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A study of relative efficiency of linear method to least-squares method in estimating the two-parameter exponential distribution

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Abstract

An attempt is made in this paper to compare the efficiency of linear method to the Ordinary Least Squares (OLS) method in estimating the parameters of two-parameter exponential distribution, a member of location-scale family of distributions. Based on computer simulated data from the study distribution OLS and linear estimates of parameters with their standard errors and two more statistics, Ellipsoid of Concentration (EC) and Generalized Variance (GV) are computed for five different sample size and nine parameter combinations. The relative efficiency of linear method with respect to OLS method is computed using the GV and EC by parameter combination and sample size. It is observed that linear method is on an average 7.85 times efficient by GV and 52.17 times efficient by EC than that of OLS method. No trend is observed for variation in sample size or parameter combinations. No variation for parameter combination is observed.

Keywords and Phrases: Ordinary Least Squares; Linear Method; Two-parameter Exponential Distribution; Ellipsoid of Concentration; Generalized Variance; Relative Efficiency.

1 Introduction

Because of its simplicity a good number of authors used Least Squares method in different ways to estimate parameter of Location-scale family of distributions. Cox and Hinkley (1968) studied the efficiency of Least-square estimates in extreme-value regression models and observed that Location parameter is 97.8% efficient in comparison to maximum likelihood estimator while the efficiency of scale parameter is only 55.3%. Gehan and Siddiqi (1973) studied several location family of distributions. They observed that least squares estimators are more or less equally efficient to maximum likelihood estimates for those distributions whose hazard function or its transformation can be expressed as linear function of parameters based on order statistics. Lloyd (1952) derived best linear unbiased estimator for location-scale family of distributions whose survival function is available in closed form. Mann and Fertig (1977) studied the efficiency of linear estimates for extreme-value distribution and observed that they are as good as the maximum likelihood estimates.

In modern day's two-parameter exponential distribution, a member of location-scale family of distributions, has extensive use in the field of life testing and reliability, survival analysis, medicine, economics and other fields. Karmokar (1998) observed that for scale parameter, OLS estimate possess the smallest standard error than the linear method for all sample size and all parameter combinations while for location parameter the situation is just reverse. Under this situation, it is not possible to draw a conclusion about the relative efficiency of the methods. So we have estimated the parameters of two-parameter exponential distribution using linear estimation based on order statistics and compare overall efficiency of this method with that of least squares method. Generalized variance and ellipsoid of concentration are employed to measure the relative efficiency of the methods.

2 Estimation of Parameters

2.1 OLS Estimation of Parameters

The survival function of two-parameter negative exponential distribution

$$f(x\,;\,\theta,\sigma) = \frac{1}{\sigma}\exp\left\{-\frac{(x-\theta)}{\sigma}\right\}, \quad x \geq \theta\,,\, \sigma > 0 \tag{1}$$

is

$$\begin{aligned} S(x) &= \int_x^\infty \frac{1}{\sigma}\exp\left\{-\frac{(u-\theta)}{\sigma}\right\} du; \quad u \geq \theta, \sigma > 0 \\ &= \exp\left\{-\frac{(x-\theta)}{\sigma}\right\} \\ \therefore\ Log_e\, S(x) &= \frac{-(x-\theta)}{\sigma} = \frac{\theta}{\sigma} - \frac{x}{\sigma} \end{aligned}$$

$$\Rightarrow Y = B_0 + B_1 x \tag{2}$$

where $B_0 = \frac{\theta}{\sigma}$, $B_1 = -\frac{1}{\sigma}$ and $Y = Log_e\, S(x)$

Let $x_1, x_2, \cdots, x_n$ be a random sample from this distribution so that the Kaplan-Mier Product-Limit estimator $\hat{S}(x_i)$ of $S(x_i)$ is $\hat{S}\left(x_i\right) = \prod_{j=0}^{i} \frac{n_j - 1}{n_j}$ if there is no ties, if there are d_i observations equal to x_i, then $\hat{S}\left(x_i\right) = \prod_{j=0}^{i} \frac{n_j - d_j}{n_j}$ where, n_j is no. at risk in jth failure

$$\Rightarrow Y_i^* = B_0 + B_1 x_i + e_i$$

where $Y^* = \hat{S}(x_i)$. The OLS estimate of (2) is

$$\hat{B}_1 = \frac{\sum_{i=1}^{n} \left(Y_i^* - \bar{Y}^*\right)\left(x_i - \bar{x}\right)}{\sum_{i=1}^{n} \left(x_i - \bar{x}\right)^2}, \quad \hat{B}_0 = \bar{Y}^* - \hat{B}_1 \bar{x}$$

Finally,

$$\hat{\sigma} = -\frac{1}{\hat{B}_1} \quad \text{and} \quad \hat{\theta} = -\frac{\hat{B}_0}{\hat{B}_1} \tag{3}$$

$$V\left(\hat{B}_1\right) = \frac{S_e^2}{\sum_{i=1}^{n} \left(x_i - \bar{x}\right)^2}, \quad S_e^2 \text{ is estimated error variance.}$$

$$V\left(\hat{B}_0\right) = S_e^2 \left\{ \frac{1}{n} + \frac{\bar{x}^2}{\sum_{i=1}^{n} \left(x_i - \bar{x}\right)^2} \right\} = \frac{S_e^2 \sum x_i^2}{n \sum_{i=1}^{n} \left(x_i - \bar{x}\right)^2}$$

$$Cov\left(\hat{B}_0, \hat{B}_1\right) = \frac{\bar{x}\, S_e^2}{\sum_{i=1}^{n} \left(x_i - \bar{x}\right)^2}$$

Using Statistical Differentials

$$V\left(\hat{\sigma}\right) = \frac{Var\left(\hat{B}_1\right)}{\left\{E\left(\hat{B}_1\right)\right\}^4} = \frac{S_e^2}{\hat{B}_1^4 \sum_{i=1}^{n} \left(x_i - \bar{x}\right)^2}$$

$$\begin{aligned} V\left(\hat{\theta}\right) &= \left[g'\left(\hat{B}_0\right)\right]^2 Var\left(\hat{B}_0\right) \\ &= \hat{\sigma}^2 Var\left(\hat{B}_0\right) \quad \text{where } g\left(\hat{B}_0\right) = \hat{B}_0\hat{\sigma} \\ &= \frac{S_e^2\hat{\sigma}^2}{n} + \frac{\bar{x}^2 S_e^2 \hat{\sigma}^2}{\sum\limits_{i=1}^{n}(x_i-\bar{x})^2} \end{aligned}$$

$$Cov\left(\hat{\theta},\hat{\sigma}\right) = -\frac{\bar{x}S_e^2\hat{\sigma}}{\hat{B}_1^2\sum(x_i-\bar{x})^2}$$

2.2 Linear Estimation

Let $y_1 \leq y_2 \leq \cdots \leq y_n$ be the ordered observation in a random sample of size n from two-parameter exponential model (1). Let $Z_i = \left(\frac{y_i-\theta}{\sigma}\right)$; $i = 1, 2, \cdots, n$ be the standardized order statistics and define, $\alpha_i = E(Z_i)$ and $v_{ij} = Cov(Z_i, Z_j)$; $i, j = 1, 2, \cdots, n$. From standardized order statistics, it follows that,

$$\begin{aligned} y_i &= \theta + \sigma Z_i + \varepsilon_i \\ E(y_i) &= \theta + \sigma Z_i \\ \therefore\ E(Y) &= A\mu \qquad (4) \\ \Rightarrow\ Y &= A\mu + \varepsilon \end{aligned}$$

Where,

$$A = \begin{bmatrix} 1 & \alpha_1 \\ 1 & \alpha_2 \\ \vdots & \vdots \\ 1 & \alpha_n \end{bmatrix}$$

$$Y = (y_1, y_2, \cdots, y_n)' \text{ and } \mu = (\theta, \sigma)'$$

Now if $E(Y) = A\mu$ and $Var(Y) = \sigma^2 V$, where $V = (v_{ij})$ then from (4) the linear estimates of θ and σ as follows:

$$\hat{\mu} = \left(A'V^{-1}A\right)^{-1} A'V^{-1}Y \qquad (5)$$

and variance-covariance matrix of $\left(\hat{\theta},\hat{\sigma}\right)'$ is $\sigma^2\left(A'V^{-1}A\right)^{-1}$. To solve equation (5) mathematically, computation of A and V matrix are necessary. For the computation of A matrix, we have,

$$\alpha_i = E(Z_i) = \sum_{j=1}^{i}\frac{1}{(n-j+1)}$$

2.3 Covariance between Z_i and Z_j

Let $T_1 = nZ_1$, $T_2 = (n-1)(Z_2 - Z_1)$, $\cdots$, $T_n = (Z_n - Z_{n-1})$. Then, $Z_i = \frac{T_1}{n} + \frac{T_2}{(n-1)} + \cdots + \frac{T_i}{n-i+1}$ and $T_1, T_2, \cdots, T_n$ are *iid* from $F(t) = 1 - \exp(-t)$. Hence, for $i \leq j$,

$$\begin{aligned} Cov\left(Z_i, Z_j\right) &= E\left[\left(T_1 - 1\right)/n + \left(T_2 - 1\right)/(n-1) + \cdots + \left(T_i - 1\right)/(n-i+1)\right] \\ &\quad \left[\left(T_1 - 1\right)/n + \left(T_2 - 1\right)/(n-1) + \cdots + \left(T_i - 1\right)/(n-i+1)\right] \\ &= Var\left(T_1\right)/n^2 + Var\left(T_2\right)/(n-1)^2 + \cdots + Var\left(T_i\right)/(n-i+1)^2 \\ &= 1/n^2 + 1/(n-1)^2 + \cdots + 1/(n-i+1)^2 \end{aligned}$$

2.4 Ellipsoid of Concentration

Let $(T_1, T_2, \cdots, T_r)$ be an unbiased estimator of $\{\tau_1(\theta), \tau_2(\theta), \cdots, \tau_r(\theta)\}$. Let $\sigma^{ij}(\theta)$ be the ijth element of the inverse of covariance matrix of $(T_1, T_2, \cdots, T_r)$, where the ijth element of the covariance is $\sigma_{ij}(\theta) = cov[T_i, T_j]$. The ellipsoid of concentration (EC) of $(T_1, T_2, \cdots, T_r)$ is defined as the interior point and boundary of the ellipsoid

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\sigma^{ij}(\theta)\left[t_i - \tau_i(\theta)\right]\left[t_j - \tau_j(\theta)\right] = (r+2)$$

Loosely speaking, the ellipsoid of concentration measures how concentrated the distribution of $(T_1, T_2, \cdots, T_r)$ is about $\{\tau_1(\theta), \tau_2(\theta), \cdots, \tau_r(\theta)\}$. The distribution of an estimator$(T_1, T_2, \cdots, T_r)$, whose ellipsoid of concentration is contained within the ellipsoid of concentration of another estimator $(T_1', T_2', \cdots, T_r')$ is more highly concentrated about $\{\tau_1(\theta), \tau_2(\theta), \cdots, \tau_r(\theta)\}$ than is the distribution of $(T_1', T_2', \cdots, T_r')$. Which implies that joint efficiency of estimators will be decreasing as the value of the ellipsoid of concentration increases.

2.5 Wilks' Generalized Variance

Let $(T_1, T_2, \cdots, T_r)$ be an unbiased estimator of $\{\tau_1(\theta), \tau_2(\theta), \cdots, \tau_r(\theta)\}$. Wilks' Generalized variance (GV) of $(T_1, T_2, \cdots, T_r)$ is defined to be the determinant of the covariance matrix of $(T_1, T_2, \cdots, T_r)$. This is an important statistics and shows how sufficiently it could be used to improve on an arbitrary unbiased estimator, generalizes to r dimension. It should be noted that like EC joint efficiency of the estimators decreases as the generalized variance increases.

3 Results and Discussions

To investigate efficiency of OLS and Linear methods of estimation from two-parameter exponential distribution, a member of location-scale family of distributions, we have worked with 4500 samples of five different size at nine different parameter combinations, (100, 15), (100, 20), (100, 25), (200, 15), (200, 20), (200, 25), (300, 15), (300, 20) and (300, 25) with 100 samples of each computation. Using same set of data for these methods 4500 pairs of values for the parameters in total along with their standard errors are computed for samples of five different size (10, 15, 20, 25, 30) at nine different parameter combinations $\{(\theta, \sigma)$: (100, 15)), (100, 20), (100, 25), (200, 15), (200, 20), (200, 25), (300, 15), (300, 20), (300, 25)$\}$ 100 estimates for each sample and each parameter combination. These results are summarized in Table-1. Further we have calculated two more statistics, viz. Generalized Variance (GV) and Ellipsoid of Concentration (EC). Estimated values of statistics for different sample size, parameter combination and methods of estimation are displayed in Table-2. The percentage of relative efficiency of linear method with respect to OLS method by sample size and parameter combination using GV and EC are given in Table-3.

4500 pairs of estimates for location and scale parameters of our study distribution along with corresponding standard errors are converted into 45 pairs averaging over 100 independent samples and displayed in Table-1 by methods of estimation, parameter combination and sample size. An examination of the figures of Table-1 shows that as the sample size increases average value of the estimated location parameter tends to be equal to the true parameter for both the methods for all values of location parameter (100,200,300). This implies that estimates are unbiased at least asymptotically. For a fixed sample size linear estimate of location parameter is found to be more close to the true parameter than OLS estimate for all values of the true location parameter and for all sample size included in this study. Consequently linear estimates are found to possess smaller standard error than OLS method in general. For a fixed value of location parameter, standard error of estimates decreases with increasing sample size for both the methods of estimation. This indicates that the estimates are consistent also. This is observed to be true for all values of location parameter and sample size. Effect of variation of scale parameter is observed to be very negligible on the estimates of location parameter.

Regarding closeness of the estimated value of scale parameter with true value, method of estimations exhibits no sampling variation neither for changing sample size nor for the parameter combination. What is observed that OLS estimates possess smaller standard error than linear method for all sample size and all value of the scale parameter studied. Changes in location parameter are observed to have no effect on the estimates of scale parameter for both the methods and for all sample size. Under this situation, it is not possible to draw a conclusion about the relative efficiency of the methods of estimation. GV and EC displayed in Table-2 are used to reach the conclusion.

An overview of Table-2 reveals that both GV and EC is minimum for linear method than OLS method irrespective of sample size and parameter combination for the set of parameters. This leads us to conclude approximately that linear method of estimation on an average is better than OLS method in estimating parameters of two-parameter

Table 1: Method-wise average value of estimators with corresponding standard errors by sample size and parameter combination.

(θ,σ)	Methods	Sample Size									
		10		15		20		25		30	
		$\hat{\theta}$ $S.E(\hat{\theta})$	$\hat{\sigma}$ $S.E(\hat{\sigma})$	$\hat{\theta}$ $S.E(\hat{\theta})$	$\hat{\sigma}$ $S.E(\hat{\sigma})$	$\hat{\theta}$ $S.E(\hat{\theta})$	$\hat{\sigma}$ $S.E(\hat{\sigma})$	$\hat{\theta}$ $S.E(\hat{\theta})$	$\hat{\sigma}$ $S.E(\hat{\sigma})$	$\hat{\theta}$ $S.E(\hat{\theta})$	$\hat{\sigma}$ $S.E(\hat{\sigma})$
(100,15)	OLS	99.528	14.506	99.846	14.699	100.248	13.822	100.229	13.848	99.894	14.392
		(12.392)	(1.578)	(7.758)	(0.982)	(5.867)	(0.712)	(4.991)	(0.590)	(4.443)	(0.559)
	LIN	99.878	15.415	99.896	15.035	100.041	14.949	99.941	14.915	100.009	14.959
		(1.581)	(5.000)	(15.916)	(4.009)	(0.769)	(3.441)	(0.612)	(3.062)	(0.509)	(2.785)
(100,20)	OLS	99.371	19.342	99.794	19.599	100.331	18.430	100.305	18.465	99.858	19.189
		(12.869)	(2.104)	(8.079)	(1.309)	(6.110)	(0.950)	(5.197)	(0.787)	(4.633)	(0.746)
	LIN	99.837	20.553	99.861	21.221	100.054	19.932	99.921	19.887	100.011	19.945
		(2.108)	(6.667)	(1.380)	(5.345)	(1.026)	(4.588)	(0.816)	(4.082)	(0.678)	(3.714)
(100,25)	OLS	99.214	24.177	99.743	24.499	100.413	23.037	100.381	23.081	99.823	23.987
		(13.352)	(2.630)	(8.404)	(1.636)	(6.356)	(1.187)	(5.407)	(0.983)	(4.827)	(0.932)
	LIN	99.796	25.691	99.827	26.526	100.068	24.915	99.902	24.858	100.014	24.931
		(2.635)	(8.333)	(1.725)	(6.682)	(1.282)	(5.735)	(1.021)	(5.103)	(0.848)	(4.642)
(200,15)	OLS	199.528	14.506	199.846	14.699	200.248	13.822	200.228	13.849	199.894	14.392
		(23.379)	(1.578)	(14.576)	(0.982)	(11.023)	(0.712)	(9.376)	(0.590)	(8.330)	(0.559)
	LIN	199.878	15.415	199.896	15.916	200.041	14.949	199.941	14.915	200.009	14.959
		(1.581)	(5.000)	(1.035)	(4.009)	(0.769)	(3.441)	(0.612)	(3.062)	(0.509)	(2.785)
(200,20)	OLS	199.371	19.341	199.794	19.599	200.331	18.430	200.305	18.465	199.859	19.189
		(23.844)	(2.104)	(14.886)	(1.309)	(11.258)	(0.950)	(9.575)	(0.787)	(8.513)	(0.746)
	LIN	199.837	20.553	199.861	21.221	200.054	19.932	199.921	19.887	200.011	19.945
		(2.108)	(6.667)	(1.380)	(5.345)	(1.026)	(4.588)	(0.816)	(4.082)	(0.678)	(3.714)
(200,25)	OLS	199.214	24.177	199.743	24.499	200.413	23.037	200.381	23.081	199.823	23.987
		(24.312)	(2.630)	(15.200)	(1.636)	(11.495)	(1.187)	(9.777)	(0.983)	(8.699)	(0.932)
	LIN	199.796	25.691	199.827	26.526	200.068	24.915	199.902	24.858	200.014	24.931
		(2.635)	(8.333)	(1.725)	(6.682)	(1.282)	(5.735)	(1.021)	(5.103)	(0.848)	(4.642)
(300,15)	OLS	299.529	14.505	299.847	14.698	300.249	13.821	300.229	13.848	299.894	14.391
		(34.378)	(1.578)	(21.402)	(0.982)	(16.186)	(0.712)	(13.767)	(0.590)	(12.224)	(0.559)
	LIN	299.878	15.415	299.896	15.916	300.041	14.949	299.941	14.915	300.009	14.959
		(1.581)	(5.000)	(1.035)	(4.009)	(0.769)	(3.441)	(0.612)	(3.062)	(0.509)	(2.785)
(300,20)	OLS	299.372	19.341	299.795	19.599	300.331	18.429	300.305	18.464	299.859	19.189
		(34.838)	(2.014)	(21.709)	(1.309)	(16.418)	(0.095)	(13.964)	(0.787)	(12.405)	(0.746)
	LIN	299.837	20.553	299.861	21.221	300.054	19.932	299.921	19.887	300.011	19.945
		(2.108)	(6.667)	(1.380)	(5.345)	(1.026)	(4.588)	(0.816)	(4.082)	(0.678)	(3.714)
(300,25)	OLS	299.214	24.176	299.743	24.499	300.414	23.036	300.381	23.080	299.823	23.986
		(35.300)	(2.630)	(22.018)	(1.636)	(16.651)	(1.187)	(14.163)	(0.983)	(12.587)	(0.932)
	LIN	299.796	25.691	299.827	26.527	300.068	24.915	299.902	24.858	300.014	24.931
		(2.635)	(8.333)	(1.725)	(6.682)	(1.282)	(5.735)	(1.021)	(5.103)	(0.848)	(4.642)

Table 2: Method-wise estimated generalized variance and ellipsoid of concentration by sample size and parameter combination.

(θ, σ)	Methods	Sample Size									
		10		15		20		25		30	
		GV	EC	GV	EC	GV	EC	GV	EC	GV	EC
(100,15)	OLS	269.011	0.055	121.372	0.150	35.421	0.135	29.417	0.104	25.415	0.078
	LIN	56.250	0.010	16.071	0.054	6.661	0.003	3.375	0.012	1.940	0.0004
(100,20)	OLS	850.632	0.055	383.633	0.150	111.791	0.135	93.115	0.104	80.237	0.078
	LIN	177.778	0.010	50.794	0.054	21.053	0.003	10.667	0.012	6.130	0.0004
(100,25)	OLS	2075.879	0.055	937.451	0.150	272.925	0.135	227.300	0.104	195.688	0.078
	LIN	434.028	0.010	124.008	0.054	51.398	0.003	26.042	0.012	14.967	0.0004
(200,15)	OLS	268.594	0.056	121.698	0.150	35.356	0.135	29.617	0.104	25.406	0.078
	LIN	56.250	0.010	16.071	0.054	6.661	0.003	3.375	0.012	1.940	0.0004
(200,20)	OLS	850.202	0.055	283.401	0.150	111.630	0.135	93.439	0.104	80.365	0.078
	LIN	177.778	0.010	50.794	0.054	21.053	0.003	10.667	0.012	6.130	0.0004
(200,25)	OLS	2,079.371	0.055	937.884	0.150	272.467	0.135	227.600	0.104	195.903	0.078
	LIN	434.028	0.010	124.008	0.054	51.398	0.003	26.042	0.012	14.967	0.0004
(300,15)	OLS	268.956	0.055	121.441	0.150	35.389	0.135	29.665	0.104	24.934	0.079
	LIN	56.250	0.010	16.071	0.054	6.661	0.003	3.375	0.012	1.940	0.0004
(300,20)	OLS	848.769	0.056	383.232	0.150	112.114	0.135	93.096	0.104	80.612	0.078
	LIN	177.778	0.010	50.794	0.054	21.053	0.003	10.667	0.012	6.130	0.0004
(300,25)	OLS	2080.598	0.055	935.428	0.150	271.535	0.135	226.680	0.104	195.159	0.078
	LIN	434.028	0.010	124.008	0.054	51.398	0.003	26.042	0.012	14.967	0.0004

Table 3: Percent relative efficiency of linear method with respect to OLS method by sample size and parameter combination using GV and EC.

(θ, σ)	Sample Size									
	10		15		20		25		30	
	GV	EC	GV	EC	GV	EC	GV	EC	GV	EC
(100,15)	478.243	561.225	755.205	277.778	531.751	4821.429	871.615	896.552	1310.250	19500
(100,20)	478.481	561.225	755.278	277.778	531.008	4821.429	872.950	896.552	1308.860	19500
(100,25)	478.283	561.225	755.961	277.778	531.003	4821.429	872.831	896.552	1307.510	19500
(200,15)	477.500	571.429	757.233	277.778	530.775	4821.429	877.541	896.552	1309.790	19500
(200,20)	478.239	561.225	557.946	277.778	530.243	4821.429	875.988	896.552	1310.950	19500
(200,25)	479.087	561.225	756.311	277.778	530.112	4821.429	873.983	896.552	1308.940	19500
(300,15)	478.144	561.225	755.634	277.778	531.271	4821.429	878.963	896.552	1285.460	19750
(300,20)	477.433	571.429	754.489	277.778	532.542	4821.429	872.772	896.552	1314.980	19500
(300,25)	479.370	561.225	754.330	277.778	528.299	4821.429	870.450	896.552	1303.970	19500

exponential distribution. Relative efficiency of linear method with respect to OLS method, using GV and EC by parameter combination and sample size shown in Table-3 indicates that linear method is on an average 7.85 times efficient than OLS method with a minimum of 4.77 to a maximum of 13.15 by GV and by EC linear method is on an average 52.17 times efficient than OLS method with a minimum of 2.78 to a maximum of 195.00. No trend for sample size and/or parameter combination is observed in the relative efficiency.

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