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Minimax Estimation for the Scale Parameter of the Gamma Distribution Under MILINEX and Quadratic Loss Functions

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Abstract

In this paper, minimax estimators for the scale parameter of the gamma distribution are considered for modified linear-exponential (MLINEX) and quadratic loss functions. The results are then interpreted in the light of two-person zero-sum game according to Wald [7]. Then these estimators are compared with the classical maximum likelihood estimator.

Keywords and Phrases: Gamma distribution, Scale parameter, MLINEX and quadratic loss functions, Two-person zero-sum game, Mean-squared error.

AMS Classification: 33B15, 91A05.

1 Introduction

The probability density function of the gamma distribution is given by

$$\begin{aligned} f(x|\alpha,\beta) &= \frac{\beta^{\alpha}}{\Gamma\alpha} e^{-\beta x} x^{\alpha-1};\ x \geq 0,\ \alpha,\ \beta > 0. \\ &= 0; \qquad\qquad \text{otherwise.} \end{aligned} \tag{1}$$

where α and β are the parameters of the distribution. Bruce [1], Hill [2] and Karns [4] considered the problem of estimation of the scale parameter of the gamma distribution by the use of M-order, L-order and one order statistics respectively. Estimation of scale parameter from the order statistics of unequal gamma components was also studied by Wilk et. al. [9] .

In here we shall estimate the scale parameter β of the gamma distribution by using the technique of minimax approach, assuming that α is known. The minimax approach is essentially a Bayesian approach. The most important element in the minimax approach is the specification of a distribution function on the parameter space, which is called prior distribution. In addition to the prior distribution, the minimax estimator for a particular model depends strongly on the loss function assumed. In Lehmann [3] it has been observed that in many cases, minimax approach is better than the classical approach for small sample sizes. In most cases symmetric loss functions are considered. But there are some real life situations where the use of the symmetrical loss functions may be inappropriate. In some cases a given positive error may be more serious than a given negative error and vice-versa.

In this paper, minimax estimators for the scale parameter β of the gamma distribution are considered for modified linear-exponential (MLINEX) and quadratic loss functions. The MLINEX loss function is asymmetric one and the quadratic loss function is symmetric. The results are then interpreted in the light of two-person zero-sum game according to Wald [7]. Then these estimators are compared with maximum likelihood estimator. The derivation depends primarily on a theorem, due to Lehmann [3] stated below.

Theorem 1.1: *Let $\tau = \{F_\theta; \theta \in \Theta\}$ be a family of distribution functions and D a class of estimators of θ. Suppose that $d^\star \in D$ is a Bayes estimator against a prior distribution $\xi^\star(\theta)$ on the parameter space Θ, and the risk function $R(d^\star, \theta) =$ constant on Θ; then $d^\star$ is a minimax estimator of θ.*

The main results of the paper are contained in Theorem 2.1 and Theorem 2.2 in the next section.

2 Main Results

Theorem 2.1: *Let $X = (X_1, X_2, \cdots, , X_n)$ be n independently and identically distributed random variables drawn from the density (1). Then*

$$\hat{\beta}_{MML} = \left(\frac{\Gamma(n\alpha)}{\Gamma(n\alpha - c)} \right)^{\frac{1}{c}} \frac{1}{\sum\limits_{i=1}^{n} X_i},$$

is the minimax estimator for the scale parameter β of the gamma distribution under MLINEX loss function of the form

$$L\left(\hat{\beta}, \beta\right) = \varpi \left[\left(\frac{\hat{\beta}}{\beta} \right)^c - c \ln \left(\frac{\hat{\beta}}{\beta} \right) - 1 \right]; \quad c \neq 0, \ \varpi > 0,$$

where $\hat{\beta}$ is the estimate of β, c and ϖ are two known parameters of the loss function.

Theorem 2.2: *Let $X = (X_1, X_2, \cdots, X_n)$ be n independently and identically distributed random variables drawn from the density (1). Then*

$$\hat{\beta}_{MEQ} = \frac{n\alpha - 2}{\sum\limits_{i=1}^{n} X_i},$$

is the minimax estimator for the scale parameter β of the gamma distribution under quadratic loss function of the form

$$L\left(\hat{\beta},\ \beta\right) = \left(\frac{\hat{\beta} - \beta}{\beta}\right)^2$$

To prove Theorems we use Lehmann's theorem, which was stated earlier. For this, at first we have to find the Bayes' estimator $\hat{\beta}$ of β. Then if we can show that the risk function of $\hat{\beta}$ is a constant, then the theorem will be proved.

We start with the likelihood function

$$\begin{aligned} l\left(\beta|x\right) &= \frac{\beta^{n\alpha}}{\left(\Gamma\alpha\right)^n} e^{-\beta \sum\limits_{i=1}^{n} x_i} \prod_{i=1}^{n} x_i^{\alpha-1} \\ &= \frac{\beta^{n\alpha}}{\left(\Gamma\alpha\right)^n} \exp\left\{-\beta \sum_{i=1}^{n} x_i + (\alpha - 1) \sum_{i=1}^{n} \log x_i\right\}, \end{aligned} \tag{2}$$

where $T = \sum\limits_{i=1}^{n} X_i$ is a complete sufficient statistic for β. Let us assume that β has Jeffrey's non-informative prior density defined as

$$\pi\left(\beta\right) \propto \frac{1}{\beta}, \quad \beta > 0. \tag{3}$$

Combining the equations (2) and (3), and using the Bayes theorem the posterior density of β for the given random sample $X = (X_1, X_2, \cdots, X_n)$ is

$$\pi\left(\beta|x\right) = \frac{\left(\sum\limits_{i=1}^{n} x_i\right)^{n\alpha}}{\Gamma\left(n\alpha\right)} e^{-\beta \sum\limits_{i=1}^{n} x_i} \beta^{n\alpha - 1}; \quad \beta \geq 0,\ \alpha > 0, \tag{4}$$

which is distributed as gamma distribution with parameters $n\alpha$ and $\sum\limits_{i=1}^{n} x_i$ i.e., $\beta \sim \Gamma\left(n\alpha,\ \sum\limits_{i=1}^{n} x_i\right)$. The mean of the posterior distribution function is $n\alpha / \sum\limits_{i=1}^{n} x_i$.

Since each X_i is distributed as gamma variate with parameters α and β, therefore the statistic $T = \sum_{i=1}^{n} X_i$ is distributed as gamma distribution with parameters $n\alpha$ and β i. e., $T \sim G(n\alpha,\ \beta)$. Hence the probability density function of T is given by

$$h(t) = \frac{\beta^{n\alpha}}{\Gamma(n\alpha)} e^{-\beta t} t^{n\alpha - 1} \quad t \geq 0,\ n,\ \alpha,\ \beta > 0.$$

Here, $E\left(\frac{1}{T}\right) = \frac{\beta}{(n\alpha-1)}$ and $E\left(\frac{1}{T^2}\right) = \frac{\beta^2}{(n\alpha-1)(n\alpha-2)}$. Now under the MLINEX loss function the Bayes estimator of β , using Wahed and Uddin [8], is given by

$$\hat{\beta}_{BML} = \left[E_\beta\left(\beta^{-c}\right)\right]^{-\frac{1}{c}},$$

where

$$\begin{aligned} E_\beta\left(\beta^{-c}\right) &= \int_0^\infty \beta^{-c} \pi(\beta|x) d\beta = \frac{\left(\sum_{i=1}^{n} x_i\right)^{n\alpha}}{\Gamma(n\alpha)} \int_0^\infty e^{-\beta \sum_{i=1}^{n} x_i} \beta^{n\alpha - c - 1} d\beta \\ &= \frac{\Gamma(n\alpha - c)}{\Gamma(n\alpha)} \left(\sum_{i=1}^{n} x_i\right)^c \end{aligned}$$

and hence $\hat{\beta}_{BML} = \frac{K}{T}$ is the Bayes estimator of β where $K = \left[\left(\frac{\Gamma(n\alpha)}{\Gamma(n\alpha-c)}\right)\right]^{\frac{1}{c}}$ and $T = \sum_{i=1}^{n} X_i$ is a complete sufficient statistic for β.

The risk function for the estimator $\hat{\beta}_{BML}$ under the MLINEX loss function is

$$R\left(\hat{\beta}_{BML}\right) = E\left[L\left(\hat{\beta}_{BML}, \beta\right)\right] = \varpi\left[\beta^{-c} E\left(\hat{\beta}_{BML}^c\right) - cE\left(\ln \hat{\beta}_{BML}\right) + c \ln \beta - 1\right]$$

Here, $E\left(\beta_{BML}^c\right) = E\left(\frac{K}{T}\right)^c = K^c E\left(T^{-c}\right) = K^c \int_0^\infty t^{-c} h(t) dt = \frac{\Gamma(n\alpha-c)}{\Gamma(n\alpha)} (K\beta)^c$, and $E\left(\ln \hat{\beta}_{BML}\right) = E\left[\ln\left(\frac{K}{T}\right)\right] = \ln K - E(\ln T)$.

$$E(\ln T) = \int_0^\infty \ln t\, h(t) dt = \frac{\beta^{n\alpha}}{\Gamma(n\alpha)} \int_0^\infty \ln t\, e^{-\beta t} t^{n\alpha-1} dt$$

Let $\beta t = y \quad \Rightarrow t = \frac{y}{\beta} \quad \therefore\ dt = \frac{1}{\beta} dy$.

Therefore,

$$\begin{aligned} E(\ln T) &= \frac{\beta^{n\alpha}}{\Gamma(n\alpha)} \int_0^\infty \ln\left(\frac{y}{\beta}\right) e^{-y} \left(\frac{y}{\beta}\right)^{n\alpha-1} \frac{1}{\beta} dy \\ &= -\ln \beta + \frac{1}{\Gamma(n\alpha)} \int_0^\infty \ln y e^{-y} y^{n\alpha-1} dy = -\ln\beta + \frac{\Gamma'(n\alpha)}{\Gamma(n\alpha)}; \end{aligned}$$

where $\Gamma'(n\alpha)$ is the first derivative of $\Gamma(n\alpha)$ with respect to $n\alpha$.

Therefore, $E\left(\ln \hat{\beta}_{BML}\right) = \ln K + \ln \beta - \frac{\Gamma'(n\alpha)}{\Gamma(n\alpha)}$. Using the results, the risk function for the estimator under MLINEX loss function is

$$R\left(\hat{\beta}_{BML}\right) = \varpi\left[\frac{c\Gamma'(n\alpha)}{\Gamma(n\alpha)} - \ln\left(\frac{\Gamma(n\alpha)}{\Gamma(n\alpha - c)}\right)\right]; \;\; c \neq 0,\; \varpi > 0,$$

which is constant with respect to β, as $n\alpha$ and c are known and independent of β.

Therefore, according to the Lehmann's theorem

$$\hat{\beta}_{BML} = \left(\frac{\Gamma(n\alpha)}{\Gamma(n\alpha - c)}\right)^{\frac{1}{c}} \frac{1}{\sum\limits_{i=1}^{n} X_i} = \hat{\beta}_{MML}$$

is the minimax estimator for the scale parameter β of the gamma distribution under MLINEX loss function and the Jefferey's non-informative prior density $\pi(\beta) \propto \frac{1}{\beta}$ is the least favourable prior density of β.

Now we are going to prove the Theorem 2.2. To prove the theorem we have to use Lehman's theorem 1.1 again. By using the posterior distribution (4), the Bayes estimator for the scale parameter β under the quadratic loss function is given by $\hat{\beta}_{BEQ} = \frac{(n\alpha-2)}{T}$, where $T = \sum\limits_{i=1}^{n} X_i$.

The risk function for the estimator $\hat{\beta}_{BEQ}$ under quadratic loss function is

$$\begin{aligned} R\left(\hat{\beta}_{BEQ}\right) &= E\left[L\left(\hat{\beta}_{BEQ}, \beta\right)\right] = \frac{1}{\beta^2} E\left[\left(\hat{\beta}_{BEQ} - \beta\right)^2\right] = \frac{1}{\beta^2} E\left[\frac{(n\alpha-2)}{T} - \beta\right]^2 \\ &= \frac{1}{\beta^2}\left[(n\alpha-2)^2 E\left(\frac{1}{T^2}\right) - 2(n\alpha-2)\beta E\left(\frac{1}{T}\right) + \beta^2\right] = \frac{1}{(n\alpha-1)}; \end{aligned}$$

which is constant with respect to β as n and α are known.

Therefore, according to the Lehmann theorem, $\hat{\beta}_{BEQ} = (n\alpha - 2)/\sum\limits_{i=1}^{n} X_i = \beta_{MEQ}$ is also a minimax estimator for the scale parameter β of the gamma distribution under quadratic loss function and the Jefferey's non-informative prior $\pi(\beta) \propto \frac{1}{\beta}$ is the least favourable prior density of β. The maximum likelihood estimator for the scale parameter β of the gamma distribution when α is known, is $\hat{\beta}_{MLE} = n\alpha/\sum\limits_{i=1}^{n} X_i$. For $\alpha = 1$, the estimators are the minimax estimators for the parameter β of the exponential distribution under MLINEX and quadratic loss functions respectively.

According to Wald [7] the following statistical problem is equivalent to some two-person zero-sum game between the Statistician (Player-II) and Nature (Player-I). Here the pure strategies of Nature are the different values of β in the interval $(0, \infty)$ and the mixed strategies of Nature are the prior densities of β in the interval $(0, \infty)$. The pure strategies of Statistician are all possible decision functions in the interval $(0, \infty)$.

Expectation of the loss function $L(\beta,\ d)$ is the risk function $R(\beta, d) = E[L(\beta, d)]$ which is the "gain" of player-I. $R(\xi, d)$ is the value of $\int_\beta R(\beta, d)d\xi(\beta)$, where $\xi(\beta)$ is the prior density of β. If the loss function is continuous in both d and β, and convex in d for each β, then there exist measures $\xi^\star$ and $d^\star$ for all β and d so that the following relation holds good

$$R\left(\xi, d^*\right) \leq R\left(\xi^*, d^*\right) \leq R\left(\xi^*, d\right)$$

. The number $R\left(\xi^*, d^*\right)$ is known to be the value of the game, ξ^* and d^* are the corresponding optimum strategies of Player-I and Player-II. In statistical terms ξ^* is the least favourable prior density of β and d^* is a minimax estimator of β. In fact, the value of the game is the loss of the Statistician.

Here it has been shown that, (i) $d^* = \hat{\beta}_{MML} = \left(\frac{\Gamma(n\alpha)}{\Gamma(n\alpha-c)}\right)^{\frac{1}{c}} \frac{1}{\sum\limits_{i=1}^{n} X_i}$ is the optimum strategy of Player-II for the MLINEX loss function and the value of the game is

$$R_M\left(\xi^*, d^*\right) = \varpi\left[\frac{c\Gamma'\left(n\alpha\right)}{\Gamma\left(n\alpha\right)} - \ln\left(\frac{\Gamma\left(n\alpha\right)}{\Gamma\left(n\alpha-c\right)}\right)\right];\ \ c \neq 0,\ \varpi > 0.$$

Again (ii) $d^* = \hat{\beta}_{MEQ} = \frac{(n\alpha-2)}{\sum\limits_{i=1}^{n} X_i}$ is the optimum strategy of Player-II for the quadratic loss function and the value of the game is $R_Q(\xi^*, d_1^*) = \frac{1}{(n\alpha-1)}$. In both the cases, $\xi^* = g\left(\beta\right) \propto \frac{1}{\beta};\ \ \beta > 0$, is the optimum strategy for Player-I.

3 Empirical Study

Mean squared-errors (MSEs) are considered to compare the different estimators for the scale parameter β of the gamma distribution under MLINEX and quadratic loss functions, and the method of maximum likelihood. The MSE of an estimator $\hat{\beta}$ is defined as

$$MSE\left(\hat{\beta}\right) = E\left(\hat{\beta} - \beta\right)^2 = Var\left(\hat{\beta}\right) + \left[Bias\left(\hat{\beta}\right)\right]^2 \tag{5}$$

It has been seen that for small sample size ($n < 25$), minimax estimator for quadratic loss function is the best among the three. Minimax estimators for MLINEX loss function are better than the maximum likelihood estimator. But for large sample sizes $n > 25$, all the three estimators have approximately the same mean-squared errors. The obtained results are demonstrated in Tables 1 to 6 and presented also in Figures 1 to 6.

Table 1: Estimated values and MSEs of different estimators for the scale parameter β of the gamma distribution when $\alpha = 1$, $\beta = 2$ and $c = 2$

Sample Size	Criteria	$\hat{\beta}_{MML}$	$\hat{\beta}_{MEQ}$	$\hat{\beta}_{MLE}$
5	Estimated value	0.298	0.258	0.431
	MSE	1.072	1.000	2.333
10	Estimated value	0.731	0.689	0.861
	MSE	0.458	0.444	0.667
15	Estimated value	1.162	1.120	1.292
	MSE	0.291	0.286	0.374
20	Estimated value	1.593	1.551	1.723
	MSE	0.213	0.211	0.257
25	Estimated value	2.024	1.981	2.154
	MSE	0.168	0.167	0.196
30	Estimated value	2.455	2.412	2.584
	MSE	0.139	0.138	0.158

Table 2: Estimated values and MSEs of different estimators for the scale parameter β of the gamma distribution when $\alpha = 1$, $\beta = 2$ and $c = 3$

Sample Size	Criteria	$\hat{\beta}_{MML}$	$\hat{\beta}_{MEQ}$	$\hat{\beta}_{MLE}$
5	Estimated value	0.207	0.215	0.359
	MSE	1.004	1.000	2.333
10	Estimated value	0.571	0.574	0.718
	MSE	0.445	0.444	0.667
15	Estimated value	0.931	0.933	1.077
	MSE	0.286	0.286	0.374
20	Estimated value	1.291	1.292	1.436
	MSE	0.211	0.211	0.257
25	Estimated value	1.650	1.651	1.795
	MSE	0.167	0.167	0.196
30	Estimated value	2.009	2.010	2.154
	MSE	0.138	0.138	0.158

Table 3: Estimated values and MSEs of different estimators for the scale parameter β of the gamma distribution when $\alpha = 1$, $\beta = 3$ and $c = 2$

Sample Size	Criteria	$\hat{\beta}_{MML}$	$\hat{\beta}_{MEQ}$	$\hat{\beta}_{MLE}$
5	Estimated value	0.414	0.358	0.597
	MSE	2.412	2.250	5.250
10	Estimated value	1.014	0.956	1.195
	MSE	1.029	1.000	1.500
15	Estimated value	1.612	1.553	1.792
	MSE	0.655	0.643	0.841
20	Estimated value	2.209	2.150	2.389
	MSE	0.480	0.474	0.579
25	Estimated value	2.806	2.747	2.986
	MSE	0.379	0.375	0.440
30	Estimated value	3.404	3.345	3.584
	MSE	0.313	0.310	0.355

Table 4: Estimated values and MSEs of different estimators for the scale parameter β of the gamma distribution when $\alpha = 1$, $\beta = 3$ and $c = 3$

Sample Size	Criteria	$\hat{\beta}_{MML}$	$\hat{\beta}_{MEQ}$	$\hat{\beta}_{MLE}$
5	Estimated value	0.246	0.256	0.426
	MSE	2.260	2.250	5.250
10	Estimated value	0.676	0.682	0.853
	MSE	1.000	1.000	1.500
15	Estimated value	1.106	1.108	1.279
	MSE	0.643	0.643	0.841
20	Estimated value	1.533	1.535	1.705
	MSE	0.474	0.474	0.579
25	Estimated value	1.960	1.961	2.131
	MSE	0.375	0.375	0.440
30	Estimated value	2.386	2.387	2.558
	MSE	0.310	0.310	0.355

Table 5: Estimated values and MSEs of different estimators for the scale parameter β of the gamma distribution when $\alpha = 2$, $\beta = 2$ and $c = 3$

Sample Size	Criteria	$\hat{\beta}_{MML}$	$\hat{\beta}_{MEQ}$	$\hat{\beta}_{MLE}$
5	Estimated value	0.604	0.607	0.379
	MSE	0.445	0.444	0.667
10	Estimated value	1.364	1.366	0.759
	MSE	0.211	0.211	0.257
15	Estimated value	2.123	2.124	1.138
	MSE	0.138	0.138	0.158
20	Estimated value	2.882	2.883	1.517
	MSE	0.103	0.103	0.113
25	Estimated value	3.641	3.641	1.897
	MSE	0.082	0.082	0.088
30	Estimated value	4.400	4.400	2.276
	MSE	0.068	0.068	0.073

Table 6: Estimated values and MSEs of different estimators for the scale parameter β of the gamma distribution when $\alpha = 2$, $\beta = 3$ and $c = 2$

Sample Size	Criteria	$\hat{\beta}_{MML}$	$\hat{\beta}_{MEQ}$	$\hat{\beta}_{MLE}$
5	Estimated value	1.053	0.993	0.621
	MSE	1.029	1.000	1.500
10	Estimated value	2.296	2.235	1.242
	MSE	0.480	0.474	0.579
15	Estimated value	3.538	3.476	1.862
	MSE	0.313	0.310	0.355
20	Estimated value	4.779	4.713	2.483
	MSE	0.232	0.231	0.255
25	Estimated value	6.021	5.959	3.104
	MSE	0.185	0.184	0.199
30	Estimated value	7.262	7.201	3.725
	MSE	0.153	0.153	0.163

Figure 1: MSEs of different estimators for the scale parameter of the gamma distribution when alpha=1, beta=2 and c=2.

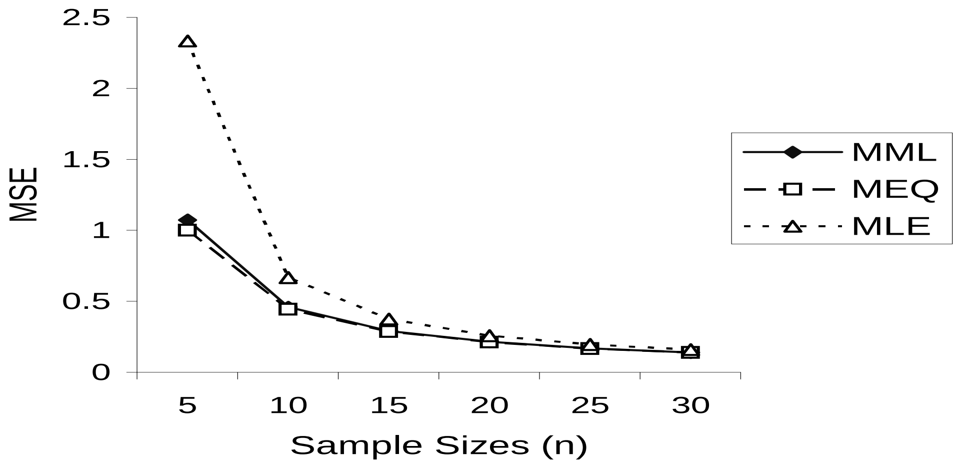


Figure 2: MSEs of different estimators for the scale parameter of the gamma distribution when alpha=1, beta=2 and c=3.

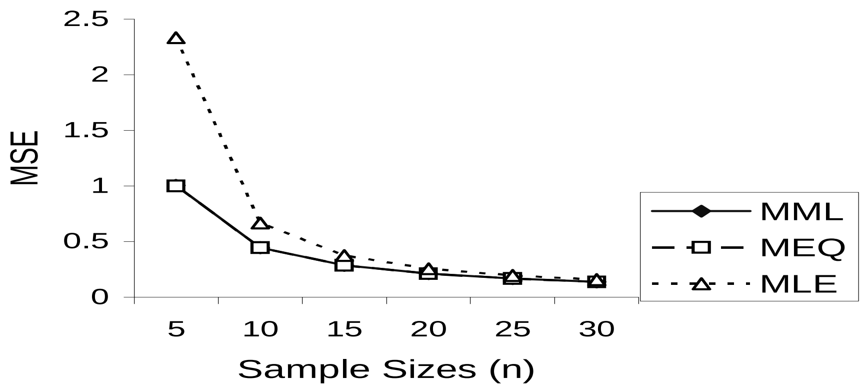


Figure 3: MSEs of different estimators for the scale parameter of the gamma distribution when alpha=1, beta=3 and c=2.

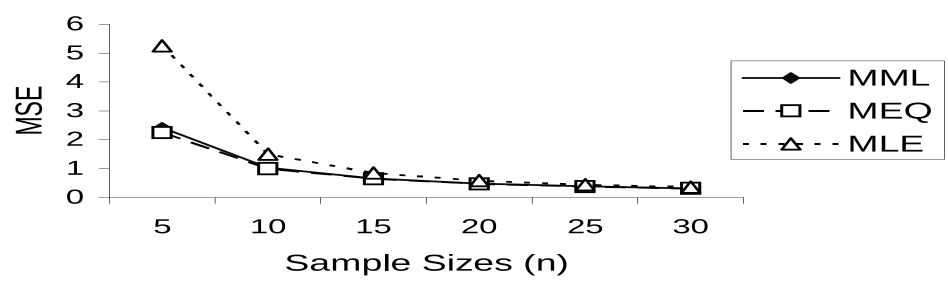


Figure 4: MSEs of different estimators for the scale parameter of the gamma distribution when alpha=1, beta=3 and c=3.

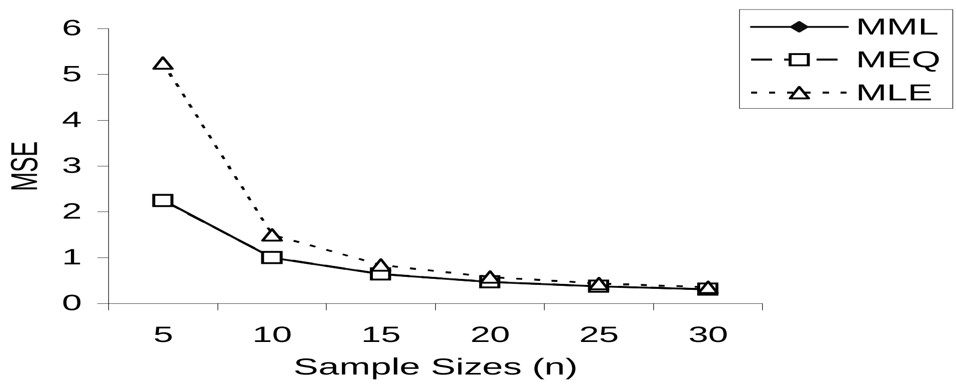


Figure 5: MSEs of different estimators for the scale parameter of the gamma distribution when alpha=2, beta=2 and c=3.

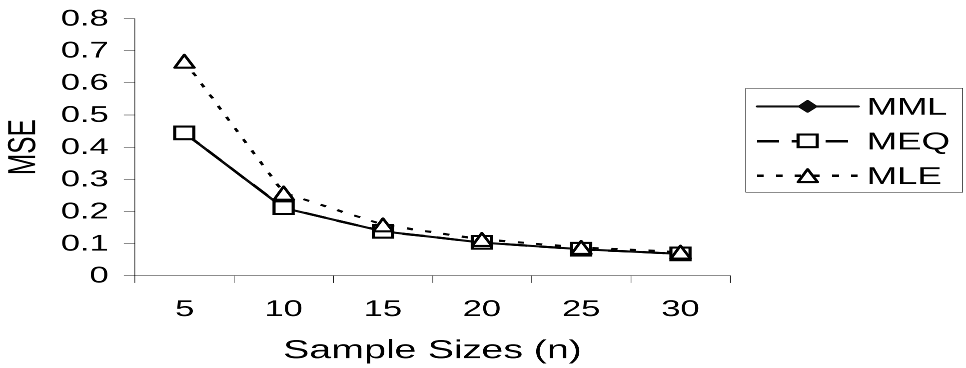
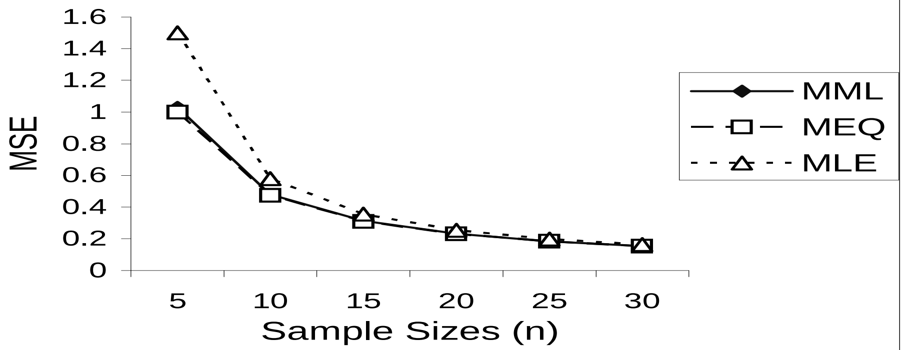


Figure 6: MSEs of different estimators for the scale parameter of the gamma distribution when alpha=2, beta=3 and c=2.



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