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A New Method of Construction of $2^2 \times 3^3$ Design with 4 Randomly Selected Blocks Each of Size 6

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Abstract

Here in this investigation, we have obtained a method of constructing a confounded asymmetrical factorial design of the type $2^2 \times 3^3$ with 4 randomly selected blocks each of size 6 in two replications. Considering different cases, different combination of replications involving simple random sampling with replacement and without replacement method, we have got different block contents. According to Das (1960), the construction of confounded asymmetrical factorial design requires, $N = \frac{P}{R} = S^k$ where S is prime or prime power, k is any positive integer, P is total number of treatment combinations and R is the number of plots per block. But by our new method, it is possible to construct the design when we have, $N = \frac{P}{R} \neq S^k$, simply the product of two primes or prime powers. The main superiority of our new method is that, selecting only 4 blocks randomly, we have obtained 50% relative information of the interaction effects taking any two generalized identity relationships out of all possible combinations of generalized identity relationships.

1 Introduction

In this present investigation, we have constructed a confounded asymmetrical factorial design of the type $2^2 \times 3^3$ with 4 randomly selected blocks each of size 6 by considering to some extent a modified method of fractional replicates of asymmetrical factorial design which was extensively developed by Das (1960). For constructing this asymmetrical factorial design, we have introduced a new method by taking fraction involving two different choices of identity relationships. In this situation, we have two replications having 36 blocks each of size 6 and by our new method we have selected only 4 blocks randomly. The usual procedure of analysis as given by Das and Giri

(1986) is being used to find the loss of information of the affected effects. The design is found to be not balanced according to the concept given by Kishen and Tyagi (1964).

2 The Method of Construction of the Confounded Design

$2^2 \times 3^3$ is an asymmetrical factorial design having 5 factors. Let the factors be denoted by A&B each at two levels (0,1) and X_1, X_2 & X_3 each at three levels (0,1,2). For construction of this design, the total number of treatment combinations, $P = 2^2 \times 3^3$ and the number of plots per block, $R = 6 = 2 \times 3$.

Here, $N = \frac{P}{R} = \frac{2^2 \times 3^3}{2 \times 3} = 2 \times 3^2$. But according to Das (1960) we have seen that the construction of confounded asymmetrical factorial design requires $\frac{P}{R} = S^k$ where S is prime or prime power, K is any positive integer, P is total number of treatment combinations and R is number of plots per block. For $2^2 \times 3^3$ design in 6 plots per block we have, $N = \frac{P}{R} \neq S^k$, simply the product of two primes or prime powers. These are mainly dependent on the level of the asymmetrical factorial designs originally given. Here the value of N contradicts the condition of Das (1960) and therefore, we can not apply that method. For the construction of the design $2^2 \times 3^3$ with 4 randomly selected blocks each of size 6, we can proceed by considering fraction of 2^2 by $\frac{1}{2}$ considering the identity relationship $I_1 = AB$ and for $\frac{1}{9}$ fraction of 3^3, we consider any one of the different choices of identity relationships I_2 which are as follows:

$$
\begin{aligned}
&I_{21} = X_1 = X_2 = X_1X_2 = X_1X_2^2 \\
&I_{22} = X_1 = X_3 = X_1X_3 = X_1X_3^2 \\
&I_{23} = X_2 = X_3 = X_2X_3 = X_2X_3^2 \\
&I_{24} = X_1X_2 = X_3 = X_1X_2X_3 = X_1X_2X_3^2 \\
&I_{25} = X_2X_3 = X_1 = X_1X_2X_3 = X_1X_2^2X_3^2 \\
&I_{26} = X_1X_3 = X_2 = X_1X_2X_3 = X_1X_2^2X_3
\end{aligned}
$$

Finally combining the first and any one of the second identity relationships by simply multiplying the individual identity relationships, we have 6 possible generalized identity relationships which are shown as below:

$$
\begin{aligned}
&I_{121} = ABX_1 = ABX_2 = ABX_1X_2 = ABX_1X_2^2 \\
&I_{122} = ABX_1 = ABX_3 = ABX_1X_3 = ABX_1X_3^2 \\
&I_{123} = ABX_2 = ABX_3 = ABX_2X_3 = ABX_2X_3^2 \\
&I_{124} = ABX_1X_2 = ABX_3 = ABX_1X_2X_3 = ABX_1X_2X_3^2 \\
&I_{125} = ABX_2X_3 = ABX_1 = ABX_1X_2X_3 = ABX_1X_2^2X_3^2 \\
&I_{126} = ABX_1X_3 = ABX_2 = ABX_1X_2X_3 = ABX_1X_2^2X_3
\end{aligned}
$$

Thus, a confounded asymmetrical factorial design of the type $2^2 \times 3^3$ in 6 plots per block in 6 replications can be obtained from above 6 generalized identity relationships. It is seen that if we consider only one replication by taking any one of the generalized identity relationship, we can not get suitable information of all affected interaction effects. Therefore, we can consider any two or more different replications by which we can get suitable information of the affected interaction effects. Now, to construct our desired design, we have used first two generalized identity relationships which gives us a confounded asymmetrical factorial design of the type $2^2 \times 3^3$ in two replications each having 18 blocks B_1, B_2, B_3, $\cdots$, B_{18} each of size 6. The procedures are same for any of the two replications.

The block contents of considered replications are displayed as follows:

Replication -1

$I_{121} = ABX_1 = ABX_2 = ABX_1X_2 = ABX_1X_2^2$

where,

$I_1 = AB$ and $I_{21} = X_1 = X_2 = X_1X_2 = X_1X_2^2$

$$\left.\begin{array}{rcl} A+B & = & 0 \\ & = & 1 \end{array}\right\}\text{Mod } 2 \qquad \left.\begin{array}{rcl} X_1 & = & 0 \\ & = & 1 \\ & = & 2 \end{array}\right\}\text{Mod } 3 \qquad \left.\begin{array}{rcl} X_2 & = & 0 \\ & = & 1 \\ & = & 2 \end{array}\right\}\text{Mod } 3$$

00	01
11	10

000	010	020	100	110	120	200	210	220
001	011	021	101	111	121	201	211	221
002	012	022	102	112	122	202	212	222

Combining above two sets of treatment combinations, we get,

B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9
00000	00010	00020	00100	00110	00120	00200	00210	00220
00001	00011	00021	00101	00111	00121	00201	00211	00221
00002	00012	00022	00102	00112	00122	00202	00212	00222
11000	11010	11020	11100	11110	11120	11200	11210	11220
11001	11011	11021	11101	11111	11121	11201	11211	11221
11002	11012	11022	11102	11112	11122	11202	11212	11222
B_{10}	B_{11}	B_{12}	B_{13}	B_{14}	B_{15}	B_{16}	B_{17}	B_{18}
01000	01010	01020	01100	01110	01120	01200	10210	01220
01001	01011	01021	01101	01111	01121	01201	01211	01221
01002	01012	01022	01102	01112	01122	01202	01212	01222
10000	10010	10020	10100	10110	10120	10200	10210	10220
10001	10011	10021	10101	10111	10121	10201	10211	10221
10002	10012	10022	10102	10112	10120	10202	10212	10222

Replication -2

$I_{122} = ABX_1 = ABX_3 = ABX_1X_3 = ABX_1X_3^2$

where,

$I_1 = AB$ and $I_{22} = X_1 = X_3 = X_1X_3 = X_1X_3^2$

$$\left.\begin{array}{rcl} A+B & = & 0 \\ & = & 1 \end{array}\right\} \text{Mod } 2 \qquad \left.\begin{array}{rcl} X_1 & = & 0 \\ & = & 1 \\ & = & 2 \end{array}\right\} \text{Mod } 3 \qquad \left.\begin{array}{rcl} X_3 & = & 0 \\ & = & 1 \\ & = & 2 \end{array}\right\} \text{Mod } 3$$

00	01
11	10

000	001	002	100	101	102	200	201	202
010	011	012	110	111	112	210	211	212
020	021	022	120	121	122	220	221	222

Combining above two sets of treatment combinations, we get,

B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9
00000	00001	00002	00100	00101	00102	00200	00201	00202
00010	00011	00012	00110	00111	00112	00210	00211	00212
00020	00021	00022	00120	00121	00122	00220	00221	00222
11000	11001	11002	11100	11101	11102	11200	11201	11202
11010	11011	11012	11110	11111	11112	11210	11211	11212
11020	11021	11022	11120	11121	11122	11220	11221	11222

B_{10}	B_{11}	B_{12}	B_{13}	B_{14}	B_{15}	B_{16}	B_{17}	B_{18}
01000	01001	01002	01100	01101	01102	01200	10201	01202
01010	01011	01012	01110	01111	01112	01210	01211	01212
01020	01021	01022	01120	01121	01122	01220	01221	01222
10000	10001	10002	10100	10101	10102	10200	10201	10202
10010	10011	10012	10110	10111	10112	10210	10211	10212
10020	10021	10022	10120	10121	10122	10220	10221	10222

2.1 $2^2 \times 3^3$ design with 4 randomly selected blocks each of size 6 obtained from two replications

2.1.1 Simple random sample with replacement method

Drawing a simple random sample of size 2 blocks from each of 2 replications with replacement method, we get B_3, B_4 for first replication and B_5 & B_{14} for second replication and denoted them by B_1', B_2' and B_3', B_4' respectively as follows :

Replication - 1	
B_1'	B_2'
00020	00020
00021	00021
00022	00022
11020	11020
11021	11021
11022	11022

Replication - 1	
B_3'	B_4'
00101	01101
00111	01111
00121	01121
11101	10101
11111	10111
11121	10121

The design is non orthogonal, the usual procedure of analysis as given by Das and Giri (1986) is as follows:

For the affected effects, let us consider R-treatments as bellow:

R- Treatments for ABX_1: $t_i = (a_1 - a_0)\ (b_1 - b_0)\ (X_1)_i\ ;\ i = 0, 1, 2.$

$= (a_1b_1 + a_0b_0 - a_1b_0 - a_0b_1)(X_1)_i$

where,

$(\mathrm{X}_1)_0$ = 000+001+002+010+011+012+020+021+022
$(\mathrm{X}_1)_1$ = 100+101+102+110+111+112+120+121+122
$(\mathrm{X}_1)_2$ = 200+201+202+210+211+212+220+221+222

R-treatments for ABX_2: $p_i = (a_1b_1 + a_0b_0 - a_1b_0 - a_0b_1)\ (X_2)_i\ ;\ i = 0, 1, 2.$

where,

$(X_2)_0$ = 000+001+002+100+101+102+200+201+202
$(X_2)_1$ = 010+011+012+110+111+112+210+211+212
$(X_2)_2$ = 020+021+022+120+121+122+220+221+222

R-treatments for ABX_3: $q_i = (a_1b_1 + a_0b_0 - a_1b_0 - a_0b_1)\ (X_3)_i;\ i = 0, 1, 2.$

where,

$(X_3)_0$ = 000+010+020+100+110+120+200+210+220
$(X_3)_1$ = 001+011+021+101+111+121+201+211+221
$(X_3)_2$ = 002+012+022+102+112+122+202+212+222

R-treatments for ABX_1X_2 : $m_i = (a_1b_1 + a_0b_0 - a_1b_0 - a_0b_1)\ (X_1X_2)_i;\ i = 0, 1, 2.$

where,

$(X_1X_2)_0$ = 000+001+002+120+121+122+210+211+212
$(X_1X_2)_1$ = 010+011+012+100+101+102+220+221+222
$(X_1X_2)_2$ = 020+021+022+110+111+112+200+201+202

R-treatments for ABX_1X_3 : $s_i = (a_1b_1 + a_0b_0 - a_1b_0 - a_0b_1)\ (X_1X_3)_i;\ i = 0, 1, 2.$

where,

$(X_1X_3)_0$ = 000+010+020+102+112+122+201+211+221
$(X_1X_3)_1$ = 001+011+021+100+110+120+202+212+222
$(X_1X_3)_2$ = 002+012+022+101+111+121+200+210+220

R-treatments for $ABX_1X_2^2$: $l_i = (a_1b_1 + a_0b_0 - a_1b_0 - a_0b_1)\ (X_1X_2^2)_i;\ i = 0, 1, 2.$
where,
$(X_1X_2^2)_0$ = 000+001+002+110+111+112+220+221+222
$(X_1X_2^2)_1$ = 020+021+022+100+101+102+210+211+212
$(X_1X_2^2)_2$ = 010+011+012+120+121+122+200+201+202

R-treatments for $ABX_1X_3^2$: $r_i = (a_1b_1 + a_0b_0 - a_1b_0 - a_0b_1)\ (X_1X_3^2)_i\ ;\ i = 0, 1, 2.$
where,
$(X_1X_3^2)_0$ = 000+010+020+101+111+121+202+212+222
$(X_1X_3^2)_1$ = 002+012+022+100+110+120+201+211+221
$(X_1X_3^2)_2$ = 001+011+021+102+112+122+200+210+220

Here the information of the affected effects ABX_1 can not be obtained because it is confounded in both the replications and hence we can not write any normal equation for ABX_1.

The normal equations for ABX_2

$$\begin{aligned}\left(4 - \frac{8}{6}\right) p_0 - \frac{8}{6}(p_1 + p_2) &= Qp_0 \\ \left(4 - \frac{8}{6}\right) p_1 - \frac{8}{6}(p_0 + p_2) &= Qp_1 \\ \left(16 - \frac{80}{6}\right) p_2 - \frac{8}{6}(p_0 + p_1) &= Qp_2\end{aligned}$$

Solving above normal equations under the restriction, $\sum_{i=0}^{2} p_i = 0$ we get, $\hat{p}_i = \frac{1}{4}Qp_i$, $i = 0, 1, 2$. Therefore, the variance of the estimate ABX_2 when it is confounded is, $V_1 = \frac{1}{4}\sigma^2$ and using proportional frequency condition the variance of the estimate when it is not confounded is -

$$V = \frac{\sum_{i=0}^{2} W_i \frac{\sigma^2}{n_i}}{\sum_{i=0}^{2} W_i} = \frac{\frac{4}{24} \cdot \frac{\sigma^2}{4} + \frac{4}{24} \cdot \frac{\sigma^2}{4} + \frac{16}{24} \cdot \frac{\sigma^2}{16}}{\frac{4}{24} + \frac{4}{24} + \frac{16}{24}} = \frac{\sigma^2}{8}$$

where w_i is the proportional frequency corresponding to the i-th level. Now the relative information is, $\frac{V}{V_1} = \frac{4}{8} = 0.50$ and the loss of information is, $(1-0.50) = 0.50$. By using similar process, we shall get the relative information 0.50 and the loss of information 0.50 for affected effect ABX_3, ABX_1X_2, ABX_3, ABX_1X_3, $ABX_1X_2^2$, $ABX_1X_3^2$.

The design is seen to be not balanced since Kishen and Taygi (1964) defined as "A factorial design in incomplete blocks confounding certain d.f. belonging to main effects and / or interactions is said to be balanced if the loss of information on each single d.f. Belonging to a particular effect is the same".

The same result follows for sampling without replacement.

Table 1: Frequency table of the R-treatment / block.

Block		ABX_1			ABX_2			ABX_3			ABX_1X_2			ABX_1X_3			$ABX_1X_2^2$			$ABX_1X_3^2$		
		t_0	t_1	t_2	p_0	p_1	p_2	q_0	q_1	q_2	m_0	m_1	m_2	s_0	s_1	s_2	l_0	l_1	l_2	r_0	r_1	r_2
Replication- 1	B'_1	6	0	0	0	0	6	2	2	2	0	0	6	2	2	2	0	6	0	2	2	2
	B'_2	6	0	0	0	0	6	2	2	2	0	0	6	2	2	2	0	6	0	2	2	2
Replication- 2	B'_3	0	6	0	2	2	2	6	0	0	2	2	2	0	6	0	2	2	2	0	6	0
	B'_4	0	-6	0	-2	-2	-2	0	-6	0	-2	-2	-2	0	0	-6	-2	-2	-2	-6	0	0
$n_{i\cdot}$					4	4	16	10	10	4	4	4	16	4	10	10	4	16	4	10	10	4
n_{ij}^2					8	8	80	44	44	8	8	8	80	8	44	44	8	80	8	44	44	8

3 Concluding Remarks

Constructing $2^2 \times 3^3$ design with 4 randomly selected blocks each of size 6 in two replications, the significant remarks can be displayed as follows:

- Considering another $({}^6c_2 - 1) = 14$ combinations of generalized identity relationships from 6 generalized identity relationships which are mentioned in (2), we can construct different confounded asymmetrical factorial design of the type $2^2 \times 3^3$ with 4 blocks each of size 6 in two replications. We can also find out the loss of information as well as the relative information of the affected interaction effects by the usual process.

- We have observed that the relative information and the loss of information of all affected effects except ABX_1 are usually same which is, 0.50 in case of simple random sample with replacement and without replacement method. For both cases, the information for the affected effect ABX_1 is not found because this affected effect is confounded in two replications simultaneously.

- The main superiority of the constructed design is that it requires the minimum possible resources to have the same information feature of the affected effects, i.e. by taking only 4 blocks randomly out of 36 blocks, we have got our expected 50% information for affected effects in both cases. Our constructed designs are seen to be not balanced.

- In our investigation, we have the freedom of selecting replications and taking blocks from replications which facilities are not available in Das (1960).

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