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On a Leslie Matrix Type of Projection Process for a Hierarchical Manpower System

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Abstract

In a hierarchical manpower system, it is sometimes desired to have a stable grade structure or there would be a management policy to bring a top-heavy grade structure to bottom-heavy consistently over time. Here we have developed a projection model of grade structure based on Leslie Matrix using wastage, recruitment and other related parameters as would be decided by the management policy.

Keywords and Phrases: Leslie matrix, Wastage, Promotion, Projection.

1 Introduction

In a hierarchical manpower system *Control theory* has often been employed for deciding an optimal grade structure. How *top-heavy* structures can gradually be reduced to *bottom-heavy* structure ultimately in a stable form has been discussed at length (Bartholomew, 1982). The present investigation is a description of the process of projection *viz.*, how a particular grade pyramid can be gradually converted to a desired form with appropriate control of the administration policies.

Population projection models of various types have been developed over the years. The majority of them are based on the projection of a population in a single region (Keyfitz,1977) simply taking account of births and deaths in the regional population. Of the late, attempts have been made to extend this technique to the multi-regional projection, by allowing movements between regions(Rogers, 1973 & 1975). Leeson (1980) develops a model on the latter type, which enable a projection of a hierarchical manpower system to include wastage and promotion intensities. His model is designed to describe how observed wastage and promotion rates will shape the internal structure of the system should these persist. The model also determines recruitment rates, necessary to achieve a desired grade structure.

In the present paper, we have attempted to develop a projection model based on Leslie Matrix using the wastage and recruitment data and the other related parameters decided by the management policy. Often this management policy is motivated to bring a *top-heavy* structure to *bottom-heavy* consistently, as it will be shown in our numerical illustration based on some specific assumptions as may be decided by the management authorities.

2 Some Basic Assumptions

(i) Wastage over different grades conform to a known stable vector independent of time.

(ii) Vacancies due to wastage are partially filled up by promotion of manpower in the lower grade to next higher grade. Also the vacancies created by personal promotion of manpower from one grade to the next are partially filled up by promotion and partially by recruitment, depending on the policy of the management.

(iii) As it will be increasingly difficult to get skilled personnel in the higher grades, therefore projection policy will be based on consistently declining rates of recruitment as grade advances. The remaining vacancies will be filled up by promotion by one grade alone.

(iv) It is also assumed that C_i proportions of persons in the i-th grade $(i = 1, 2, \cdots, l)$ who have been promoted to the $(i+1)$-th grade are absorbed in the $i-th$ grade only. C_i's are so chosen such that a *bottom-heavy* structure can be maintained (or achieved).

(v) The system has a provision for promotion from grade i to $(i+1)$ with a probability $\pi_{i,i+1}$, Which will decrease with increasing value. However the vacancies created by personal promotion from grade i to $(i+1)$ is increasingly filled up by personal promotion and less by direct recruitment because of the reason stated in (iv).

3 Notations

(i) $n_i^{(t)}$ $\equiv$Size of the manpower in the i-th grade in the t-th year $(i = 1, 2, \cdots, l)$.

(ii) $W = \left(w_1^{(t)}, w_2^{(t)}, \cdots, w_l^{(t)}\right)' = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_l \end{bmatrix}$ $\equiv$ Wastage vector, assuming stability in the wastage pattern in various grades over time.

(iii) $r_i^{(t)} = r_i \equiv$ Probability of fresh recruitment in the i-th grade.

(iv) $\pi_{i,i+1} \equiv$ Probability of promotion from grade i to $i+1$.

4 The Projection Process

Following the above assumptions, we have the system of equation as follows:

$$\begin{array}{rcl}
n_1^{(t)} & = & C_1\pi_{12}n_1^{(t-1)} + (1-\pi_{12})n_1^{(t-1)} + n_1^{(t-1)}w_1r_1 \\
n_2^{(t)} & = & C_2\pi_{23}n_2^{(t-1)} + \pi_{12}n_1^{(t-1)} + n_2^{(t-1)}w_2r_2 \\
n_3^{(t)} & = & C_3\pi_{34}n_3^{(t-1)} + \pi_{23}n_2^{(t-1)} + n_3^{(t-1)}w_3r_3 \\
\vdots & = & \vdots \\
n_{l-1}^{(t)} & = & C_{l-1}\pi_{l-1,l}n_{l-1}^{(t-1)} + \pi_{l-2,l-1}n_{l-2}^{(t-1)} + n_{l-1}^{(t-1)}w_{l-1}r_{l-1} \\
n_l^{(t)} & = & \pi_{l-1,l}n_{l-1}^{(t-1)} + n_l^{(t-1)}w_lr_l
\end{array} \tag{1}$$

Putting these equations in the matrix form, we have

$$\begin{bmatrix} n_1^{(t)} \\ n_2^{(t)} \\ \vdots \\ n_l^{(t)} \end{bmatrix}_{l\times 1} = \begin{bmatrix} C_1\pi_{12}+(1-\pi_{12})+w_1r_1 & 0 & 0 & \cdots & 0 \\ \pi_{12} & C_2\pi_{23}+w_2r_2 & 0 & \cdots & 0 \\ 0 & \pi_{23} & C_3\pi_{34}+w_3r_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \pi_{l-1,l} & w_lr_l \end{bmatrix}_{l\times l} \times \begin{bmatrix} n_1^{(t-1)} \\ n_2^{(t-1)} \\ \vdots \\ n_l^{(t-1)} \end{bmatrix}_{l\times 1}$$

or using vector notations, we have

$$\mathbf{N^{(t)}} = \mathbf{LN^{(t-1)}} \tag{2}$$

Where $\mathbf{N^{(t)}}$ and $\mathbf{N^{(t-1)}}$ represent the grade structure in the t-th and $(t-1)$-th year respectively and $\mathbf{L}$ is the Leslie Matrix (evolved by P.H. Leslie: 1945, 1948), consisting elements which are assumed to be time-independent.

Thus with the time independent Leslie Matrix

$$\begin{array}{rcl}
\mathbf{N^{(t)}} & = & \mathbf{LN^{(t-1)}} \\
& = & \mathbf{L^2N^{(t-2)}} \\
\cdots & & \cdots \\
& = & \mathbf{L^tN^{(0)}} \\
\Rightarrow \mathbf{N^{(t)}} & = & \mathbf{L^tN^{(0)}}
\end{array} \tag{3}$$

Now assuming **L** to be non-singular and possessing distinct characteristic root λ_i $(i = 1, 2, \cdots, l)$, we have by spectral decomposition,

$$L^t = \frac{\lambda_1^t X_1 Y_1'}{\partial_1} + \frac{\lambda_2^t X_2 Y_2'}{\partial_2} + \cdots + \frac{\lambda_l^t X_l Y_l'}{\partial_l} \tag{4}$$

where $\delta_i = \mathbf{X_i Y_i}$, $\mathbf{X_i}$ and $\mathbf{Y_i}$ being the right and left charateristic vectors respectively corresponding to the characteristic root λ_i $(i = 1, 2, \cdots, l)$.

Again since **L** is non-singular, there exits a non-singular matrix $\mathbf{Q} \ni$

$$\mathbf{Q^{-1}LQ} = \mathbf{\Lambda} = \mathrm{diag}(\lambda_\mathbf{1}, \lambda_\mathbf{2}, \cdots, \lambda_\mathbf{l})$$

$$\Rightarrow \ \mathbf{L^t} = \mathbf{Q\Lambda^t Q^{-1}} \tag{5}$$

Below we present a result without proof:

Result 1: *Leslie Matrix* **L** *has only one positive eigen value and the rest of the eigen values are either complex or negative. Further if λ_1 is the positive eigen value of* **L**, *then $|\lambda_i| \leq \lambda_l$, where $\lambda_i (i = 1, 2, \cdots, l)$ are complex or negative eigen values of* **L**. *In other words, λ_l is most dominant.*

Now from (3) and (5), we get

$$\frac{N^{(t)}}{\lambda_l^t} = \frac{\mathbf{Q\Lambda^t Q^{-1} N^{(0)}}}{\lambda_l^t} \tag{6}$$

Let $\mathbf{Q^{-1}N^{(0)}} = \begin{bmatrix} c \\ * \\ \vdots \\ * \end{bmatrix}_{\mathbf{l\times 1}}$, i.e., first element of $\mathbf{Q^{-1}N^{(0)}}$ is c, where as other elements may be arbitrary and denoted by $*$.

$$\Rightarrow \frac{N^{(t)}}{\lambda_l^t} = \frac{\mathbf{Q\Lambda^t}}{\lambda_l^t} \begin{bmatrix} c \\ * \\ \vdots \\ * \end{bmatrix} = \mathbf{Q}\mathrm{diag}(\mathbf{1}, \mathbf{0}, \cdots, \mathbf{0}) \begin{bmatrix} c \\ * \\ \vdots \\ * \end{bmatrix}, \ \text{ as } \mathbf{t} \to \infty.$$

[since $\lambda_1 \geq |\lambda_i|$, $i = 2, 3, \cdots, l$, $\frac{\lambda_i^t}{\lambda_l^t} \to 0$ as $t \to 0$, $i = 2, 3, \cdots, l$]

$\therefore$ For large t,

$$\frac{N^{(t)}}{\lambda_l^t} = \frac{\mathbf{Q}}{\lambda_l^t} \begin{bmatrix} c \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

and

$$\frac{N^{t+1}}{\lambda^{t+1}} = \frac{N^{t+2}}{\lambda^{t+2}} = \cdots = \mathbf{Q} \begin{bmatrix} c \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \text{ as } \mathbf{t} \to \infty.$$

$$\therefore \; N^{(t+1)} = \lambda_1 N^{(0)}. \tag{7}$$

This result holds only when t is sufficiently large. Suppose we go on with $N^{(0)}$ and get $N^{(1)}$ by using the most dominant characteristic root λ_1 (by *power method*). $N^{(1)}$ will differ from $N^{(0)}$ normally. But while repeating the process when $N^{(t)} \cong N^{(t-l)}$, we shall designate t to be the minimum time, when the *bottom-heavy* desirable grade structure will be realised.

5 Determination of Elements of the Projection Matrix

As explained earlier, the projection process will lead to the *bottom-heavy* structure and so with a view to substantiate the same, we propose to have the following devices:

(i) In the lower grade, manpower will be mostly filled-up by direct recruitment and less by promotion.

(ii) We assume that C_i's are monotonically increasing number and that r_i's are monotonically decreasing and $\pi_{12} < \pi_{23} < \cdots < \pi_{l-1,l}$.

(iii) $w_1, w_2, \cdots, w_l$ are known and r_i $(i = 1, 2, \cdots, l)$ and c_i $(i = 1, 2, \cdots, l-1)$ are decided by the management polocy of the organization.

Suppose specifically,

(a) $\pi_{j,j+1} = \frac{1}{2^{j-1}} \pi_{j-1,j}$; $j \neq 1$, $j = 2, 3, \cdots, l-1$. π_{12} is pre-assigned.

(b) $C_j = C_{j-1} + m$; $j \neq 1$; $j = 2, 3, \cdots, l-1$, where m is any positive number.

(c) r_j's are usually chosen arbitrarily by the Manpower Planners keeping in view of decreasing level of skilled manpower at higher grades. However, a reasonable choice conforming this pattern may be as follows:

$$r_j = \left[\frac{1}{2}\right]^{j-1} r_{j-1};\; j \neq 1,\; j = 2, 3, ..., l.$$

Therefore having the values of π_{12}, C_1, m and r_1 it is possible to get the entire projection matrix $\mathbf{L}$. One may note that for $C_1 > 1$, the system is expanding, for $C_1 < 1$, it is contracting and if $C_1 = 1$, the system is called a fixed size system.

6 Numerical Examples

Let us consider a system with $l = 3$ grades. Then the Leslie Matrix will be given by

$$\begin{bmatrix} n_1^{(t)} \\ n_2^{(t)} \\ n_3^{(t)} \end{bmatrix} = \begin{bmatrix} C_1\pi_{12} + (1-\pi_{12}) + w_1 r_1 & 0 & 0 \\ \pi_{12} & C_2\pi_{23} + w_2 r_2 & 0 \\ 0 & \pi_{23} & w_3 r_3 \end{bmatrix} \begin{bmatrix} n_1^{(t-1)} \\ n_2^{(t-1)} \\ n_3^{(t-1)} \end{bmatrix}$$

Example 1: Let our initial grade vector is given by

$$\mathbf{N}^{(0)} = (0.52, 0.37, 0.11)'$$
$$\pi_{12} = 0.35, \ \ \pi_{23} = 0.5(0.35) = 0.175$$
$$C_1 = 0.45, \ \ C_2 = 0.75, \ \ m = 0.30$$
$$\mathbf{w} = (0.1, 0.1, 0.2)', \ \ \mathbf{r} = (0.158, 0.210, 0.632)'$$

Here $\mathbf{r}$ is also arbitrarily chosen.

Then the projection matrix $\mathbf{L}$ is given by $\mathbf{L} = \begin{bmatrix} 0.8233 & 0 & 0 \\ 0.3500 & 0.152 & 0 \\ 0 & 0.175 & 0.126 \end{bmatrix}$ and from (3), for $t = 1$, $N^{(1)} = (0.428, 0238, 0.075)'$ which gives the grade structure in the year $t = 1$. One may note that although this system is contracting, ($C_1 < 1$), the of manpower in different grades are same as that of $N^{(0)}$ i.e., about 57%, 32% and 11% of the total manpower in the grade, I, II, and III respectively.

Example 2: Next let us take the recruitment vector in a specified form (in geometric progression)

$\mathbf{r} = (0.62, 0.31, 0.16)$, and $\pi_{12} = 0.35$, $\pi_{23} = 0.175$,
$C_1 = 0.45$, $m = 0.20$, $C_2 = 0.65$, $w = (0.2, 0.1, 0.1)'$
$N^{(0)} = (0.58, 0.35, 0.07)'$

$$\therefore \quad \mathbf{L} = \begin{bmatrix} 0.9315 & 0 & 0 \\ 0.3500 & 0.145 & 0 \\ 0 & 0.175 & 0.016 \end{bmatrix} \text{ and}$$

$$\begin{array}{llll} N^{(1)} = (0.54, & 0.25, & 0.06)' & \\ \quad\ \ [65\%, & 29\%, & 7\%] & \end{array} \qquad \begin{array}{llll} N^{(2)} = (0.50, & 0.23, & 0.05)' & \\ \quad\ \ [65\%, & 30\%, & 5\%] & \end{array} \quad \text{etc.}$$

These results show how the initial grade structure becomes more bottom-heavy as time advances (in the unit of year) given the above promotion as well as recruitment policy.

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