International Journal of Statistical Sciences *ISSN 1683–5603*
Vol. 1, 2002, pp 25–35

Some New Adjusted Test Procedures for RBD Assuming Error Variances Vary From Cell to Cell

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[Received March 7, 2002; Revised August 8, 2002; Accepted October 3, 2002]

Abstract

It is well known that usual ANOVA- F tests are not valid for experimental data having some sort of heteroscedasticity. Some new adjusted test procedures are developed in this paper for RBD assuming error variances vary from cell to cell.

1 Introduction

In this paper, we have considered the case where error variances (σ_{ij}^2's) vary from cell to cell of a Randomized Block Design (RBD) having r observations per cell. On the assumption that the error variances are known Weighted Analysis of Variance (WANOVA) are derived. Such WANOVA χ^2 or F tests can be easily studentized using suitable unbiased estimators of the error variances. New test procedures are derived from such studentized χ^2 or F tests using the well known Meier's theorem (1953). Talukder (1976) and Sen (1984) had shown that such adjusted studentized test procedures are good for practical uses.

2 Heteroscedastic Two-way Model

In heteroscedastic two-way classification, the observations are classified according to two criteria or factors, say different treatments (A) and different blocks (B). Suppose there are p treatments and q blocks in such an experiment. The observations in such an experiment may be arranged in an two-way table. Let there be r observations in each of the pq cells of the table. Let y_{ijk} be the k-th observation in the $(i.j)$-th cell. Then the additive two way fixed effect model for "RBD" may be taken as follows:

$$y_{ijk} = \mu + \alpha_i + \beta_j + e_{ijk} \text{ for } i = 1, 2, \cdots, p; \; j = 1, 2, \cdots, q; \; k = 1, 2, \cdots, r \quad (1)$$

with the constraints $\sum_i V_{ij}\alpha_i = \sum_j V_{ij}\beta_j = 0; \forall i\&j$, where $V_{ij} = \frac{1}{\sigma_{ij}^2}$
$\mu =$ the general mean,
$\alpha_i =$ effect due to the $i-th$ treatment (A)
$\beta_j =$ effect due to the $j-th$ block (B)
and e_{ijk}= the random error term assumed to be independently normally distributed with mean zero and variances σ_{ij}^2's which vary from cell to cell. Our main object is to test the following hypotheses about the fixed effects of the factors:

$$H_A : \alpha_1 = \alpha_2 = \cdots = \alpha_p \quad \text{against} \quad H_{0A} : \text{ all } \alpha_i\text{'s are not equal.} \tag{2}$$

$$H_B : \beta_1 = \beta_2 = \cdots = \beta_q \quad \text{against} \quad H_{0B} : \text{ all } \beta_j\text{'s are not equal.} \tag{3}$$

3 Weighted Analysis of Variance (WANOVA)

Let us assume, to start with that the weights, $V_{ij} = \frac{1}{\sigma_{ij}^2}$, the reciprocal of the error variances (σ_{ij}^2's) for $i = 1, 2, \cdots, p$ and $j = 1, 2, \cdots, q$ are known. To get weighted least square (WLS) estimators of the parameters in the model (1), we are to minimize $\sum_i \sum_j \sum_k V_{ij}\left(y_{ijk} - \mu - \alpha_i - \beta_j\right)^2$ with respect to μ, α_i, β_j and the normal equations are as follows:

$$\begin{aligned}
\mu \, &: r\hat{\mu}\sum_i\sum_j V_{ij} + r\sum_i\sum_j V_{ij}\hat{\alpha}_i + r\sum_i\sum_j \hat{V}_{ij}\hat{\beta}_j = \sum_i\sum_j\sum_k V_{ij}y_{ijk} \\
\alpha_i \, &: r\hat{\mu}\sum_j V_{ij} + r\hat{\alpha}_i\sum_j V_{ij} + r\sum_j V_{ij}\hat{\beta}_j = \sum_j\sum_k V_{ij}y_{ijk} \\
\beta_j \, &: r\hat{\mu}\sum_i V_{ij} + r\sum_i V_{ij}\hat{\alpha}_i + r\hat{\beta}_j\sum_i V_{ij}\hat{\beta}_j = \sum_i\sum_k V_{ij}y_{ijk}
\end{aligned}$$

Since the equations are dependent, the following WLS estimators are obtained by using the constraints $\sum_i V_{ij}\hat{\alpha}_i = \sum_j V_{ij}\hat{\beta}_j = 0$

$$\begin{aligned}
\hat{\mu} &= \sum_i\sum_j\sum_k V_{ij}y_{ijk}/rV_{\cdot\cdot} = \tilde{y}_{\cdots} \text{ (say) with } V_{\cdot\cdot} = \sum_i\sum_j V_{ij} \\
\hat{\alpha}_i &= \sum_j\sum_k V_{ij}y_{ijk}/rV_{i\cdot} - \hat{\mu} = \tilde{y}_{i\cdot\cdot} - \tilde{y}_{\cdots} \\
\text{and } \hat{\beta}_j &= \sum_i\sum_k V_{ij}y_{ijk}/rV_{\cdot j} - \hat{\mu} = \tilde{y}_{\cdot j\cdot} - \tilde{y}_{\cdots}
\end{aligned}$$

the bar ($\sim$) over $y_{\cdots}$ implies that it is a function of σ_{ij}^2's and this convention of notation will be followed through out this paper.

Here the WLS estimators of α_i's and β_j's are unbiased and hence the H_A and H_B are testable as follows in usual ANOVA method:

Now substituting the values of $\hat{\mu}$, $\hat{\alpha_i}$ and $\hat{\beta_j}$ in (1) we get,

$$\begin{aligned} \hat{e}_{ijk} &= y_{ijk} - \hat{\mu} - \hat{\alpha}_i - \hat{\beta}_j \\ &= y_{ijk} - \tilde{y}_{...} - \tilde{y}_{i..} + \tilde{y}_{...} - \tilde{y}_{.j.} + \tilde{y}_{...} \\ &= y_{ijk} - \tilde{y}_{i..} - \tilde{y}_{.j.} + \tilde{y}_{...} \end{aligned}$$

Now from (1) we have

$$\begin{aligned} y_{ijk} &= \hat{\mu} + \hat{\alpha}_i + \hat{\beta}_j + \hat{e}_{ijk} \\ &= \tilde{y}_{...} + (\tilde{y}_{i..} - \tilde{y}_{...}) + (\tilde{y}_{.j.} - \tilde{y}_{...}) + (y_{ijk} - \tilde{y}_{i..} - \tilde{y}_{.j.} + \tilde{y}_{...}) \\ \Rightarrow (y_{ijk} - \tilde{y}_{...}) &= (\tilde{y}_{i..} - \tilde{y}_{...}) + (\tilde{y}_{.j.} - \tilde{y}_{...}) + (y_{ijk} - \tilde{y}_{i..} - \tilde{y}_{.j.} + \tilde{y}_{...}) \end{aligned}$$

Squaring and multiplying both sides by V_{ij} and then summing over i, j and k we have,

$$\begin{aligned} \sum_i \sum_j \sum_k V_{ij}(y_{ijk} - \tilde{y}_{...})^2 &= r \sum_i V_{i.}(\tilde{y}_{i..} - \tilde{y}_{...})^2 + rp \sum_j V_{.j}(\tilde{y}_{.j.} - \tilde{y}_{...})^2 \\ &\quad + \sum_i \sum_j \sum_k V_{ij}(y_{ijk} - \tilde{y}_{i..} - \tilde{y}_{.j.} + \tilde{y}_{...})^2 \end{aligned}$$

as product terms are zero.
or, $SST(V) = SSA(V) + SSB(V) + SSE(V)$ [in usual notation]
with $(pqr - 1)$ d.f. $= (p - 1)$ d.f. $+ (q - 1)$ d.f. $+ (pqr - p - q + 1)$ d.f.

The corresponding WANOVA table obtained is given below:

Table 1: WANOVA table

Sources	d.f.	SS	E(SS)
Treatment	$(p-1)$	SSA(V)	$(p-1)+r\sum_i V_{i.}\alpha_i^2$, where $V_{i.} = \sum_j V_{ij}$
Block	$(q-1)$	SSB(V)	$(q-1)+r\sum_i V_{.j}\beta_j^2$, where $V_{.j} = \sum_i V_{ij}$
Error	(pqr-p-q+1)	SSE(V)	$(pqr-p-q+1)$
Total	$(pqr-1)$	SST(V) by adding	- - - - - - - -

By the general theorem of Sen (1991), we can conclude that SSE(V) are central χ^2 with $(pqr - p - q + 1)$ d.f. and SSA(V) and SSB(V) are non-central χ^2 with $(p - 1)$, $(q-1)$ and $(pqr-p-q+1)$ d.f. and with non-centrality parameters $\lambda_{AV} = r\sum_i V_{i.}\alpha_i^2$ and $\lambda_{BV} = r\sum V_{.j}\beta_j^2$ respectively, which becomes zero under H_B and H_{0B}; and also they are independent. Hence the WANOVA F-test for H_A is provided by,

$$\begin{aligned} F(V) &= \frac{SSA(V)/(p-1)}{SSE(V)/(pqr-p-q+1)} \\ &= \frac{r\sum_i \sum_j V_{ij}(\tilde{y}_{i..} - \tilde{y})^2/(p-1)}{\sum_i \sum_j \sum_k V_{ij}(y_{ijk} - \hat{\tilde{y}}_{i..} - \tilde{y}_{.j.} + \tilde{y}_{...})^2/(pqr-p-q+1)} \end{aligned}$$

which has a central F-distribution with $(p-1)$ and $(pqr-p-q+1)$ d.f. under H_A and a non-central F-distribution under H_{0A}.

Likewise, the WANOVA F-test for H_B is provided by,

$$F(V) = \frac{SSB(V)/(q-1)}{SSE(V)/(pqr-p-q+1)}$$
$$= \frac{rp\sum_j V_{\cdot j}\tilde{y}_{\cdot j\cdot} - \tilde{y}_{\cdots})^2/(p-1)}{\sum_i\sum_j\sum_k V_{ij}(y_{ijk} - \tilde{y}_{i\cdot\cdot} - \tilde{y}_{\cdot j\cdot} + \tilde{y}_{\cdots})^2/(pqr-p-q+1)}$$

which has a central F-distribution with $(q-1)$ and $(pqr-p-q+1)$ d.f. under H_B and a non-central F distribution under H_{0B}. Since the variance components are assumed to be known, to test H_A, we can also define WANOVA- χ^2 as follows: $\chi^2(V) = SSA(V) = r\sum_i\sum_j V_{ij}(\tilde{y}_{i\cdot\cdot} - \tilde{y}_{\cdots})$ which follows central χ^2 with $(p-1)$ d.f.

To test H_B, we can similarly define WANOVA as:

$\chi^2(V) = SSB(V) = rp\sum_j V_{\cdot j\cdot}(\tilde{y}_{\cdot j\cdot} - \tilde{y}_{\cdots})^2$ which follows central χ^2 with $(q-1)$ d.f. Thus, the above test procedures are quite complete when and only when unequal error variances are known. But in practice, the error variances are not known to experimenters and hence the test statistic $F(V)$ and $\chi^2(V)$ which are a function of σ_{ij}^2's are not of any use for practical applications. It is therefore clear that as error variances are unequal and unknown neither classical ANOVA nor WANOVA can be derived for practical uses. Of course, suitable studentized test criteria from such WANOVA exact tests can provides us very useful test procedures for practical uses when σ_{ij}^2's are unknown and unequal. This will be discussed in the following sections.

4 Estimation of Error Variances (σ_{ij}^2)

As there are more than one observation per cell we can find Naive Unbiased Estimators (NUE) of σ_{ij}^2's for which we consider

$$E\left[\sum_{k=1}^{r}(y_{ijk} - y_{ij\cdot})^2\right] = E\left[\sum_{k=1}^{r}(e_{ijk} - \bar{e}_{ij\cdot})^2\right] = (r-1)\sigma_{ij}^2$$

So,

$$E\left[\sum_{k=1}^{r}(y_{ijk} - y_{ij\cdot})^2/(r-1) = S_{ij}^2 \quad \text{(say)}\right] \tag{4}$$

which is unbiased estimator for σ_{ij}^2 and clearly under normality assumption of y_{ijk}, $(r-1)S_{ij}^2/\sigma_{ij}^2$ follows exact χ^2 distribution with $(r-1)$ d.f. and also such χ^2's are independently distributed.

5 Adjustment of the Test Statistics using Estimated Weights

WANOVA-χ^2 or WANOVA-F test statistics involve actual weights, the reciprocals of the error variances. If the estimators of error variances are used in place of actual ones in these test statistics then bias will be introduced. It is difficult to obtain the magnitudes of these biases analytically. But, since the estimators of error variances are independent, bias of some order can be eliminated by adjusting these statistics with the help of a theorem due to Meier (1953), stated as follows: If $X_i, i = 1, 2, \cdots, t$ are independently mean χ^2 distributed random variables with n_i d.f. and $R(X_1, X_2, \cdots, X_t)$ is a rational function with no singularities for $0 < X_1, X_2, \ldots, X_t < \infty$ then $E[R(X_1, X_2, \cdots, X_t)]$ can be expanded in an asymptotic series in the $(1/n_i)$. In particular

$$E[R(X_1, X_2, \cdots, X_t)] = R[1, \cdots, 1] + \sum_{i=1}^{t} \frac{1}{n_i} \left[\frac{\partial^2 R}{\partial X_i^2}\right] + 0 \times \left(\sum \frac{1}{n_i^2}.\right)$$

Here All $X_i = 1$. This theorem implies that the adjusted statistic

$$R[X_1, X_2, \cdots, X_t] - \sum_{i=1}^{t} \frac{1}{n_i} \left[\frac{\partial^2 R}{\partial X_i^2}\right] \frac{1}{n_i},$$

All $X_i = 1$ being free from terms of order $\sum \frac{1}{n_i}$, approximate the actual value, $R[1, \cdots, 1]$ of the function more closely than $R(X_1, X_2, \cdots, X_t)$ itself.

To test H_A and H_B most desirable estimator of σ_{ij}^2 is S_{ij}^2 of (4). Since $\frac{(r-1)S_{ij}^2}{\sigma_{ij}^2}$ follows χ^2 distribution with $(r-1)$ d.f. so $\frac{S_{ij}^2}{\sigma_{ij}^2} = X_{ij}$ (say) follows mean χ^2 so that $S_{ij}^2 = X_{ij}\sigma_{ij}^2$. Now the estimated weight is $\hat{V}_{ij} = \frac{1}{\hat{\sigma}_{ij}^2} = \frac{1}{S_{ij}^2} = \frac{1}{X_{ij}\sigma_{ij}^2}$ for all i and j.

6 New Test Procedures for H_A

$$\begin{aligned} SSA(\hat{V}) &= r \sum_i \hat{V}_{i\cdot}(\hat{\bar{y}}_{i\cdot\cdot} - \hat{\bar{y}}_{\cdot\cdot\cdot})^2 \\ &= \frac{r\sum_i(\hat{\bar{y}}_{i\cdot\cdot} - \hat{\bar{y}}_{\cdot\cdot\cdot})^2}{x_{ij}\sigma_{ij}^2} + \sum_{m\neq i}\sum_{l\neq j} \frac{r}{X_{ml}\sigma_{ml}^2}(\bar{y}_{m\cdot\cdot} - \hat{\bar{y}}_{\cdot\cdot\cdot})^2 \end{aligned} \qquad (5)$$

where, $\hat{V}_{ij} = \frac{1}{X_{ij}\sigma_{ij}^2}$

Now, we have,

$$\frac{\partial(\hat{\bar{y}}_{\cdot j\cdot} - \hat{\bar{y}}_{\cdot\cdot\cdot})^2}{\partial X_{ij}} = \frac{2(\hat{\bar{y}}_{i\cdot\cdot} - \hat{\bar{y}}_{\cdot\cdot\cdot})}{X_{ij}^2\sigma_{ij}^2}\left\{\frac{1}{\hat{V}_{i\cdot}}(\hat{\bar{y}}_{i\cdot\cdot} - \bar{y}_{ij\cdot}) + \frac{1}{\hat{V}_{\cdot\cdot}}(y_{ij\cdot} - \hat{\bar{y}}_{\cdot\cdot\cdot})\right\}$$

and

$$\frac{\partial(\bar{y}_{m\cdot\cdot} - \hat{\bar{y}}_{\dots})^2}{\partial X_{ij}} = \frac{2}{X_{ij}^2 \sigma_{ij}^2 V}(\bar{y}_{m\cdot\cdot} - \hat{\bar{y}}_{\dots})(\bar{y}_{ij\cdot} - \hat{\bar{y}}_{\dots})$$

Thus

$$\frac{\partial}{\partial X_{ij}}\left[SSA(\hat{V})\right] = -\frac{r}{X_{ij}^2\sigma_{ij}^2}(\hat{\bar{y}}_{i\cdot\cdot} - \hat{\bar{y}}_{\dots})^2 + \frac{2r(\hat{\bar{y}}_{i\cdot\cdot} - \hat{\bar{y}}_{\dots})}{X_{ij}^3\sigma_{ij}^4}\left[\frac{(\hat{\bar{y}}_{i\cdot\cdot} - \bar{y}_{ij\cdot})}{\hat{V}_{i\cdot}} + \frac{(\bar{y}_{ij\cdot} - \hat{\bar{y}})}{\hat{V}_{\cdot\cdot}}\right]$$
$$+ \sum_{m\neq i}\sum_{l\neq j}\frac{2r(\hat{\bar{y}}_{m\cdot\cdot} - \hat{\bar{y}}_{\dots})(\bar{y}_{ij\cdot} - \hat{\bar{y}}_{\dots})}{X_{ml}\sigma_{ml}^2 X_{ij}^2\sigma_{ij}^2\hat{V}} \qquad (6)$$

Taking partial derivative of this again and putting $X_i = 1$ for all i and simplifying we get,

$$\left[\frac{\partial^2(SSA(\hat{V}))}{\partial X_i^2}\right]_{\text{all } X_i=1} = -\frac{4r\hat{V}_{ij}^2(\hat{\bar{y}}_{i\cdot\cdot} - \hat{\bar{y}}_{\dots})(\hat{y}_{ij\cdot} - \hat{\bar{y}}_{\dots})}{\hat{V}_{\cdot\cdot}} - \frac{8r\hat{V}_{ij}^2(\hat{\bar{y}}_{i\cdot\cdot} - \bar{y}_{ij\cdot})}{\hat{V}_{i\cdot}}$$
$$+\frac{4r\hat{V}_{ij}^3(\hat{\bar{y}}_{i\cdot\cdot} - \hat{\bar{y}}_{\dots})(\hat{\bar{y}}_{i\cdot\cdot} - \bar{y}_{ij\cdot})}{\hat{V}_{i\cdot}^2} + \frac{4r\hat{V}_{ij}^3(\hat{\bar{y}} - \bar{y}_{ij\cdot})(y_{ij\cdot} - \hat{\bar{y}}_{\dots})}{\hat{V}_{i\cdot}\hat{V}_{\cdot\cdot}}$$
$$+\frac{4r\hat{V}_{ij}^2(\bar{y}_{ij\cdot} - \hat{\bar{y}}_{\dots})}{\hat{V}_{\cdot\cdot}^2}\sum_i\sum_j \hat{V}_{ij}(\hat{\bar{y}}_{i\cdot\cdot} - \hat{\bar{y}}_{\dots})$$
$$-\frac{4r\hat{V}_{ij}(\bar{y}_{ij\cdot} - \hat{\bar{y}}_{\dots})}{\hat{V}_{\cdot\cdot}}\sum_i\sum_j \hat{V}_{ij}(\hat{\bar{y}}_{i\cdot\cdot} - \hat{\bar{y}}_{\dots}) + 2r\hat{V}_{ij}(\hat{\bar{y}}_{i\cdot\cdot} - \hat{\bar{y}}_{\dots})^2$$
$$+\frac{2r\hat{V}_{ij}^3(\hat{\bar{y}}_{i\cdot\cdot} - \bar{y}_{ij\cdot})^2}{\hat{V}_{i\cdot}^2} + \frac{2r\hat{V}_{ij}^2(\bar{y}_{ij\cdot} - \hat{\bar{y}}_{\dots})^2}{\hat{V}_{\cdot\cdot}} \qquad (7)$$

Hence, by the theorem of Meier (1953),

$$SSA(\hat{V})\,(\text{adjusted}) = r\sum_i\sum_j \hat{V}_{ij}(\hat{\bar{y}}_{i\cdot\cdot} - \hat{\bar{y}}_{\dots}) - \sum_i\sum_j\Big[\frac{4r\hat{V}_{ij}^3(\hat{\bar{y}}_{i\cdot\cdot} - \bar{y}_{ij\cdot})(\bar{y}_{ij\cdot} - \hat{\bar{y}}_{\dots})}{\hat{V}_{i\cdot}\hat{V}_{\cdot\cdot}}$$
$$+\frac{4r\hat{V}_{ij}^3(\hat{\bar{y}}_{i\cdot\cdot} - \bar{y}_{\dots})(\hat{\bar{y}}_{i\cdot\cdot} - \bar{y}_{ij\cdot})}{\hat{V}_{i\cdot}^2} - \frac{4r\hat{V}_{ij}^2(\hat{\bar{y}}_{i\cdot\cdot} - \hat{\bar{y}}_{\dots})(\bar{y}_{ij\cdot} - \hat{\bar{y}}_{\dots})}{\hat{V}_{\cdot\cdot}}$$
$$-\frac{8r\hat{V}_{ij}^2(\hat{\bar{y}}_{i\cdot\cdot} - \hat{\bar{y}}_{\dots})(\hat{\bar{y}}_{i\cdot\cdot} - \bar{y}_{ij\cdot})}{\hat{V}_{i\cdot}} + 2r\hat{V}_{ij}(\hat{\bar{y}}_{i\cdot\cdot} - \bar{y}_{\dots})^2$$
$$+\frac{4r\hat{V}_{ij}^2(\bar{y}_{ij\cdot} - \hat{\bar{y}}_{\dots})(\hat{\bar{y}}_{i\cdot\cdot} - \hat{\bar{y}}_{\dots})pq}{\hat{V}_{\cdot\cdot}}\left(\frac{\hat{V}_{ij}}{\hat{V}_{\cdot\cdot}} - 1\right) + \frac{2r\hat{V}_{ij}^3(\hat{\bar{y}}_{i\cdot\cdot} - \bar{y}_{ij\cdot})^2}{\hat{V}_{i\cdot}^2}$$

$$
\begin{aligned}
&+\frac{2r\hat{V}_{ij}^2(\bar{y}_{ij\cdot}-\hat{\bar{y}}_{ij\cdot})^2}{\hat{V}_{\cdot\cdot}}]/(r-1)\\
=\ & r\sum_i\sum_j \hat{V}_{ij}[(\hat{\bar{y}}_{i\cdot\cdot}-\hat{\bar{y}}_{\cdot\cdot\cdot})^2-\frac{2}{(r-1)}\{\frac{2\hat{V}_{ij}^2(\bar{y}_{ij\cdot}-\hat{\bar{y}}_{\cdot\cdot\cdot})(\hat{\bar{y}}_{i\cdot\cdot}-\bar{y}_{ij\cdot})}{\hat{V}_{i\cdot}\hat{V}_{\cdot\cdot}}\\
&+\frac{2\hat{V}_{ij}^2(\hat{\bar{y}}_{i\cdot\cdot}-\hat{\bar{y}}_{\cdot\cdot\cdot})(\hat{\bar{y}}_{i\cdot\cdot}-\bar{y}_{ij\cdot})}{\hat{V}_{i\cdot}^2}-\frac{2\hat{V}_{ij}(\hat{\bar{y}}_{i\cdot\cdot}-\hat{\bar{y}}_{\cdot\cdot\cdot})(\bar{y}_{ij\cdot}-\hat{\bar{y}}_{\cdot\cdot\cdot})}{\hat{V}_{\cdot\cdot}}\\
&-\frac{4\hat{V}_{ij}(\hat{\bar{y}}_{i\cdot\cdot}-\hat{\bar{y}}_{\cdot\cdot\cdot})(\hat{\bar{y}}_{i\cdot\cdot}-\bar{y}_{ij\cdot})}{\hat{V}_{i\cdot}}+(\hat{\bar{y}}_{i\cdot\cdot}-\hat{\bar{y}}_{\cdot\cdot\cdot})^2+\frac{\hat{V}_{ij}^2(\hat{\bar{y}}_{i\cdot\cdot}-\bar{y}_{ij\cdot})^2}{\hat{V}_{i\cdot}}\\
&+\frac{\hat{V}_{ij}(\bar{y}_{ij\cdot}-\hat{\bar{y}}_{\cdot\cdot\cdot})^2}{\hat{V}_{\cdot\cdot}}+\frac{2\hat{V}_{ij}(\bar{y}_{ij\cdot}-\hat{\bar{y}}_{\cdot\cdot\cdot})(\bar{y}_{i\cdot\cdot}-\hat{\bar{y}}_{\cdot\cdot\cdot})pq}{\hat{V}_{\cdot\cdot}}(\frac{\hat{V}_{ij}}{\hat{V}_{\cdot\cdot}}-1)\}]\\
=\ & \chi_1^2 \text{ (say)}
\end{aligned}
\tag{8}
$$

which follows approx. χ^2 with $(p-1)$ d.f. following Talukder (1976). The approximation is being free from first order, that is, free from terms of order $(\frac{1}{r-1})$.

7 New Test Procedure for H_B

The weighted treatment SS under H_B using estimated weights is,

$$
\begin{aligned}
SSB(\hat{V}) &= r\sum_i\sum_j \hat{V}_{ij}(\hat{\bar{y}}_{\cdot j\cdot}-\hat{\bar{y}}_{\cdot\cdot\cdot})^2\\
&= \frac{r(\hat{\bar{y}}_{\cdot j\cdot}-\hat{\bar{y}}_{\cdot\cdot\cdot})^2}{X_{ij}\sigma_{ij}^2}+\sum_{m\neq i}\sum_{l\neq j}\frac{r}{X_{ml}\sigma_{ml}^2}(\bar{y}_{\cdot l\cdot}-\hat{\bar{y}}_{\cdot\cdot\cdot})^2
\end{aligned}
\tag{9}
$$

where, $\hat{V}_{ij}=\frac{1}{X_{ij}\sigma_{ij}^2}$.

Now, we have,

$$
\frac{\partial(\hat{\bar{y}}_{\cdot j\cdot}-\hat{\bar{y}}_{\cdot\cdot\cdot})^2}{\partial X_{ij}}=\frac{2(\hat{\bar{y}}_{\cdot j\cdot}-\hat{\bar{y}}_{\cdot\cdot\cdot})}{X_{ij}^2\sigma_{ij}^2}\left\{\frac{1}{\hat{V}_{\cdot j}}(\hat{\bar{y}}_{\cdot j}-\bar{y}_{ij\cdot})+\frac{1}{\hat{V}_{\cdot\cdot}}(y_{ij\cdot}-\hat{\bar{y}}_{\cdot\cdot\cdot})\right\}
$$

and

$$
\frac{\partial(\bar{y}_{\cdot l\cdot}-\hat{\bar{y}}_{\cdot\cdot\cdot})^2}{\partial X_{ij}}=\frac{2}{X_{ij}^2\sigma_{ij}^2\hat{V}}(\bar{y}_{\cdot l\cdot}-\hat{\bar{y}}_{\cdot\cdot\cdot})(\bar{y}_{ij\cdot}-\hat{\bar{y}}_{\cdot\cdot\cdot}).
$$

Thus

$$
\frac{\partial}{\partial X_{ij}}\left[SSB(\hat{V})\right]=-\frac{r}{X_{ij}^2\sigma_{ij}^2}(\hat{\bar{y}}_{\cdot j\cdot}-\hat{\bar{y}}_{\cdot\cdot\cdot})^2+\frac{2r(\hat{\bar{y}}-\hat{\bar{y}}_{\cdot\cdot\cdot})}{X_{ij}^3\sigma_{ij}^4}\left[\frac{(\hat{\bar{y}}_{\cdot j\cdot}-\bar{y}_{ij\cdot})}{\hat{V}_{\cdot j}}+\frac{(\bar{y}_{ij\cdot}-\hat{\bar{y}}_{\cdot\cdot\cdot})}{\hat{V}_{\cdot\cdot}}\right]
$$

$$+\sum_{m\neq i}\sum_{l\neq j}\frac{2r(\bar{y}_{.l.}-\hat{\bar{y}}_{...})(\bar{y}_{ij.}-\hat{\bar{y}}_{...})}{X_{ml}\sigma^2_{ml}X^2_{ij}\sigma^2_{ij}\hat{V}_{..}}. \tag{10}$$

Taking partial derivative of this again and putting $X_{ij}=1$ for all i and j and simplifying we get,

$$\begin{aligned}\left[\frac{\partial^2(SSB(\hat{V}))}{\partial X^2_{ij}}\right]_{\text{all }X_i=1} = & -\frac{4r\hat{V}^2_{ij}(\hat{\bar{y}}_{.j.}-\hat{\bar{y}}_{...})(\bar{y}_{ij.}-\hat{\bar{y}}_{...})}{\hat{V}_{..}}-\frac{8r\hat{V}^2_{ij}(\hat{\bar{y}}_{.j.}-\hat{\bar{y}}_{...})(\hat{\bar{y}}_{.j.}-\bar{y}_{ij.})}{\hat{V}_{.j}}\\ & +\frac{4r\hat{V}^3_{ij}(\hat{\bar{y}}_{.j.}-\hat{\bar{y}}_{...})(\hat{\bar{y}}_{.j.}-\bar{y}_{ij.})}{\hat{V}^2_{.j}}+\frac{4r\hat{V}^3_{ij}(\hat{\bar{y}}_{.j.}-\bar{y}_{ij.})(\bar{y}_{ij.}-\hat{\bar{y}}_{...})}{\hat{V}_{.j}\hat{V}_{..}}\\ & +\frac{4r\hat{V}^2_{ij}(\bar{y}_{ij.}-\hat{\bar{y}}_{...})}{\hat{V}^2_{..}}\sum_i\sum_j\hat{V}_{ij}(\hat{\bar{y}}_{.j.}-\hat{\bar{y}}_{...})\\ & -\frac{4r\hat{V}_{ij}(\bar{y}_{ij.}-\hat{\bar{y}}_{...})}{\hat{V}_{..}}\sum_i\sum_j\hat{V}_{ij}(\hat{\bar{y}}_{.j.}-\hat{\bar{y}}_{...})+2r\hat{V}_{ij}(\hat{\bar{y}}_{.j.}-\hat{\bar{y}}_{...})^2\\ & +\frac{2r\hat{V}^3_{ij}(\hat{\bar{y}}_{.j.}-\bar{y}_{ij.})^2}{\hat{V}^2_{.j}}+\frac{2r\hat{V}^2_{ij}(\bar{y}_{ij.}-\hat{\bar{y}}_{...})^2}{\hat{V}_{..}}.\end{aligned} \tag{11}$$

Hence, by the theorem of Meier (1953),

$$\begin{aligned}SSB(\hat{V})\,(\text{adjusted}) = & \; r\sum_i\sum_j\hat{V}_{ij}(\hat{\bar{y}}_{.j.}-\hat{\bar{y}}_{...})^2-\sum_i\sum_j[\frac{4r\hat{V}^3_{ij}(\hat{\bar{y}}_{.j.}-\bar{y}_{ij.})(\bar{y}_{ij.}-\hat{\bar{y}}_{...})}{\hat{V}_{.j}\hat{V}_{..}}\\ & +\frac{4r\hat{V}^3_{ij}(\hat{\bar{y}}_{.j.}-\hat{\bar{y}}_{...})(\hat{\bar{y}}_{.j.}-\bar{y}_{ij.})}{\hat{V}^2_{.j}}-\frac{4r\hat{V}^2_{ij}(\hat{\bar{y}}_{.j.}-\hat{\bar{y}}_{...})(\bar{y}_{ij.}-\hat{\bar{y}}_{...})}{\hat{V}_{..}}\\ & -\frac{8r\hat{V}^2_{ij}(\hat{\bar{y}}_{.j.}-\hat{\bar{y}}_{...})(\hat{\bar{y}}_{.j.}-\bar{y}_{ij.})}{\hat{V}_{.j}}+2r\hat{V}_{ij}(\hat{\bar{y}}_{.j.}-\hat{\bar{y}}_{...})^2\\ & +\frac{4r\hat{V}^2_{ij}(\bar{y}_{ij.}-\hat{\bar{y}}_{...})(\hat{\bar{y}}_{.j.}-\hat{\bar{y}}_{...})pq}{\hat{V}_{..}}\left(\frac{\hat{V}_{ij}}{\hat{V}_{..}}-1\right)+\frac{2r\hat{V}^3_{ij}(\hat{\bar{y}}_{.j.}-\bar{y}_{ij.})^2}{\hat{V}^2_{.j}}\\ & +\frac{2r\hat{V}^2_{ij}(\bar{y}_{ij.}-\hat{\bar{y}}_{...})^2}{\hat{V}_{..}}]\frac{1}{(r-1)}\\ = & \; r\sum_i\sum_j\hat{V}_{ij}[(\hat{\bar{y}}_{.j.}-\hat{\bar{y}}_{...})^2-\frac{2}{(r-1)}\{\frac{2\hat{V}^2_{ij}(\hat{\bar{y}}_{.j.}-\bar{y}_{ij.})(\bar{y}_{ij.}-\hat{\bar{y}}_{...})}{\hat{V}_{.j}\hat{V}_{..}}\\ & +\frac{2\hat{V}^2_{ij}(\hat{\bar{y}}_{.j.}-\hat{\bar{y}}_{...})(\hat{\bar{y}}_{.j.}-\bar{y}_{ij.})}{\hat{V}^2_{.j}}-\frac{2\hat{V}_{ij}(\hat{\bar{y}}_{.j.}-\hat{\bar{y}}_{...})(\bar{y}_{ij.}-\hat{\bar{y}}_{...})}{\hat{V}_{..}}\end{aligned}$$

$$-\frac{4\hat{V}_{ij}(\hat{\bar{y}}_{\cdot j\cdot}-\hat{\bar{y}}_{\cdots})(\hat{\bar{y}}_{\cdot j\cdot}-\bar{y}_{ij\cdot})}{\hat{V}_{\cdot j}}+(\hat{\bar{y}}_{\cdot j\cdot}-\hat{\bar{y}}_{\cdots})^2+\frac{\hat{V}_{ij}^2(\hat{\bar{y}}_{\cdot j\cdot}-\bar{y}_{ij\cdot})^2}{\hat{V}_{\cdot j}^2}$$
$$+\frac{\hat{V}_{ij}(\bar{y}_{ij\cdot}-\hat{\bar{y}}_{\cdots})^2}{\hat{V}_{\cdot\cdot}}+\frac{\hat{V}_{ij}(\bar{y}_{ij\cdot}-\hat{\bar{y}}_{\cdots})(\hat{\bar{y}}_{\cdot j\cdot}-\hat{\bar{y}}_{\cdots})pq}{\hat{V}_{\cdot\cdot}}(\frac{\hat{V}_{ij}}{\hat{V}_{\cdot\cdot}}-1)\}]$$
$$=\chi_2^2 \text{ (say)} \qquad (12)$$

which follows approx. χ^2 with $(q-1)$ d.f. following Talukder (1976).

8 Alternative New Test Procedure for H_A and H_B

Here, the weighted error SS using estimated weights is,

$$\begin{aligned} SSE(\hat{V}) &= \sum_i\sum_j\sum_k V_{ij}(y_{ijk}-\hat{\bar{y}}_{i\cdot\cdot}-\hat{\bar{y}}_{\cdot j\cdot}+\hat{\bar{y}}_{\cdots})^2 \\ &= \sum_i\sum_j\sum_k\left(\frac{1}{X_{ij}\sigma_{ij}^2}\right)(y_{ijk}-\hat{\bar{y}}_{i\cdot\cdot}-\hat{\bar{y}}_{\cdot j\cdot}+\hat{\bar{y}}_{\cdots})^2 \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial X_{ij}}\left[SSE(\hat{V})\right] &= \sum_k[\frac{2(y_{ijk}-\hat{\bar{y}}_{i\cdot\cdot}-\hat{\bar{y}}_{\cdot j\cdot}+\hat{\bar{y}}_{\cdots})}{X_{ij}^3\sigma_{ij}^4} \\ &\left\{\frac{(\bar{y}_{ij\cdot}-\hat{\bar{y}}_{i\cdot\cdot})}{\hat{V}_{i\cdot}}+\frac{(\bar{y}_{ij\cdot}-\hat{\bar{y}})}{\hat{V}_{\cdot j\cdot}}+\frac{\hat{\bar{y}}-\bar{y}_{ij\cdot}}{\hat{V}_{\cdot\cdot}}\right\} \\ &+\sum_{m\neq i}\sum_{l\neq j}\frac{2(\hat{\bar{y}}_{\cdots}-\bar{y}_{ij\cdot})(y_{mlk}-\bar{y}_{m\cdot\cdot}-\bar{y}_{\cdot l\cdot}+\hat{\bar{y}}_{\cdots})}{X_{ml}\sigma_{ml}^2X_{ij}^2\sigma_{ij}^2\hat{V}_{\cdot\cdot}} \\ &-\frac{(y_{ijk}-\hat{\bar{y}}_{i\cdot\cdot}-\hat{\bar{y}}_{\cdot j\cdot}+\hat{\bar{y}}_{\cdots})^2}{X_{ij}^2\sigma_{ij}^2}] \end{aligned} \qquad (13)$$

Taking partial derivative of this again and putting $X_{ij}=1$ for all i and j and simplifying we get,

$$\begin{aligned} \left[\frac{\partial^2(SSE(\hat{V}))}{\partial X_{ij}^2}\right]_{\text{all } X_{ij}=1} &= \sum_k[2V_{ij}^3\left\{\frac{(\bar{y}_{ij\cdot}-\hat{\bar{y}}_{i\cdot\cdot})}{\hat{V}_{i\cdot}}+\frac{(\bar{y}_{ij\cdot}-\hat{\bar{y}}_{\cdot j\cdot})}{\hat{V}_{\cdot j}}+\frac{\hat{\bar{y}}_{\cdots}-\bar{y}_{ij\cdot}}{\hat{V}_{\cdot\cdot}}\right\}^2 \\ &+4\hat{V}_{ij}^3\left(y_{ijk}-\hat{\bar{y}}_{i\cdot\cdot}-\hat{\bar{y}}_{\cdot j\cdot}+\hat{\bar{y}}_{\cdots}\right) \\ &\left\{\frac{(\bar{y}_{ij\cdot}-\hat{\bar{y}}_{i\cdot\cdot})}{\hat{V}_{i\cdot}^2}+\frac{(\bar{y}_{ij\cdot}-\hat{\bar{y}}_{\cdot j\cdot})}{\hat{V}_{\cdot j}^2}+\frac{pq}{\hat{V}_{\cdot\cdot}}\right\} \end{aligned}$$

$$-4\hat{V}_{ij}^2\left(y_{ijk}-\hat{\bar{y}}_{i..}-\hat{\bar{y}}_{.j.}+\hat{\bar{y}}_{...}\right)$$
$$+\left\{\frac{2(\bar{y}_{ij.}-\hat{\bar{y}}_{i..})}{\hat{V}_{i.}}+\frac{2(\bar{y}_{ij.}-\hat{\bar{y}}_{.j.})}{\hat{V}_{.j}}+\frac{(\hat{\bar{y}}_{...}-\bar{y}_{ij.})pq}{\hat{V}_{..}}\right\}$$
$$+2\hat{V}_{ij}\left(y_{ijk}-\hat{\bar{y}}_{i..}-\hat{\bar{y}}_{.j.}+\hat{\bar{y}}_{...}\right)$$
$$+\frac{2\hat{V}_{ij}^2(\hat{\bar{y}}_{...}-\bar{y}_{ij.})^2}{\hat{V}_{..}}(1-\frac{\hat{V}_{ij}}{\hat{V}_{..}})]. \tag{14}$$

Hence by the theorem of Meier (1953)

$$\begin{aligned}SSE(\hat{V})\,(\text{adj}) &= \sum_k\sum_i\sum_j\hat{V}_{ij}\left(y_{ijk}-\hat{\bar{y}}_{i..}-\hat{\bar{y}}_{.j.}+\hat{\bar{y}}_{...}\right)^2\\
&-\sum\sum\sum[\frac{2\hat{V}_{ij}^3}{(r-1)}\left\{\frac{(\bar{y}_{ij.}-\hat{\bar{y}}_{i..})}{\hat{V}_{i.}}+\frac{(\bar{y}_{ij.}-\hat{\bar{y}}_{.j.})}{\hat{V}_{.j}}+\frac{(\hat{\bar{y}}_{...}-\bar{y}_{ij.})}{\hat{V}_{..}}\right\}^2\\
&+4\hat{V}_{ij}^3\left(y_{ijk}-\hat{\bar{y}}_{i..}-\hat{\bar{y}}_{.j.}+\hat{\bar{y}}_{...}\right)\left\{\frac{(\bar{y}_{ij.}-\hat{\bar{y}}_{i..})}{\hat{V}_{i.}^2}+\frac{(\bar{y}_{ij.}-\hat{\bar{y}}_{.j.})}{\hat{V}_{.j}^2}+\frac{pq}{\hat{V}_{..}^2}\right\}\\
&-4\hat{V}_{ij}^3\left(\bar{y}_{ij.}-\hat{\bar{y}}_{i..}-\hat{\bar{y}}_{.j.}+\hat{\bar{y}}_{...}\right)\\
&\left\{\frac{2(\bar{y}_{ij.}-\hat{\bar{y}}_{i..})}{\hat{V}_{i.}}+\frac{2(\bar{y}_{ij.}-\hat{\bar{y}}_{.j.})}{\hat{V}_{.j}}+\frac{(\hat{\bar{y}}_{...}-\bar{y}_{ij.})pq}{\hat{V}_{..}}\right\}\\
&+2\hat{V}_{ij}\left(\bar{y}_{ij.}-\hat{\bar{y}}_{i..}-\hat{\bar{y}}_{.j.}+\hat{\bar{y}}_{...}\right)^2+\frac{2\hat{V}_{ij}^2(\hat{\bar{y}}_{...}-\bar{y}_{ij.})^2}{\hat{V}_{..}}(1-\frac{\hat{V}_{ij}}{\hat{V}_{..}})]\\
&= \chi_3^2\ (\text{say})\end{aligned} \tag{15}$$

which follows approx. χ^2 with $(pqr-p-q+1)$ d.f. following Talukder (1976).

Now, from (8) and (15) we get,

$$\hat{F}(\hat{V})(\text{adj})=\frac{(pqr-p-q+1)\chi_1^2}{(p-1)\chi_3^2}$$

with $(p-1)$ and $(pqr-p-q+1)$ d.f. which is approximate F-distribution for testing H_A if error variances vary from cell to cell.

And from (15) and (12) we get,

$$\hat{F}(\hat{V})(\text{adj})=\frac{(pqr-p-q+1)\chi_2^2}{(q-1)\chi_3^2}$$

with $(q-1)$ and $(pqr-p-q+1)$ d.f. which is approximate F-distribution for testing H_B if error variances vary from cell to cell.

Thus, we can use adjusted χ^2 as well as adjusted F test procedures for testing H_A or H_B when error variances are unequal and unknown.

9 Conclusion

The WANOVA-χ^2 or F tests derived in this paper are complete if the error variances (σ_{ij}^2's) are known. If σ_{ij}^2 are unknown, the above WANOVA-χ^2 or F-tests can not be used for practical purposes. In such situation the adjusted test procedures developd in this paper can be used for practical purpose.

References

Meier, P. (1953). Variances of a Weighted Mean, *Biometrics*, 9, 59–73.

Sen, K. (1984). *Some Contributions to Heteroscedastic Analysis of Variance*, Unpublished Ph.D. thesis University of Madras.

Sen, K. (1991). On a General Theorem of WANOVA With The General Heteroscedastic Model Not of Full Rank, *Jou. of Bang. Ac of Science*, 15(2), 149–151.

Talukder, M. A. H. (1976). *Analysis of Variance with Unequal Group Variances*, Unpublished Ph.D. thesis, University of Kent at Canterbury, England.