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Concomitants of Order Statistics from Bivariate Pareto II Distribution

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Abstract

The probability density function (pdf) of the r^{th}, $1 \leq r \leq n$ and the joint pdf of the r^{th} and s^{th}, $1 \leq r < s \leq n$, concomitants of order statistics are derived for bivariate Pareto II distribution introduced by Sankaran and Nair (1993) and their moments and product moments are obtained.

Keywords and Phrases: Order statistics, Concomitants of order statistics, Single and Product moments.

AMS Classification: 62 E 15, 62 G 30.

1 Introduction

Let (X_i, Y_i), $i = 1, 2, \cdots, n$, be n pairs of independent random variables from some bivariate population with distribution function $F(x, y)$. If we arrange the X-variates in ascending order as $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$, then the Y-variates paired (not necessarily in ascending order) with these ordered statistics are called the concomitants of order statistics and are denoted by $Y_{[1:n]}, Y_{[2:n]}, \cdots, Y_{[n:n]}$. The probability density function(pdf) of $Y_{[r:n]}$, the r^{th} concomitant of order statistics, is given by David (1981)

$$g_{[r:n]}(y) = \int_x f(y|x) f_{r:n}(x) dx \tag{1}$$

where $f_{r:n}(x)$ is the pdf of $X_{r:n}$. That is,

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!}[F(x)]^{r-1}[1 - F(x)]^{n-r} f(x) \tag{2}$$

The joint pdf of the $Y_{[r:n]}$and $Y_{[s:n]}$, $1 \le r < s \le n$ is

$$g_{[r,s:n]}(y_1, y_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{x_2} f(y_1|x_1) f(y_2|x_2) f_{r,s:n}(x_1, x_2) dx_1 dx_2 \tag{3}$$

where $f_{r,s:n}(x_1, x_2)$ is the joint pdf of $(x_{r:n}, x_{s:n})$. That is,

$$f_{r,s:n}(x_1, x_2) = C_{r,s:n}[F(x_1)]^{r-1}[F(x_2) - F(x_1)]^{s-r-1}[1 - F(x_2)]^{n-s} f(x_1) f(x_2) \tag{4}$$

with $C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}$.

Here we consider bivariate Pareto II distribution (Sankaran and Nair, 1993) with distribution function(df)

$$\begin{aligned} F(x, y) \;\; &= \;\; 1 - (1 + ax)^{-p} - (1 + by)^{-p} + (1 + ax + by + cxy)^{-p}; \\ &\qquad x, y > 0, \quad p, a, b > 0, \quad 0 \le c \le (p + 1)ab \end{aligned} \tag{5}$$

and probability density function (pdf)

$$\begin{aligned} f(x, y) \;\; &= \;\; \frac{p[p(a + cy)(b + cx) + ab - c]}{(1 + ax + by + cxy)^{p+2}}; \\ &\qquad x, y > 0, \quad p, a, b > 0, \quad 0 \le c \le (p + 1)ab \end{aligned} \tag{6}$$

Then the conditional pdf of Y, given X will be

$$f(y|x) = \frac{[p(a + cy)(b + cx) + ab - c](1 + ax)^{p+1}}{a(1 + ax + by + cxy)^{p+2}}; \qquad y > 0 \tag{7}$$

and the marginal df of X is

$$F(x) \;\; = \;\; 1 - (1 + ax)^{-p}; \qquad x > 0 \tag{8}$$

Expressions for the single and joint pdf for the r^{th} and s^{th} concomitants of order statistics from bivariate Pareto II distribution are obtained and their single and product moments are derived. For applications of concomitants of order statistics, one may refer to David (1982) & David and Nagaraja (1998).

Putting $c = 0$ in (5), then it reduces to the Lindley-Singpurwalla model (Lindley and Singpurwalla, 1986). Begum and Khan (1998) have also considered the distribution form in (5) at $c = 0$.

2 Probability Density Function of $Y_{[r:n]}$

For bivariate Pareto II distribution with distribution function (5), the pdf of the 1^{st} concomitant of the order statistics, in view of (1) and (2) for $r = 1$ is,

$$\begin{aligned} g_{[1:n]}(y) \;\; &= \;\; \int_0^{\infty} \frac{[p(a + cy)(b + cx) + ab - c](1 + ax)^{p+1}}{a(1 + ax + by + cxy)^{p+2}} . \frac{npa}{(1 + ax)^{np+1}} dx \\ &= \;\; \int_0^{\infty} np(1 + ax)^{-(np-p)}(1 + ax + by + cxy)^{-(p+2)} \\ &\qquad [p(a + cy)(b + cx) + ab - c] dx \end{aligned} \tag{9}$$

Let $ax + cxy = t$, then the RHS of (9) reduces to

$$\begin{aligned} &= np(a+cy)^{np-p-1}a^{-(np-p)}\Big\{[p(ab+bcy)+ab-c] \\ &\int_0^\infty \Big[\frac{a+cy}{a}+t\Big]^{-(np-p)}(t+1+by)^{-(p+2)}dt \\ &+pc\int_0^\infty t\Big[\frac{a+cy}{a}+t\Big]^{-(np-p)}(t+1+by)^{-(p+2)}dt\Big\} \end{aligned} \tag{10}$$

Noting that, (Erdélyi *et al.*, 1954)

$$\int_0^\infty x^{\nu-1}(a+x)^{-\mu}(x+y)^{-\rho}dx = \frac{\Gamma(\nu)\Gamma(\mu-\nu+\rho)y^{\nu-\rho}}{\Gamma(\mu+\rho)a^\mu}{}_2F_1\left(\begin{matrix}\mu, & \nu\\ \mu+\rho\end{matrix};1-\frac{y}{a}\right); \tag{11}$$
$$|\arg\ a| < \pi,\ \ \mathrm{Re}\,\nu > 0,\ \ |\arg\ y| < \pi,\ \ \mathrm{Re}\,\rho > \mathrm{Re}\,(\nu-\mu)$$

we get, after simplification,

$$\begin{aligned} g_{[1:n]}(y) &= \frac{np}{(a+cy)}\Bigg\{[p(ab+bcy)+ab-c]\frac{\Gamma(np+1)(1+by)^{-p-1}}{\Gamma(np+2)} \\ &{}_2F_1\left(\begin{matrix}np-p, & 1\\ np+2\end{matrix};\ 1-\frac{a(1+by)}{a+cy}\right)+pc\frac{\Gamma(np)(1+by)^{-p}}{\Gamma(np+2)} \\ &{}_2F_1\left(\begin{matrix}np-p, & 2\\ np+2\end{matrix};\ 1-\frac{a(1+by)}{a+cy}\right)\Bigg\} \end{aligned} \tag{12}$$

$$\begin{aligned} \Rightarrow g_{[1:n]}(y) &= \frac{np}{(a+cy)}\Bigg\{[pb(a+cy)+ab-c]\frac{1}{(np+1)(1+by)^{p+1}} \\ &{}_2F_1\left(\begin{matrix}np-p, & 1\\ np+2\end{matrix};\ \frac{(c-ab)y}{a+cy}\right)+\frac{pc}{np(np+1)(1+by)^p} \\ &{}_2F_1\left(\begin{matrix}np-p, & 2\\ np+2\end{matrix};\ \frac{(c-ab)y}{a+cy}\right)\Bigg\} \end{aligned} \tag{13}$$

where,

$${}_pF_q\left(\begin{matrix}\alpha_1,\cdots,\alpha_p\\ \beta_1,\cdots,\beta_q\end{matrix};\ z\right) = \sum_{k=0}^\infty \frac{(\alpha_1)_k\ \dots\ (\alpha_p)_k}{(\beta_1)_k\ \dots\ (\beta_q)_k}\ \frac{z^k}{k!} \tag{14}$$

is generalized hypergeometric series and

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)};\qquad \lambda \neq 0,-1,-2,\cdots \tag{15}$$

is Pochhammer symbol (Prudnikov *et al.*, 1986).

Using the relation (Prudnikov *et al.*, 1986)

$$
\begin{aligned}
{}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix};\ z\right) &= [c-2a-(b-a)z]^{-1}\Bigg[a(z-1)\,{}_2F_1\left(\begin{matrix} a+1,\ b \\ c \end{matrix};\ z\right) \\
&\quad +(c-a)\,{}_2F_1\left(\begin{matrix} a-1,\ b \\ c \end{matrix};\ z\right)\Bigg] \qquad (16)
\end{aligned}
$$

and then ${}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix};\ 0\right) = 1$ in the last hypergeometric series of the equation (13), we obtain, after simplification,

$$
\begin{aligned}
g_{[1:n]}(y) &= \frac{np}{(a+cy)}\Bigg[[pb(a+cy)+ab-c]\frac{1}{(np+1)(1+by)^{p+1}} \\
&\quad {}_2F_1\left(\begin{matrix} np-p, & 1 \\ np+2 & \end{matrix};\ \frac{(c-ab)y}{a+cy}\right) + \frac{pc}{np(np+1)(1+by)^{p}} \\
&\quad \left\{\frac{np(a+cy)-(np-p-1)(c-ab)y}{-a(1+by)}\,{}_2F_1\left(\begin{matrix} np-p, & 1 \\ np+2 & \end{matrix};\ \frac{(c-ab)y}{a+cy}\right)\right. \\
&\quad \left.+\frac{(np+1)(a+cy)}{a(1+by)}\right\}\Bigg] \\
&= \frac{npc}{na(1+by)^{p+1}} + \frac{np}{(np+1)(a+cy)(1+by)^{p+1}} \\
&\quad {}_2F_1\left(\begin{matrix} np-p, & 1 \\ np+2 & \end{matrix};\ \frac{(c-ab)y}{a+cy}\right)\left[\frac{(p+1)(ab-c)(na+cy)}{na}\right]
\end{aligned}
$$

$$
\begin{aligned}
\therefore\ g_{[1:n]}(y) &= \frac{np}{na(1+by)^{p+1}}\Bigg[c+\frac{(p+1)(ab-c)(na+cy)}{(np+1)(a+cy)} \\
&\quad {}_2F_1\left(\begin{matrix} np-p, & 1 \\ np+2 & \end{matrix};\ \frac{(c-ab)y}{a+cy}\right)\Bigg]; \qquad y>0 \qquad (17)
\end{aligned}
$$

It can be seen that $\int g_{[1:n]}(y)dy = 1$ for $c = 0$ and $c = ab$.

It is well-known that the distribution functions of order statistics are connected by the relation (David, 1981).

$$
F_{r:n}(x) = \sum_{i=n-r+1}^{n} (-1)^{i-n+r-1}\binom{i-1}{n-r}\binom{n}{i}F_{1:i}(x); \qquad 1\le r\le n \qquad (18)
$$

The relation in (18) is also clearly true interms of pdfs of concomitants of order statistics.

Thus the pdf of $Y_{[r:n]}$ is

$$\begin{aligned} g_{[r:n]}(y) &= \sum_{i=n-r+1}^{n} (-1)^{i-n+r-1} \binom{i-1}{n-r} \binom{n}{i} g_{[1:i]}(y) \\ &= \sum_{i=n-r+1}^{n} (-1)^{i-n+r-1} \binom{i-1}{n-r} \binom{n}{i} \frac{ip}{ia(1+by)^{p+1}} \Big[c + \\ & \quad \frac{(p+1)(ab-c)(ia+cy)}{(ip+1)(a+cy)} {}_2F_1 \left(\begin{matrix} ip-p, & 1 \\ ip+2 & \end{matrix} ; \frac{(c-ab)y}{a+cy} \right) \Big]; \ y > 0 \quad (19) \end{aligned}$$

Also, using (1) directly, we obtain,

$$\begin{aligned} g_{[r:n]}(y) &= \frac{p}{a(1+by)^{p+1}} \Big[c + \frac{n!}{(r-1)!(n-r)!} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \\ & \quad \frac{(p+1)(ab-c)(na-ra+ia+a+cy)}{(a+cy)(np-rp+ip+p+1)(n-r+i+1)} \\ & \quad {}_2F_1 \left(\begin{matrix} np-rp+ip\,, & 1 \\ np-rp+ip+p+2 & \end{matrix} ; \frac{(c-ab)y}{(a+cy)} \right) \Big]; \ y > 0 \quad (20) \end{aligned}$$

Putting $c = 0$ in equations (17), (19) and (20), then its become the pdfs of the 1^{st} and r^{th} concomitants of order statistics from the Lindley-Singpurwalla model (Lindley and Singpurwalla, 1986). The above results were obtained by Begum and Khan (1998).

Sankaran and Nair (1993) find out the particular case of bivariate Pareto II distribution when $c = ab$. Further, setting $c = ab$ in equations (17), (19) and (20), then we obtain the pdfs of 1^{st} and r^{th} concomitants of order statistics from the particular case of bivariate Pareto II distribution introduced by Sankaran and Nair (1993). It is an interesting case that the pdfs of 1^{st} and r^{th} concomitants of order statistics both are equal to the pdf of Y, i.e,

$$f(y) = \frac{bp}{(1+by)^{p+1}}; \qquad y > 0$$

which is known as univariate Pareto II distribution.

3 Moments of $Y_{[r:n]}$

Let us denote $\mu_{[r:n]}^{(k)} = E(Y_{[r:n]}^k)$ as the k^{th} moment of $Y_{[r:n]}$. Then for $c = 0$ we get,

$$\mu_{[1:n]}^{(k)} = \frac{n}{(np-k)b^k} \frac{\Gamma(p-k+1)\Gamma(k+1)}{\Gamma p}; \ \ p > k \quad (21)$$

Thus, the k^{th} moment of $Y_{[r:n]}$ is

$$\begin{aligned}\mu_{[r:n]}^{(k)} &= \sum_{i=n-r+1}^{n} (-1)^{i-n+r-1} \binom{i-1}{n-r} \binom{n}{i} \mu_{[1:i]}^{(k)} \\ &= \sum_{i=n-r+1}^{n} (-1)^{i-n+r-1} \binom{i-1}{n-r} \binom{n}{i} \frac{i}{(ip-k)b^k} \frac{\Gamma(p-k+1)\Gamma(k+1)}{\Gamma p};\ p > k\end{aligned}$$

Begum and Khan (1998) have already obtained these results.

Also, using the equation (20), after simplification, we get,

$$\mu_{[r:n]}^{(k)} = \frac{1}{b^k} \frac{\Gamma(p-k+1)\Gamma(k+1)}{\Gamma(p+1)} \frac{\Gamma(n+1)}{\Gamma(n+1-r)} \frac{\Gamma(n+1-\frac{k}{p}-r)}{\Gamma(n+1-\frac{k}{p})};\ \ p > k \tag{22}$$

A recurrence relation satisfied by the moments is

$$\mu_{[r+1:n]}^{(k)} = \frac{(n-r)}{(n-\frac{k}{p}-r)} \mu_{[r:n]}^{(k)}$$

which can be used to evaluate the successive moments.

It may be noted here that,

$$\sum_{r=1}^{n} \mu_{[r:n]}^{(1)} = n \frac{1}{b(p-1)} = nE(Y); \quad p > 1 \tag{23}$$

and

$$\sum_{r=1}^{n} \mu_{[r:n]}^{(2)} = n \frac{2}{b^2(p-1)(p-2)} = nE(Y^2); \quad p > 2 \tag{24}$$

In particular,

$$\frac{\mu_{[2:n]}^{(1)}}{\mu_{[1:n]}^{(1)}} = \frac{(n-1)p}{(n-1)p-1} \tag{25}$$

and hence,

$$p = \frac{\mu_{[2:n]}^{(1)}}{(n-1)[\mu_{[2:n]}^{(1)} - \mu_{[1:n]}^{(1)}]} = \frac{E(Y_{[2:n]}^{(1)})}{(n-1)E\{Y_{[2:n]}^{(1)} - Y_{[1:n]}^{(1)}\}} \tag{26}$$

Thus, an estimate of p can be proposed as

$$\hat{p} = \frac{Y_{[2:n]}^{(1)}}{(n-1)\{Y_{[2:n]}^{(1)} - Y_{[1:n]}^{(1)}\}}$$

in terms of the first two concomitants. Being a ratio it is obviously biased, but can be used a quick estimate of p. The estimate of b is then

$$\hat{b} = \frac{n}{(n\hat{p} - 1)Y^{(1)}_{[1:n]}}.$$

The parameters p and b can be expressed interms of μ and σ^2 as follows

$$p = \frac{2\sigma^2}{\sigma^2 - \mu^2} \quad \text{and} \quad b = \frac{\sigma^2 - \mu^2}{\mu(\sigma^2 + \mu^2)}.$$

Mean of the concomitant of order statistics from the bivariate Pareto II distribution have been shown in Begum and Khan (1998).
For $c = ab$, $\mu^{(k)}_{[1:n]}$ and $\mu^{(k)}_{[r:n]}$ both are equal to the moments of Y which is

$$\frac{\Gamma(p-k)\Gamma(k+1)}{b^k \Gamma p}; \quad p > k.$$

4 Joint Distribution of Two Concomitants $Y_{[r:n]}$ and $Y_{[s:n]}$

For the bivariate pareto II distribution with pdf (6) for $c = 0$, we have,

$$\begin{aligned} g_{[r,s:n]}(y_1, y_2) &= C_{r,s:n} b^2 p^2 (p+1)^2 \sum_{i=0}^{r-1} \sum_{j=0}^{s-r-1} (-1)^{i+j} \binom{r-1}{i} \binom{s-r-1}{j} \frac{1}{(d\gamma)} \\ & \quad \frac{1}{(by_1)^\beta (by_2)^\beta} F^{1:2;1}_{1:1;0} \left(\begin{array}{lllll} \gamma & : \beta \,,\, d \,; & \beta \,; & -\frac{1}{by_2} \,, -\frac{1}{by_1} \\ \gamma+1 & : \; d+1 \,; & - \,; & \end{array} \right); \\ & \quad 0 < y_1, y_2 < \infty \end{aligned} \tag{27}$$

with $\beta = p+2$, $d = sp-np-jp+1$, $\gamma = rp-ip-np+p+2$ and $C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}$ where,

$$F^{p\,:\,q\,;\,k}_{\ell\,:\,m\,;\,n} \left(\begin{array}{llll} (a_p) & : (b_q) \,; & (c_k) \,; & x, \; y \\ (\alpha_\ell) & : (\beta_m) \,; & (\gamma_n) \,; & \end{array} \right) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_{r+s} \prod_{j=1}^{q} (b_j)_r \prod_{j=1}^{k} (c_j)_s}{\prod_{j=1}^{\ell} (\alpha_j)_{r+s} \prod_{j=1}^{m} (\beta_j)_r \prod_{j=1}^{n} (\gamma_j)_s} \frac{x^r}{r!} \frac{y^s}{s!} \tag{28}$$

is known as kampé de Fériet's series (Srivastava and Karlsson, 1985). This joint pdf was obtained by Begum and Khan (1998).

For $c = ab$, the joint distribution of r^{th} and s^{th} concomitants of order statistics is product of two univariate Pareto II distribution, i.e

$$g_{[r,s:n]}(y_1, y_2) = \frac{p^2 b^2}{(1+by_1)^{p+1}(1+by_2)^{p+1}}.$$

We have checked that $\int\int g_{[r,s:n]}(y_1, y_2) dy_1 dy_2 = 1$.

5 Product Moments of Two Concomitants $Y_{[r:n]}$ and $Y_{[s:n]}$

The product moments of two concomitants $Y_{[r:n]}$ and $Y_{[s:n]}$ for $c = 0$ is given by

$$\begin{aligned} E\Big[Y^u_{[r:n]}Y^v_{[s:n]}\Big] &= \int_0^\infty \int_0^\infty y_1^u y_2^v g_{[r,s:n]}(y_1, y_2) dy_1 dy_2 \\ &= C_{r,s:n} p^2 (p+1)^2 \sum_{i=0}^{r-1} \sum_{j=0}^{s-r-1} (-1)^{i+j} \binom{r-1}{i} \binom{s-r-1}{j} \\ & \quad \frac{1}{b^{u+v}} \frac{\Gamma(p-u+1)\Gamma(u+1)}{\Gamma(p+2)} \frac{\Gamma(p-v+1)\Gamma(v+1)}{\Gamma(p+2)} \\ & \quad \frac{1}{(sp-np-jp-p+v)(rp-ip-np-p+u+v)} \end{aligned} \tag{29}$$

This result was given by Begum and Khan (1998). Here, if we put $u = 0$ or $v = 0$, then we obtain the single moments of concomitant of ordered statistics.

For $u = 1$ and $v = 1$, we get,

$$\begin{aligned} E\Big[Y_{[r:n]}Y_{[s:n]}\Big] &= \frac{1}{b^2} C_{r,s:n} \sum_{i=0}^{r-1} \sum_{j=0}^{s-r-1} (-1)^{i+j} \binom{r-1}{i} \binom{s-r-1}{j} \\ & \quad \frac{1}{(sp-np-jp-p+1)(rp-np-ip-p+2)} \end{aligned} \tag{30}$$

It may be noted here that

$$\begin{aligned} \sum_{r=1}^{n} \sum_{s=1}^{n} E\Big[Y_{[r:n]}Y_{[s:n]}\Big] &= n\Big[\frac{2}{b^2(p-1)(p-2)}\Big] + n(n-1)\Big[\frac{1}{b(p-1)}\Big]^2 \\ &= nE(Y^2) + n(n-1)E^2(Y). \end{aligned}$$

The product moments between two concomitants of order statistics have been shown in Begum and Khan(1998). For $c = ab$, the $(u, v)^{th}$ product moments between two concomitants of order statistics is the product of two single moments of concomitants of order statistics.

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