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Glimpses of Inequalities in Probability and Statistics

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Abstract

In this article we compile all basic and well-known inequalities useful in probability and statistics. Proofs are given only for two less known inequalities.

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1 Introduction

Rao (1996) wrote an article entitled, "Seven inequalities in statistical estimation theory", that has appeared in the journal, Student[6]. The present article is almost a complement of Professor Rao's article. Professor Rao's inequalities were on estimation theory, whereas our inequalities are primarily on probability theory. Most of the inequalities are well-known, but some of the inequalities may not be known to all. This compilation of basic inequalities is expected to serve as reference to most researches in probability and statistics.

2 Basic Inequalities

In this section we state fifteen basic well-known inequalities.

2.1 Markov's Inequality.

Suppose that X is a nonnegative random variable with finite mean, $E(X)$. Then for any $t > 0$,

$$P(X \geq t) \leq \frac{E(X)}{t}.$$

2.2 Chebyshev's Inequality.

Suppose that X is a random variable with finite mean μ and variance σ^2. Then for any $t > 0$,

$$P\left(|X - \mu| \geq t\right) \leq \frac{\sigma^2}{t^2}$$

or

$$P\left(|X - \mu| < t\right) \geq 1 - \frac{\sigma^2}{t^2}.$$

2.3 One-sided Chebyshev's inequality.

Let X be a random variable with mean μ and variance σ^2. Then for any $t > 0$,

$$(i)\ P(X \geq \mu + t) \leq \frac{\sigma^2}{\sigma^2 + t^2}$$

$$(ii)\ P(X \leq \mu - t) \leq \frac{\sigma^2}{\sigma^2 + t^2}.$$

2.4 Chernoff bounds.

Let X be any r.v. with moment generating function $M_X(t)$. Then,

$$(i)\ P(X \geq s) \leq \frac{M_X(t)}{e^{ts}} \quad \text{for all } t > 0$$

$$(ii)\ P(X \leq s) \leq \frac{M_X(t)}{e^{ts}} \quad \text{for all } t < 0.$$

2.5 Basic Inequality.

Let g be a nonnegative and nondecreasing function on R on the range of the random variable X. Further assume that *ess.sup* $g(X) < \infty$. Then

$$\frac{E\{g(X)\} - g(a)}{\text{ess. sup}\, g(X)} \leq P(X \geq a) \leq \frac{E\{g(X)\}}{g(a)}.$$

In addition, if g is an even function, replace $P(X \geq a)$ by $P(|X| \geq a)$ in the above. Here *ess. sup* $g(x)$ is the least among constants below which almost all values of $g(x)$ lie.

2.6 Corollary to the Basic Inequality.

Let $g(X) = |X|^r$ for $r > 0$. Then by basic inequality, we have

$$\frac{E(|X|^r) - a^r}{ess.sup\, g(X)} \leq P(|X| \geq a) \leq \frac{E(|X|^r)}{a^r}.$$

(i) For $r = 1$ right hand side becomes the Markov's inequality.
(ii) For $r = 2$ right hand side becomes the Chebyshev's inequality.

2.7 Cauchy-Schwarz Inequality.

Let X and Y be any two random variables. Then

$$[E(XY)]^2 \leq E(X^2)E(Y^2),$$

or equivalently,

$$|E(XY)| \leq [E(X^2)]^{\frac{1}{2}}[E(Y^2)]^{\frac{1}{2}}.$$

2.8 C_r-inequality.

Let X and Y be any two random variables. Then for $r \geq 0$,

$$E\{|X + Y|^r\} \leq C_r[E\{|X|^r\} + E\{|X|^r\}],$$

where

$$C_r = \begin{cases} 1 & \text{for } r \leq 1 \\ 2^{r-1} & \text{for } r > 1. \end{cases}$$

2.9 Holder's inequality.

Let X and Y be any two random variables. Then for $r > 1$, $s > 1$ with $\frac{1}{r} + \frac{1}{s} = 1$

$$E\{|XY|\} \leq [E(|X|^r)]^{\frac{1}{r}}[E(|Y|^s)]^{\frac{1}{s}}.$$

2.10 Lyapounov's inequality.

For a random variable X

$$(E(|X|^\alpha))^{1/\alpha} \leq \left(E(|X|^\beta)\right)^{1/\beta}, \ \ 0 < \alpha \leq \beta.$$

2.11 Minkowski Inequality.

Let X and Y be any two random variables. Then for $r \geq 1$,

$$[E(|X+Y|^r)]^{1/r} \leq [E(|X|^r)]^{1/r} + [E(|Y|^r)]^{1/r}.$$

2.12 Jensen's inequality.

Let g be a convex function and $E(X)$ be finite. Then

$$E(g(X)) \geq g(E(X)).$$

2.13 An Inequality on the Expectation of $E|X|^\nu$.

Let X be a random variable such that $E|X|^\beta < \infty$ for some $\beta > 0$. Then for $0 \leq \alpha \leq \beta$

$$E|X|^\alpha \leq 1 + E|X|^\beta$$

and

$$E|X|^\alpha < \infty.$$

2.14 Bernstein Inequality (Binomial case).

Let S_n follow binomial distribution with parameters n and p. Then for $\epsilon > 0$

$$P(|S_n - np| \geq n\epsilon) \leq 2\, e^{-n\epsilon^2/2}.$$

That is, the probability that S_n exceeds its expected value np more than a multiple $n\epsilon$ of n, converges to zero exponentially fast as $n \to \infty$. Here S_n may also be viewed as the sums of n independent Bernoulli trials.

2.15 Hoeffding's Inequality.

We propose to give a more general version of this important inequality. Let X_1, X_2, $\cdots$, X_n be independently distributed random variables with finite means $E(X_i) = \mu_i$ and $a_i \leq X_i \leq b_i$ for all i. Also, let $S_n = \sum_{i=1}^n X_i$. Then for $\epsilon > 0$

$$P(|S_n - E(S_n)| \geq n\epsilon) \leq 2\, e^{-2n^2\epsilon^2/\sum_{i=1}^n (b_i - a_i)^2}.$$

3 Further Inequalities

In this section we state two additional less known inequalities on quantiles and the expectation of harmonic mean. In both cases we supply the proofs.

3.1 An Inequality on Quantiles.

Let ζ_p be the pth quantile of the random variable X that has mean μ and variance σ^2. Then

$$\mu - \sigma\sqrt{\frac{1-p}{p}} \le \zeta_p \le \mu + \sigma\sqrt{\frac{p}{1-p}}.$$

For $0 < p < 1$, the quantity ζ_p is said to be the pth quantile of the random variable X if $P(X \le \zeta_p) \ge p$ and $P(X \ge \zeta_p) \ge 1 - p$.

Proof. By definition, ζ_p satisfies the inequality

$$p \le P(X \le \zeta_p) = P\left(\frac{X-\mu}{\sigma} \le \frac{\zeta_p - \mu}{\sigma}\right) = P(Z \le t),$$

where $Z = \frac{X-\mu}{\sigma}$ which has mean 0 and variance 1, and $t = \frac{\zeta_p - \mu}{\sigma}$. Now for $\zeta_p < \mu$, we have $t < 0$, and consequently using one-sided Chebyshev's inequality on $P(Z \le t)$ we have

$$p \le P(Z \le t) \le \frac{1}{1+t^2},$$

i.e.

$$p \le \frac{1}{1+t^2} = \frac{1}{1 + \left(\frac{\xi_p - \mu}{\sigma}\right)^2}$$

i.e.

$$-\sqrt{\frac{1-p}{p}} \le \frac{\zeta_p - \mu}{\sigma}$$

$$\mu - \sigma\sqrt{\frac{1-p}{p}} \le \zeta_p.$$

Again by definition, ζ_p satisfies the inequality

$$1 - p \le P(X \ge \zeta_p) = P\left(\frac{X-\mu}{\sigma} \ge \frac{\zeta_p - \mu}{\sigma}\right) = P(Z \ge t).$$

Now for $\zeta_p > \mu$, we have $t > 0$, and consequently using one-sided Chebyshev's inequality on $P(Z \ge t)$ we have

$$1 - p \le P(Z \ge t) \le \frac{1}{1+t^2},$$

i.e.

$$1 - p \le \frac{1}{1+t^2} = \frac{1}{1 + \left(\frac{\xi_p - \mu}{\sigma}\right)^2}$$

i.e.

$$\frac{\zeta_p - \mu}{\sigma} \le \sqrt{\frac{p}{1-p}}$$

or equivalently

$$\zeta_p \le \mu + \sigma\sqrt{\frac{1-p}{p}}\,.$$

Now combining both limits, we have

$$\mu - \sigma\sqrt{\frac{1-p}{p}} \le \zeta_p \le \mu + \sigma\sqrt{\frac{p}{1-p}}.$$

$\square$

Corollary. The lower and upper bounds on $\zeta_{\frac{1}{2}}$ (median) are given by

$$\mu - \sigma \le \zeta_{\frac{1}{2}} \le \mu + \sigma.$$

3.2 An Inequality on the Expectation of Harmonic Mean (see Rao, 1996).

Let $X > 0$ and $Y > 0$ be random variables. Then

$$E\left(\frac{XY}{X+Y}\right) \le \frac{E(X)E(Y)}{E(X+Y)}.$$

Proof (in a special case). Let $\frac{E(X)}{E(Y)} = \frac{r}{s}$ be rational, and take $X_1 = \frac{X}{r}$, $Y_1 = \frac{Y}{s}$ so that $E(X_1) = E(Y_1) = \mu > 0$. Since geometric mean is greater than or equal to harmonic mean, thus we have

$$\frac{r+s}{\frac{r}{Y_1} + \frac{s}{X_1}} \le Y_1^{r/(r+s)} X_1^{s/(r+s)}.$$

Now taking expectations on both sides we get

$$E\left(\frac{r+s}{\frac{r}{Y_1} + \frac{s}{X_1}}\right) \le E\left(Y_1^{r/(r+s)} X_1^{s/(r+s)}\right).$$

Note that $\frac{XY}{X+Y} = \frac{rs}{r+s} \cdot \frac{r+s}{\frac{r}{Y_1} + \frac{s}{X_1}}$. Thus we have

$$E\left(\frac{XY}{X+Y}\right) = \frac{rs}{r+s}\,E\left(\frac{r+s}{\frac{r}{Y_1} + \frac{s}{X_1}}\right) \le \frac{rs}{r+s}\,E\left(Y_1^{r/(r+s)} X_1^{s/(r+s)}\right).$$

Using Holder's inequality, we obtain

$$E\left(Y_1^{r/(r+s)} X_1^{s/(r+s)}\right) \leq (E(Y_1))^{r/(r+s)} (E(X_1))^{s/(r+s)} = \mu.$$

Finally, we establish

$$E\left(\frac{XY}{X+Y}\right) \leq \frac{rs}{r+s}\mu = \frac{E(X)E(Y)}{E(X+Y)}.$$

$\square$

4 Concluding remarks

The inequalities given in this article are useful for solving problems and proving results. These inequalities are a useful tool even for solving research problems. The proofs of the inequalities in section 2 are omitted since these are available in standard text books cited below. The collection of all these inequalities in a compact form becomes handy to students and teachers for reference purposes.

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