International Journal of Statistical Sciences *ISSN 1683–5603*
Vol. 3 (Special Issue), 2004, pp 119–125

Inference on Reliability $P(Y < X)$ in Two-Parameter Exponential Distribution

M. Masoom Ali
George and Frances Ball Distinguished Professor of Statistics and Professor of Mathematical Sciences
Ball State University
Muncie, IN 47306 USA

Jungsoo Woo
Professor, Department of Mathematics and Statistics
Yeungnam University, Gyongsan,
South Korea

Manisha Pal
Professor, Department of Statistics
University of Calcutta, Kolkata
India

[Received April 12, 2004; Accepted June 23, 2004]

Abstract

In this paper, we consider the point and interval estimation of $P(Y < X)$ in two-parameter exponential distribution when the scale parameter is either known or unknown including a test of hypothesis.

Keywords and Phrases: Reliability; Right-tail probability; MSE.

AMS Classification: Primary: 62F30; Secondary: 62N05.

1 Introduction

A two-parameter exponential distribution is given by

$$f(x;\mu,\sigma) = \frac{1}{\sigma}e^{-(x-\mu)/\sigma},\ x > \mu,\ \sigma > 0,\ \mu \in R. \tag{1}$$

The problem of estimating and of drawing inferences about the probability that a random variable Y is less than an independent random variable X arises in reliability studies. When Y represents the random value of a stress that a device will be subjected to in-service and X represents the strength that varies from item to item in the population of devices, then the reliability R, i.e., the probability that a randomly selected device functions successfully is equal to $P(Y < X)$. The same problem also arises in the context of statistical tolerence where Y represents, say, the diameter of a shaft and X the diameter of a bearing that is to be mounted on the shaft. The probability that the bearing fits without interference is then $P(Y < X)$.

In biometry, Y represents a patient's remaining years of life if treated with drug A and X represents the patient's remaining years when treated with drug B. If the choice of drug is left to the patient, person's deliberations will center on whether $P(Y < X)$ is less than or greater than 1/2.

The probability that a Weibull random variable Y is less than another independent Weibull random variable X was considered by McCool (1991). Many other authors have considered the problem for a number of other distributions.

In this paper, we consider the inference problem on $P(Y < X)$ in two-parameter exponential distribution when the scale parameters are known or not known, which includes point and interval estimation and a test of hypothesis.

2 Inference on $P(Y < X)$

Let X and Y be independent two-parameter exponential random variables with parameters (μ_x, σ_x) and (μ_y, σ_y), respectively. Then

$$\begin{aligned} P(Y < X) &= \int\int_{\mu_y < y < x} f_Y(y; \mu_y, \sigma_y) f_X(x; \mu_x, \sigma_x) dy dx \\ &= \begin{cases} \frac{\rho}{\rho+1} e^{-\delta/\sigma_x}, & \delta \geq 0, \\ 1 - \frac{1}{\rho+1} e^{\delta/\sigma_y}, & \delta < 0, \end{cases} \end{aligned} \tag{2}$$

where $\rho = \sigma_x/\sigma_y$, $\delta = \mu_y - \mu_x$, and $f_X(x)$ and $f_Y(y)$ are the pdf's of X and Y, respectively.

To consider inference on $P(Y < X)$, assume $X_1, X_2, \ldots, X_m$ and $Y_1, Y_2, \ldots, Y_n$ be two independent random samples from two-parameter exponential distributions with parameters (μ_x, σ_x) and (μ_y, σ_y), respectively. Then the MLE $\hat{\delta}$ of δ is

$$\hat{\delta} = \hat{\mu}_y - \hat{\mu}_x = Y_{(1)} - X_{(1)} \equiv D, \tag{3}$$

where $X_{(1)}$ and $Y_{(1)}$ are the first order statistics of X_i's and Y_i's, respectively.

By the results in Johnson *et al* (1970), we have the following Fact 1.

Fact 1: (a) $X_{(1)}$ follows an exponential distribution with a location parameter μ_x and scale parameter σ_x/m.

(b) If $X_1, X_2, \ldots, X_m$ are iid exponential distributions with a scale parameter σ_x and a location parameter μ_x, then $\sum_{i=1}^{m}(X_i - X_{(1)})$ follows a gamma distribution with a shape $m-1$ and scale σ_x.
(c) If a random variable X follows a gamma distribution with a shape α and a scale σ, then $E\left(\frac{1}{X^k}\right) = \frac{\Gamma(\alpha-k)}{\Gamma(\alpha)\sigma^k}$, for $\alpha > k$.

From Fact 1(a), we can obtain the expectation and variance of $\hat{\delta} \equiv D$:

$$\begin{aligned} E(D) &= \delta + \frac{\sigma_y}{n} - \frac{\sigma_x}{m} \\ Var(D) &= \frac{\sigma_x^2}{m^2} + \frac{\sigma_y^2}{n^2}. \end{aligned} \tag{4}$$

Then we can obtain the pdf of D as follows.

$$f_D(d) = \begin{cases} \frac{mn}{n\sigma_x + m\sigma_y} e^{-\frac{n}{\sigma_y}(d-\delta)}, & \text{if } d \geq \delta \\ \frac{mn}{n\sigma_x + m\sigma_y} e^{-\frac{m}{\sigma_x}(\delta-d)}, & \text{if } d < \delta \end{cases} \tag{5}$$

2.1 When the scale parameters $\sigma_x = \sigma_y = \sigma_0$ is known

From the result (2),

$$R \equiv P(Y < X) = \begin{cases} \frac{1}{2}e^{-\delta/\sigma_0}, & \delta \geq 0 \\ 1 - \frac{1}{2}e^{\delta/\sigma_0}, & \delta < 0. \end{cases}$$

Then the probability depends on δ only. Because R is a monotone function in δ, inference on δ is equivalent to inference on R. We hereafter confine attention to the parameter δ (see McCool (1991)).

When the scale parameters $\sigma_x = \sigma_y = \sigma_0$ and σ_0 is known, let $T = D - \delta$. Then from the pdf (5) of D, we have the following pdf of T.

$$f_T(t) = \begin{cases} \frac{m}{m+n} \cdot \frac{n}{\sigma_0} e^{-\frac{n}{\sigma_0}t}, & \text{if } t \geq 0 \\ \frac{n}{m+n} \cdot \frac{m}{\sigma_0} e^{\frac{m}{\sigma_0}t}, & \text{if } t < 0. \end{cases} \tag{6}$$

Based on a pivotal quantity T, we shall consider an $(1 - p_1 - p_2)100\%$ confidence interval of δ. For a given p_1, there exists a b_1 such that

$$p_1 = \int_{-\infty}^{b_1} \frac{n}{m+n} \cdot \frac{m}{\sigma_0} e^{\frac{m}{\sigma_0}t} dt$$

and hence

$$b_1 = \frac{\sigma_0}{m} \ln[(m+n)p_1/n]. \tag{7}$$

For another $0 < p_2 < 1$, there exists a b_2 such that

$$p_2 = \int_{b_2}^{\infty} \frac{m}{m+n} \cdot \frac{n}{\sigma_0} e^{-\frac{n}{\sigma_0}t} dt$$

and hence

$$b_2 = -\frac{\sigma_0}{n} \ln[(m+n)p_2/m]. \tag{8}$$

Next we wish to test the null hypothesis $H_0 : \mu_x = \mu_y$ against $H_1 : \mu_x \neq \mu_y$. Let $\Theta = \{(\mu_x, \mu_y) | \mu_x \in R, \mu_y \in R\}$ and $\theta = (\mu_x, \mu_y)$. Then the joint pdf of $(X_1, X_2, \ldots, X_m, Y_1, Y_2, \ldots, Y_n)$ is

$$L(\theta) = f_\theta(x, y) = \prod_{i=1}^{m} \frac{1}{\sigma_0} e^{-(x_i - \mu_x)/\sigma_0} \prod_{i=1}^{n} \frac{1}{\sigma_0} e^{-(y_i - \mu_y)/\sigma_0}, \text{ for all } x_i > \mu_x,\ y_i > \mu_y.$$

From the likelihood function, we can obtain the MLE's of μ_x and μ_y as

$$\hat{\mu}_x = X_{(1)} \text{ and } \hat{\mu}_y = Y_{(1)}.$$

If $\mu_x = \mu_y = \mu$, then the MLE of μ is

$$\hat{\mu} = \min(X_{(1)}, Y_{(1)}) = (Y_{(1)} + X_{(1)} - |Y_{(1)} - X_{(1)}|)/2.$$

From definition of likelihood ratio test (Rohatgi (1976)), the likelihood ratio test function can be obtained as

$$\Lambda(x, y) = \exp(-|D|(\frac{m}{2\sigma_0} + \frac{n}{2\sigma_0}) + D(\frac{m}{2\sigma_0} - \frac{n}{2\sigma_0})), \text{ where } D = Y_{(1)} - X_{(1)}.$$

Therefore,

$$\Lambda(x, y) < \lambda \Leftrightarrow D < b_1 \text{ or } D > b_2. \tag{9}$$

Under $H_0 : \mu_x = \mu_y$, i.e., $\delta = 0$, $T = D - \delta = D$, and hence, for given $0 < \alpha < 1$, we can find b_1 and b_2 of (9) through the results (7) and (8) by substituting $p_1 = p_2 = \alpha/2$. This yields $b_1 = \frac{\sigma_0}{m} \ln[(m+n)\alpha/2n]$ and $b_2 = -\frac{\sigma_0}{n} \ln[(m+n)\alpha/2m]$.

2.2 When the scale parameters σ_x and σ_y are unknown

We first develop a test for the null hypothesis $H_0 : \sigma_x = \sigma_y = \sigma$, σ unknown against the alternative hypothesis $H_1 : \sigma_x \neq \sigma_y, \mu_x \in R, \mu_y \in R$. Let $\Theta = \{(\sigma_x, \sigma_y, \mu_x, \mu_y) | \sigma_x > 0, \sigma_y > 0, \mu_x \in R, \mu_y \in R\}$ and $\theta = (\sigma_x, \sigma_y, \mu_x, \mu_y)$.
Then the joint pdf of $(X_1, X_2, \ldots, X_m, Y_1, Y_2, \ldots, Y_n)$ is

$$L(\theta) = f_\theta(x, y) = \prod_{i=1}^{m} \frac{1}{\sigma_x} e^{-(x_i - \mu_x)/\sigma_x} \prod_{i=1}^{n} \frac{1}{\sigma_y} e^{-(y_i - \mu_y)/\sigma_y}, \text{ for all } x_i > \mu_x,\ y_i > \mu_y.$$

Differentiating the likelihood function with respect to σ_x and σ_y, we obtain the MLE's as

$$\hat{\sigma}_x = \frac{1}{m}\sum_{i=1}^{m}(X_i - X_{(1)}),\ \hat{\sigma}_y = \frac{1}{n}\sum_{i=1}^{n}(Y_i - Y_{(1)}),\ \text{and}\ \hat{\mu}_x = X_{(1)},\ \hat{\mu}_y = Y_{(1)}.$$

If $\sigma_x = \sigma_y = \sigma$, then the MLE of σ is

$$\hat{\sigma} = \frac{1}{n+m}\left(\sum_{i=1}^{m}(X_i - X_{(1)}) + \sum_{i=1}^{n}(Y_i - Y_{(1)})\right). \tag{10}$$

From the definition of likelihood ratio test, the likelihood ratio test function can be obtained as

$$\Lambda(x,y) = \left(\frac{\hat{\sigma}_x}{\hat{\sigma}}\right)^m \cdot \left(\frac{\hat{\sigma}_y}{\hat{\sigma}}\right)^n = \left(\frac{m+n}{m}\right)^m \cdot \left(\frac{m+n}{n}\right)^n \cdot \left(\frac{1}{1+1/U}\right)^m \cdot \left(\frac{1}{1+U}\right)^n,$$

where $U \equiv \frac{\sum_{i=1}^{m}(X_i - X_{(1)})}{\sum_{i=1}^{n}(Y_i - Y_{(1)})}$. Therefore,

$$\Lambda(x,y) < c \Leftrightarrow U < u_1 \ \text{or}\ U > u_2. \tag{11}$$

From Fact 1(b) and Rohatgi (1976), we have the following Fact 2.

Fact 2: (a) $Z \equiv \frac{2\sum_{i=1}^{m}(X_i - X_{(1)})}{\sigma_x}$ and $W \equiv \frac{2\sum_{i=1}^{n}(Y_i - Y_{(1)})}{\sigma_y}$ follow chi-square distributions with $2(m-1)$ and $2(n-1)$ degrees of freedom, respectively. (b) The random variables Z and W are independent.

Under $H_0 : \sigma_x = \sigma_y = \sigma$, From Fact 2, $U = \frac{\sum_{i=1}^{m}(X_i - X_{(1)})/(m-1)}{\sum_{i=1}^{n}(Y_i - Y_{(1)})/(n-1)}$ follows an F-distribution with $2(m-1)$ and $2(n-1)$ degrees of freedom. Hence for a given $0 < \alpha < 1$, from (11),

$$u_2 = F_{\alpha/2}(2(m-1), 2(n-1)) \ \text{and}\ u_1 = 1/F_{\alpha/2}(2(n-1), 2(m-1)).$$

If the null hypothesis is accepted, then $R = P(Y < X)$ is a monotone function of $\beta \equiv \delta/\sigma$ and is given by the following.

$$R \equiv P(Y < X) \ = \ \begin{cases} \frac{1}{2}e^{-\delta/\sigma}, & \delta \geq 0 \\ 1 - \frac{1}{2}e^{\delta/\sigma}, & \delta < 0. \end{cases}$$

Then the plug-in estimator of β, using results in (3) and (10) is obtained as

$$\hat{\beta} = D/\hat{\sigma} = \frac{(m+n)(Y_{(1)} - X_{(1)})}{\sum_{i=1}^{m}(X_i - X_{(1)}) + \sum_{i=1}^{n}(Y_i - Y_{(1)})}.$$

From the results (4) and Fact 1(c), we obtain the following.

$$E(\hat{\beta}) = \beta + \frac{3}{m+n-3}\beta + \frac{m^2 - n^2}{mn(m+n-3)}$$

and

$$Var(\hat{\beta}) = \frac{(m+n)^2(m^2+n^2)}{m^2n^2(m+n-3)^2(m+n-4)}\beta^2.$$

Hence, a large sample test for R can be suggested in terms of β in a routine manner.

2.3 When $\mu_x = \mu_y = \mu_0$ and μ_0 is known

From the result (2), $R = P(Y < X) = \rho/(1+\rho)$ is a monotone functione of ρ. Hence, inference on ρ is equivalent to the inference on R (see, McCool (1991)).

The MLE of ρ is given by

$$\hat{\rho} = \frac{n}{m}\frac{\sum_{i=1}^{m}(X_i - \mu_0)}{\sum_{i=1}^{n}(Y_i - \mu_0)}.$$

From Fact 1, we can obtain the expectation and variance of $\hat{\rho}$ as

$$E(\hat{\rho}) = \frac{n}{n-1}\rho \text{ and } Var(\hat{\rho}) = \frac{n}{m(n-1)^2(n-2)}\rho^2,\ n > 2.$$

Define

$$\tilde{\rho} = \frac{n-1}{m}\frac{\sum_{i=1}^{m}(X_i - \mu_0)}{\sum_{i=1}^{n}(Y_i - \mu_0)}.$$

Then $\tilde{\rho}$ is an unbiased estimator of ρ and $Var(\tilde{\rho}) = \frac{1}{mn(n-2)}\rho^2$, $n > 2$.

Then we have the following Fact 3.

Fact 3: An unbiased estimator $\tilde{\rho}$ has less MSE than the MLE $\hat{\rho}$ for $n > 2$.

Next, we wish to test the following hypothesis $H_0 : \sigma_x = \sigma_y (= \sigma)$ against the alternative $H_1 : \sigma_x \neq \sigma_y$. As in the preceding section, the MLE's of σ_x and σ_y are $\hat{\sigma}_x = \frac{1}{m}\sum_{i=1}^{m}(X_i - \mu_0)$ and $\hat{\sigma}_y = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \mu_0)$, respectively. If the null hypothesis is true, then the MLE of σ is given by $\hat{\sigma} = \frac{1}{m+n}(\sum_{i=1}^{m}(X_i - \mu_0) + \sum_{i=1}^{n}(Y_i - \mu_0))$. From the likelihood ratio test and using the same argument of the preceding section, the critical region for the test is given by $\Lambda(x, y) < c \Leftrightarrow W < w_1$, or $W > w_2$, where $W = \sum_{i=1}^{m}(X_i - \mu_0)/\sum_{i=1}^{n}(Y_i - \mu_0)$. If the null hypothesis is true, then the test statistic W follows an F-distribution with $(2m, 2n)$ degrees of freedom. Therefore, for a given $0 < \alpha < 1$, the critical points are given by $w_1 = 1/F_{\alpha/2}(2n, 2m)$ and $w_2 = F_{\alpha/2}(2m, 2n)$.

3 Concluding Remarks

In this article, we have developed inference procedures for R without explicitly deriving the form of UMVUE for R. We propose to undertake this and other related studies in a forthcoming article.

Acknowledgement

The authors thank an anonymous referee for carefully reading the manuscript and suggesting some necessary changes.

References

[1] Johnson, N.L. and Kotz, S. (1970). *Continuous Univariate Distributions - 2.* Houghton Mifflin Company, Boston.

[2] McCool, J.I. (1991). Inference on $P(X < Y)$ in the Weibull case. *Commun. Statist.-Simula.*, 20(1), 129-148.

[3] Rohatgi, V.K. (1976). *An Introduction to Probability Theory and Mathematical Statistics.* John Wiley and Sons, Inc., New York.

BLANK PAGE