}$$

Since

$$F_{(n+1)}(x) - F_{(n)}(x) = \frac{(H(x))^n}{\Gamma(n+1)} F(x) = f_{(n+1)}(x) \frac{F(x)}{f(x)} \tag{18}$$

Substituting (18) in (17), we obtain on simplification

$$\frac{f(x)}{F(x)} = \frac{1}{x}, \quad \text{for all } x,\ 0 < x < 1. \tag{19}$$

Hence $F(x) = x$, $x \in (0, 1]$.

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