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On a comparison of three estimators of binomial variance by multiple criteria decision making method

Satinee Lertprapai *
Department of Mathematics
Faculty of Science, Mahidol University
Bangkok, Thailand

Montip Tiensuwan †
Department of Mathematics
Faculty of Science, Mahidol University
Bangkok, Thailand

Bimal K. Sinha ‡
Department of Mathematics and Statistics
University of Maryland
Baltimore County, USA

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Abstract

In this paper we consider the problem of estimation of $\theta\,(1-\theta)$ based on $X \sim B(n,\theta)$, n being known and $0<\theta<1$, θ being unknown. We compare three standard estimators $T_1 = \frac{X}{n}\left(1-\frac{X}{n}\right)$, $T_2 = \frac{X(n-X)}{n\,(n-1)}$, and $T_3 = \frac{X(n-X)+n\sqrt{n}/2+n/4}{(n+\sqrt{n})^2}$ on the basis of Multiple Criteria Decision Making (MCDM) procedure. MCDM is a novel statistical procedure to compare several competing estimators of a parameter. It turns out that our preference is mostly for T_1.

Keywords and Phrases: Binomial distribution, variance, minimax, multiple criteria decision making.

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1 Introduction

We consider the problem of estimation of $\theta(1-\theta)$ based on $X \sim B(n,\theta)$. Here n is known and $0 < \theta < 1$, θ being unknown. It is well known that there are three standard estimators of $\theta(1-\theta)$, namely, $T_1 = \frac{X}{n}\left(1-\frac{X}{n}\right)$, the maximum likelihood estimate [1], $T_2 = \frac{X(n-X)}{n(n-1)}$, the minimum variance unbiased estimate, and $T_3 = \frac{X(n-X)+n\sqrt{n}/2+n/4}{(n+\sqrt{n})^2}$, based on the minimax estimate of θ. In this paper we compare T_1, T_2 and T_3 on the basis of Multiple Criteria Decision Making (MCDM) method. This method is briefly described in Section 2 and Section 3 contains the main results of this paper. It turns out that most often T_1 is the preferred choice. For detailed discussions on MCDM, we refer to Zeleny [5].

2 A brief description of MCDM procedure

In the context of a '*discrete*' data matrix $X = (x_{ij}) : K \times N$ where $x_{ij}'s$ represent '*risk*' of ith '*source*' for jth '*category*', and we need to compare the K rows simultaneously with respect to all the N columns, MCDM is a novel statistical procedure to integrate the multiple indicators $(x_{i1}, \ldots, x_{iN})$ for row i across all indicators into a single meaningful and overall index. This is done by defining an Ideal Row (IDR) with the smallest observed value for each column as

$$IDR = (\min_i x_{i1}, \ldots, \min_i x_{iN}) = (u_1, \ldots, u_N), \text{ say}$$

and a Negative-ideal Row (NIDR) with the largest observed value for each column as

$$NIDR = (\max_i x_{i1}, \ldots, \max_i x_{iN}) = (v_1, \ldots, v_N), \text{ say.}$$

For any given row$i$,we now compute the distance of each row from Ideal row and from Negative Ideal row based on a suitably chosen norm. Under L_1-norm, we compute

$$L_1(i, IDR) = \sum_{j=1}^{N} |x_{ij}-u_j|\, w_j = \sum_{j=1}^{N} [x_{ij}-u_j]\; w_j$$

$$L_1(i, NIDR) = \sum_{j=1}^{N} |x_{ij}-v_j|\, w_j = \sum_{j=1}^{N} [v_j-x_{ij}]\; w_j$$

where $w_j's$ are appropriate weights. The various rows are now compared based on an overall index computed as

$$L_1(Index_i) = \frac{L_1(i, IDR)}{L_1(i, IDR) + L_1(i, NIDR)}, \quad i = 1, \ldots, K. \qquad (1)$$

Similarly, under L_2-norm, we compute

$$L_2(i\,,IDR) \quad = \quad [\sum_{j=1}^{N} (x_{ij} - u_j)^2 w_j]^{1/2}$$

$$L_2(i\,,NIDR) \quad = \quad [\sum_{j=1}^{N} (x_{ij} - v_j)^2 w_j]^{1/2}$$

and compare the rows based on

$$L_2(Index_{\,i}) \quad = \quad \frac{L_2(i\,,\,IDR)}{L_2(i\,,\,IDR)\ +\ L_2(i\,,\,NIDR)}, \quad i\ =\ 1\,,\ \ldots\ ,K. \tag{2}$$

A '*continuous*' version of this setup would involve $x_{ij}\,'s$ where the index j would vary '*continuously*'. In the context of our problem of comparing T_1 , T_2 and T_3 for estimation of $\theta\,(1-\theta\,)$, obviously $K = 3$, $x_{ij}\,'s$ are chosen to represent the mean squared errors of T_1 , T_2 and T_3 for various values of θ, and L_1-norm and L_2-norm would be redefined as

$$L_1(i\,,IDR) \quad = \quad \int_0^1 [x_i(\theta) - u(\theta)]\ w\,(\theta)\,d\theta \tag{3}$$

$$L_1(i\,,NIDR) \quad = \quad \int_0^1 [v\,(\theta) - x_i(\theta)]\ w\,(\theta)\,d\theta \tag{4}$$

$$L_2(i\,,IDR) \quad = \quad \sqrt{\int_0^1 (x_i(\theta) - u(\theta)\,)^2 w\,(\theta)\,d\theta} \tag{5}$$

$$L_2(i\,,NIDR) \quad = \quad \sqrt{\int_0^1 (x_i(\theta) - v\,(\theta)\,)^2 w\,(\theta)\,d\theta} \tag{6}$$

where $u(\theta) = \min_i \{x_i(\theta)\}$ and $v(\theta) = \max_i \{x_i(\theta)\}$.

3 Main Results

We first start with the mean squared errors of T_1, $T_2$and T_3, given below. For details of derivation, we refer to Technical Report [3].

$$\text{MSE}\,(T_1) \quad = \quad B_1(n)\,\theta + C_1(n)\,\theta^2\ + D_1(n)\,\theta^3\ +\ E_1(n)\,\theta^4 \tag{7}$$

where $B_1(n) = \dfrac{n\,(n-1)^2}{n^4}$, $C_1(n) = \dfrac{(5n-7)\,(n-n^2)+n^2}{n^4}$,

$D_1(n) = \dfrac{(2n-3)\,(4n^2-4n)-2n^2}{n^4}$ and $E_1(n) = \dfrac{(2n-3)\,(2n-2n^2)+n^2}{n^4}$.

$$\text{MSE}\,(T_2) = B_2(n)\,\theta + C_2(n)\,\theta^2 + D_2(n)\,\theta^3 + E_2(n)\,\theta^4 \qquad (8)$$

where $B_2(n) = \dfrac{1}{n}$, $C_2(n) = \dfrac{(7-5n)}{n\,(n-1)}$, $D_2(n) = \dfrac{4(2n-3)}{n\,(n-1)}$ and $E_2(n) = \dfrac{-2\,(2n-3)}{n\,(n-1)}$.

$$\text{MSE}\,(T_3) = A_3(n) + B_3(n)\,\theta + C_3(n)\,\theta^2 + D_3(n)\,\theta^3 + E_3(n)\,\theta^4 \qquad (9)$$

where $A_3(n) = \dfrac{(n\sqrt{n}/2+n/4)^2}{(n+\sqrt{n})^4}$,

$B_3(n) = \dfrac{-2(2n+2n\sqrt{n})(n\sqrt{n}/2+n/4)+n\,(n-1)^2}{(n+\sqrt{n})^4}$,

$C_3(n) = \dfrac{2(2n+2n\sqrt{n})(n\sqrt{n}/2+n/4)+(2n+2n\sqrt{n})^2-n\,(5n-7)(n-1)}{(n+\sqrt{n})^4}$,

$D_3(n) = \dfrac{-2\,(2n+2n\sqrt{n})^2+4n(2n-3)(n-1)}{(n+\sqrt{n})^4}$

and $E_3(n) = \dfrac{(2n+2n\sqrt{n})^2-2n\,(2n-3)(n-1)}{(n+\sqrt{n})^4}$.

Writing $x_1(\theta) = \text{MSE}(T_1)$, $x_2(\theta) = \text{MSE}(T_2)$ and $x_3(\theta) = \text{MSE}(T_3)$, we present in Figure 1 their graphical patterns for n = 5,10,15,20. It is interesting to note the bimodal nature of $x_1(\theta)$and $x_2(\theta)$, and convex nature of $x_3(\theta)$.

Since $0 < \theta < 1$, the intersection of three graphs can separate the interval of θ into seven intervals $(0 < c_1(n) < c_2(n) < c_3(n) < c_4(n) < c_5(n) < c_6(n) < 1)$. Obviously, MSE($T_1$)= MSE($T_2$) holds whenever $\theta = c_3(n),\ c_4(n)$ where

$$c_3(n) = \frac{6-17n+9n^2-\sqrt{12-64n+109n^2-62n^3+9n^4}}{2\,(6-17n+9n^2)}$$

and

$$c_4(n) = \frac{6-17n+9n^2+\sqrt{12-64n+109n^2-62n^3+9n^4}}{2\,(6-17n+9n^2)}.$$

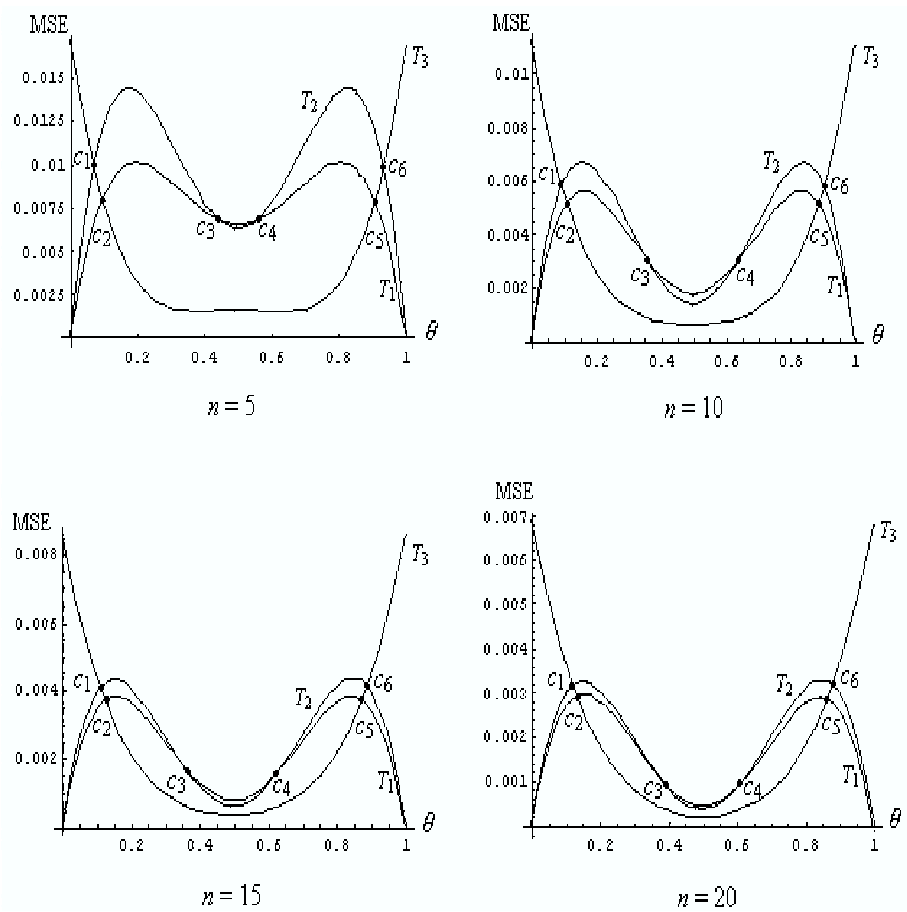

Figure 1: Graphical illustration of mean squared errors for n = 5,10,15, 20.

Likewise, MSE(T_1)= MSE(T_3) holds whenever $\theta = c_2(n),\ \ c_5(n)$ where

$$
\begin{aligned}
c_2(n) \quad = \quad & 24+48\sqrt{n}+4n-88n^{3/2}-72n^2+32n^{5/2}+44n^3+8n^{7/2}- \\
& \Big((-24-48\sqrt{n}-4n+88n^{3/2}+72n^2-32n^{5/2}-44n^3-8n^{7/2})^2- \\
& 4(24+48\sqrt{n}+4n-88n^{3/2}-72n^2+32n^{5/2}+44n^3+8n^{7/2}) \\
& (2+4\sqrt{n}-8n^{3/2}-6n^2+4n^{5/2}+6n^3+2n^{7/2}- \\
& \Big(4+16\sqrt{n}+16n-32n^{3/2}-88n^2-32n^{5/2}+114n^3+ \\
& 128n^{7/2}-37n^4-124n^{9/2}-26n^5+52n^{11/2}+25n^6\Big)^{\ 1/2})\Big)^{1/2}\ / \\
& (\,2\,(24+48\sqrt{n}+4n-88n^{3/2}-72n^2+32n^{5/2}+44n^3+8n^{7/2}))
\end{aligned}
$$

and

$$\begin{aligned} c_5(n) &= 24+48\sqrt{n}+4n-88n^{3/2}-72n^2+32n^{5/2}+44n^3+8n^{7/2}+ \\ &\Big((-24-48\sqrt{n}-4n+88n^{3/2}+72n^2-32n^{5/2}-44n^3-8n^{7/2})^2- \\ &4(24+48\sqrt{n}+4n-88n^{3/2}-72n^2+32n^{5/2}+44n^3+8n^{7/2}) \\ &(2+4\sqrt{n}-8n^{3/2}-6n^2+4n^{5/2}+6n^3+2n^{7/2}- \\ &\Big(4+16\sqrt{n}+16n-32n^{3/2}-88n^2-32n^{5/2}+114n^3+ \\ &128n^{7/2}-37n^4-124n^{9/2}-26n^5+52n^{11/2}+25n^6\Big)^{1/2})\Big)^{1/2} / \\ &(2\,(24+48\sqrt{n}+4n-88n^{3/2}-72n^2+32n^{5/2}+44n^3+8n^{7/2})). \end{aligned}$$

Lastly, MSE(T_2)= MSE(T_3) holds whenever $\theta = c_1(n),\ \ c_6(n)$ where

$$\begin{aligned} c_1(n) &= -48-56\sqrt{n}+24n+40n^{3/2}+8n^2-((48+56\sqrt{n}-24n-40n^{3/2}-8n^2)^2 \\ &-4(-48-56\sqrt{n}+24n+40n^{3/2}+8n^2)(-4-5\sqrt{n}+2n+5n^{3/2}+2n^2 \\ &-\sqrt{16+34\sqrt{n}-16n-78n^{3/2}-24n^2+44n^{5/2}+24n^3}\,))^{1/2} / \\ &(2\,(-48-56\sqrt{n}+24n+40n^{3/2}+8n^2)) \end{aligned}$$

and

$$\begin{aligned} c_6(n) &= -48-56\sqrt{n}+24n+40n^{3/2}+8n^2+((48+56\sqrt{n}-24n-40n^{3/2}-8n^2)^2 \\ &-4(-48-56\sqrt{n}+24n+40n^{3/2}+8n^2)(-4-5\sqrt{n}+2n+5n^{3/2}+2n^2 \\ &-\sqrt{16+34\sqrt{n}-16n-78n^{3/2}-24n^2+44n^{5/2}+24n^3}\,))^{1/2} / \\ &(2\,(-48-56\sqrt{n}+24n+40n^{3/2}+8n^2)). \end{aligned}$$

Moreover, the Ideal row and Negative-ideal row are as follows :

$$IDR\,:\,u(\theta\,)=\{\,x_1(\theta)\,:\,\theta<c_2(n)\,,\ x_3(\theta)\,:\,c_2(n)<\theta<c_5(n),\ x_1(\theta)\,:\,\theta>c_5(n)\,\}\,. \tag{10}$$

$$\begin{aligned} NIDR\,:\,v(\theta\,) &= \{\,x_3(\theta):\theta<c_1(\,n),\ x_2(\theta)\,:\,c_1(n)<\theta<c_3(n)\,, \\ &\quad x_1(\theta)\,:\,c_3(n)<\theta<c_4(n), x_2(\theta)\,:\,c_4(n)<\theta<c_6(n)\,, \\ &\quad x_3(\theta):\theta>c_6(n)\}\,. \end{aligned} \tag{11}$$

Since we are dealing with a continuous parameter θ, $0<\theta<1$, a proper formulation of the MCDM procedure as described earlier in (3)-(6) can be given as follows.

3.1 Analysis based on L_1-norm

For $i = 1$, applying equations (3) and (4), we get

$$L_1(1, IDR) = \int_{c_2(n)}^{c_5(n)} (x_1(\theta) - x_3(\theta))\, w(\theta)\, d\theta\,,$$

$$L_1(1, NIDR) = \int_{\theta<c_1(n)\cup\theta>c_6(n)} (x_3(\theta) - x_1(\theta))\, w(\theta)\, d\theta \;+$$

$$\int_{c_1(n)}^{c_3(n)} (x_2(\theta) - x_1(\theta))\, w(\theta)\, d\theta + \int_{c_4(n)}^{c_6(n)} (x_2(\theta) - x_1(\theta))\, w(\theta)\, d\theta\,.$$

For $i = 2$, applying equations (3) and (4), we obtain

$$L_1(2, IDR) = \int_{\theta<c_2(n)\cup\theta>c_5(n)} (x_2(\theta) - x_1(\theta))\, w(\theta)\, d\theta \;+$$

$$\int_{c_2(n)}^{c_5(n)} (x_2(\theta) - x_3(\theta))\, w(\theta)\, d\theta,$$

$$L_1(2, NIDR) = \int_{\theta<c_1(n)\cup\theta>c_6(n)} (x_3(\theta) - x_2(\theta))\, w(\theta)\, d\theta + \int_{c_3(n)}^{c_4(n)} (x_1(\theta) - x_2(\theta))\, w(\theta)\, d\theta.$$

For $i = 3$, applying equations (3) and (4), we obtain

$$L_1(3, IDR) = \int_{\theta<c_2(n)\cup\theta>c_5(n)} (x_3(\theta) - x_1(\theta))\, w(\theta)\, d\theta\,,$$

$$L_1(3, NIDR) = \int_{c_1(n)}^{c_3(n)} (x_2(\theta) - x_3(\theta))\, w(\theta)\, d\theta + \int_{c_4(n)}^{c_6(n)} (x_2(\theta) - x_3(\theta))\, w(\theta)\, d\theta \;+$$

$$\int_{c_3(n)}^{c_4(n)} (x_1(\theta) - x_3(\theta))\, w(\theta)\, d\theta.$$

The overall index can then be computed from equation (1). It is clear that for the purpose of comparison of the three estimates, we can work with

$$L_1(Index_i) = \frac{L_1(i\,,\, IDR)}{L_1(i\,,\, IDR) + L_1(i\,,\, NIDR)}, \quad i = 1\,,\, 2, 3.$$

3.2 Analysis based on L$_2$-norm

For $i = 1$, applying equations (5) and (6), we get

$$L_2(1, IDR) = \sqrt{\int_{c_2(n)}^{c_5(n)} (x_1(\theta) - x_3(\theta))^2 \; w(\theta) \; d\theta} \,,$$

$$L_2(1, NIDR) = \sqrt{\begin{array}{l} \int_{\theta<c_1(n) \cup \theta>c_6(n)} (x_3(\theta) - x_1(\theta))^2 \, w(\theta) \, d\theta + \\ \int_{c_1(n)}^{c_3(n)} (x_2(\theta) - x_1(\theta))^2 \; w(\theta) \; d\theta + \\ \int_{c_4(n)}^{c_6(n)} (x_2(\theta) - x_1(\theta))^2 \, w(\theta) \, d\theta \end{array}} \,.$$

For $i = 2$, applying equations (5) and (6), we obtain

$$L_2(2, IDR) = \sqrt{\int_{\theta<c_2(n) \cup \theta>c_5(n)} (x_2(\theta) - x_1(\theta))^2 \, w(\theta) \, d\theta + \int_{c_2(n)}^{c_5(n)} (x_2(\theta) - x_3(\theta))^2 w(\theta) \, d\theta},$$

$$L_2(2, NIDR) = \sqrt{\int_{\theta<c_1(n) \cup \theta>c_6(n)} (x_3(\theta) - x_2(\theta))^2 \, w(\theta) \, d\theta + \int_{c_3(n)}^{c_4(n)} (x_1(\theta) - x_2(\theta))^2 \, w(\theta) \, d\theta}.$$

For $i = 3$, applying equations (5) and (6), we obtain

$$L_2(3, IDR) = \sqrt{\int_{\theta<c_2(n) \cup \theta>c_5(n)} (x_3(\theta) - x_1(\theta))^2 \; w(\theta) \; d\theta} \,,$$

$$L_2(3, NIDR) = \sqrt{\begin{array}{l} \int_{c_1(n)}^{c_3(n)} (x_2(\theta) - x_3(\theta))^2 \; w(\theta) \; d\theta + \int_{c_4(n)}^{c_6(n)} (x_2(\theta) - x_3(\theta))^2 \; w(\theta) \; d\theta + \\ \int_{c_3(n)}^{c_4(n)} (x_1(\theta) - x_3(\theta))^2 \; w(\theta) \; d\theta \end{array}} \,.$$

Under L_2-norm also, the overall index can be computed from equation (2) for each value of n.

3.3 Choice of weight functions

Our first weight function $w_1(\theta)$ is defined by $w_1(\theta) = \dfrac{\theta^{\alpha-1}(1-\theta)^{\beta-1}}{B(\alpha,\beta)}$ for some α, $\beta > 0$, which is a conjugate prior for the binomial parameter θ. Following Filar et al. [2], we also consider two additional choices of $w(\theta)$. The first one, denoted by $w_2(\theta)$, is based on the notion of entropy among $x_1(\theta)$, $x_2(\theta)$ and $x_3(\theta)$ for various values of θ, and the second one, denoted by $w_3(\theta)$, is based on the coefficient of variation of $x_1(\theta)$, $x_2(\theta)$ and $x_3(\theta)$ for various values of θ (Vide [4]). It turns out that

$$w_2(\theta) = \frac{1-\phi(\theta)}{\int_{\underline{\theta}}^{\bar{\theta}} [1-\phi(\theta)]\, d\theta} \tag{12}$$

where $\phi(\theta) = -\dfrac{1}{\log 3} \displaystyle\sum_{i=1}^{3} \left\{ \frac{x_i(\theta)}{\sum_{i=1}^{3} x_i(\theta)} \,.\, \log \left[\frac{x_i(\theta)}{\sum_{i=1}^{3} x_i(\theta)} \right] \right\}$,

and

$$w_3(\theta) = \frac{\sqrt{2\,(x_1^2(\theta) + x_2^2(\theta) + x_3^2(\theta) - x_1(\theta)x_2(\theta) - x_1(\theta)x_3(\theta) - x_2(\theta)x_3(\theta))}}{x_1(\theta)+x_2(\theta)+x_3(\theta)}. \tag{13}$$

For details of above derivation, we refer to Technical Report [3]. These expressions can be readily computed using the functions $x_1(\theta)$, $x_2(\theta)$ and $x_3(\theta)$ given in (3.1)-(3.3).

3.4 Comparison of estimators

We report in Table 1 the ranks of the three estimators when compared on the basis of the weight function $w_1(\theta)$. In Table 2, we provide the ranks for the two other weight functions $w_2(\theta)$ and $w_3(\theta)$.

Table 1: Rank of three estimators using weight $w_1(\theta)^*$

		L_1		L_2	
		rank ($\alpha = \beta = 1$)	rank ($\alpha = \beta = \sqrt{n}/2$)	rank ($\alpha = \beta = 1$)	rank ($\alpha = \beta = \sqrt{n}/2$)
n =5	T_1	2	2	2	2
	T_2	3	3	3	3
	T_3	1	1	1	1
n=10	T_1	2	2	1	2
	T_2	3	3	2	3
	T_3	1	1	3	1
n=15	T_1	1	2	1	2
	T_2	3	3	3	3
	T_3	2	1	2	1
n=20	T_1	1	2	1	2
	T_2	2	3	2	3
	T_3	3	1	3	1

* Rank 1 = best, Rank 3 = worst

Table 2: Rank of three estimators using weights $w_2(\theta)$and $w_3(\theta)$*

		L_1		L_2	
		$w_2(\theta)$	$w_3(\theta)$	$w_2(\theta)$	$w_3(\theta)$
n=5	T_1	1	2	1	2
	T_2	2	3	2	3
	T_3	3	1	3	1
n=10	T_1	1	1	1	1
	T_2	2	3	2	2
	T_3	3	2	3	3
n=15	T_1	1	1	1	1
	T_2	2	2	2	2
	T_3	3	3	3	3
n=20	T_1	1	1	1	1
	T_2	2	2	2	2
	T_3	3	3	3	3

* Rank 1 = best, Rank 3 = worst

4 Conclusion

Based on the above analysis under L_1- and L_2- norms, we conclude that, for small values of n, our preference is uniformly for T_1. Under the weight function $w_1(\theta)$, T_3 also has some advantages. Of the three estimators studied in this paper, it turns out that T_2 is improper since $T_2(x) > \frac{1}{4}$ whenever $\frac{n-\sqrt{n}}{2} < x < \frac{n+\sqrt{n}}{2}$. On the other hand, both T_1 and T_3 are seen to be proper estimators. Therefore, one should use the truncated version T_2^* of T_2 and compute its mse and then compare it with the other two. This will improve the performance of T_2 and possibly make it preferable over T_1 and T_3. We propose to undertake this study in future.

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Appendix

Derivation of equations (3.1)-(3.3).
Recall that

$$T_1 = \frac{X}{n}\left(1 - \frac{X}{n}\right),\ T_2 = \frac{X(n-X)}{n\,(n-1)},\ T_3 = \frac{X(n-X\,) + n\sqrt{n}/2 + n/4}{(n+\sqrt{n})^2}$$

Let $\delta_\theta = \theta(1-\theta)$. It is easy to verify that $E(X) = n\theta$, $E(X^2) = n\theta(1+n\theta-\theta)$ $E(X^3) = n\theta(1+3n\theta+n^2\theta^2-3n\theta^2-3\theta+2\theta^2)$, and $E(X^4) = n\theta(1-7\theta+7n\theta+12\theta^2-18n\theta^2+6n^2\theta^2-6\theta^3+11n\theta^3-6n^2\theta^3+n^3\theta^3)$. Moreover, $V(X) = n\theta(1-\theta)$, and one can easily show $V(X^2) = n\theta(1-7\theta+6n\theta+12\theta^2-16n\theta^2+4n^2\theta^2-6\theta^3+10n\theta^3-4n^2\theta^3)$, and $Cov(X,X^2) = n\theta\,(1+2n\theta-2n\theta^2+2\theta^2-3\theta)$.
Note that $E(T_1) = \dfrac{\theta(1-\theta)(n-1)}{n}$, $E(T_2) = \theta(1-\theta)$, and
$E(T_3) = \left\{\theta(1-\theta)(n-1)+\dfrac{\sqrt{n}}{2}+\dfrac{1}{4}\right\}\cdot\dfrac{n}{(n+\sqrt{n})^2}$.
Hence, $(E(T_1)-\delta_\theta)^2 = \dfrac{1}{n^4}(n^2\theta^4-2n^2\theta^3+n^2\theta^2)$, $(E(T_2)-\delta_\theta)^2 = 0$, and
$(E(T_3)-\delta_\theta)^2 = \dfrac{1}{(n+\sqrt{n})^4}\left(\left(\dfrac{n\sqrt{n}}{2}+\dfrac{n}{4}\right)-2\,\theta\,(1-\theta)\,n\,(1+\sqrt{n})\right)^2$.
Furthermore,

$$\begin{aligned}
V(T_1) &= V\left(\frac{X(n-X)}{n^2}\right) = \frac{1}{n^4}\left(n^2\,V(X)+V(X^2)-2\,n\,Cov(X,X^2)\right)\\
&= \frac{1}{n^4}\left(n\,(n-1)^2\theta+(5n-7)(n-n^2)\,\theta^2+(2n-3)(4n^2-4n)\,\theta^3\right)+\\
&\quad \frac{1}{n^4}\left((2n-3)(2n-2n^2)\,\theta^4\right),
\end{aligned}$$

$$\begin{aligned}
V(T_2) &= V\left(\frac{X(n-X)}{n\,(n-1)}\right) = \frac{1}{n^2(n-1)^2}\left(n^2\,V(X)+V(X^2)-2\,n\,Cov(X,X^2)\right)\\
&= \frac{1}{n^2(n-1)^2}\left(n(n-1)^2\theta+(5n-7)(n-n^2)\theta^2+(2n-3)(4n^2-4n)\theta^3\right)+\\
&\quad \frac{1}{n^2(n-1)^2}\left((2n-3)(2n-2n^2)\theta^4\right),
\end{aligned}$$

$$\begin{aligned}
V(T_3) &= V\left(\frac{X(n-X)+n\sqrt{n}/2+n/4}{(n+\sqrt{n})^2}\right)\\
&= \frac{1}{(n+\sqrt{n})^4}\left(n^2V(X)+V(X^2)-2nCov(X,X^2)\right)\\
&= \frac{1}{(n+\sqrt{n})^4}\left(n(n-1)^2\theta+(5n-7)(n-n^2)\theta^2+(2n-3)(4n^2-4n)\theta^3\right)+\\
&\quad \frac{1}{(n+\sqrt{n})^4}\left((2n-3)(2n-2n^2)\theta^4\right).
\end{aligned}$$

We are now in a position to compute the mean squared errors of the three estimators.
1. $\mathrm{MSE}(T_1) \; = \; E(\,T_1 - \delta_\theta)^2 \; = \; V(T_1) + (E(T_1) - \delta_\theta)^2$

$$= \frac{1}{n^4}\left\{n(n-1)^2\theta + \left((5n-7)(n-n^2)+n^2\right)\theta^2 + \left((2n-3)(4n^2-4n)-2n^2\right)\theta^3 + \left((2n-3)(2n-2n^2)+n^2\right)\theta^4\right\}.$$

2. $\mathrm{MSE}(T_2) \; = \; E(\,T_2 - \delta_\theta)^2 \; = \; V(T_2)$

$$= \frac{1}{n^2(n-1)^2}\{n(n-1)^2\theta - n(5n-7)(n-1)\theta^2 + 4n(2n-3)(n-1)\theta^3 - 2n(2n-3)(n-1)\theta^4\}.$$

3. $\mathrm{MSE}(T_3) \; = \; E(\,T_3 - \delta_\theta)^2 \; = \; V(T_3) \; + \; (E(T_3) - \delta_\theta)^2$

$$\begin{aligned} = \; & \frac{1}{(n+\sqrt{n})^4}\{[(2n+2n\sqrt{n})^2 - 2n(2n-3)(n-1)]\theta^4 \\ & +[-2(2n+2n\sqrt{n})^2 + 4n(2n-3)(n-1)]\theta^3 \\ & +[2(2n+2n\sqrt{n})(n\sqrt{n}/2+n/4) + (2n+2n\sqrt{n})^2 - n(5n-7)(n-1)]\theta^2 \\ & +[-2(2n+2n\sqrt{n})(n\sqrt{n}/2+n/4) + n(n-1)^2]\theta \; + \; (n\sqrt{n}/2+n/4)^2\}. \end{aligned}$$

Finally, a justification of (13) follows from the following observation. Referring to [2] and [4], note that

$$w_3(\theta) \quad = \quad \frac{s}{\bar{x}} \quad = \quad \frac{\sqrt{\frac{1}{3}\sum_{i=1}^{3}(\,x_i(\theta) - \bar{x}(\theta)\,)^2}}{\frac{1}{3}\sum_{i=1}^{3}x_i(\theta)}\,.$$

Since
$$\sum_{i=1}^{3}(x_i(\theta)-\bar{x}(\theta))^2 = \frac{2}{3}\left[x_1^2(\theta)+x_2^2(\theta)+x_3^2(\theta)-x_1(\theta)x_2(\theta)-x_1(\theta)x_3(\theta)-x_2(\theta)x_3(\theta)\right],$$
(13) follows.

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