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Gamma Mixture of Normal Moment Distribution

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Abstract

In this paper, gamma mixture of normal moment distribution has been studied. Different moments, characteristic function, shape characteristics of the distribution and estimates of the parameters have been provided.

Keywords and Phrases: Mixture distributions, normal moment distribution, Gamma distribution, and characteristic function.

1 Introduction

Mixture of gamma distributions were studied by a number of authors. Wasilewski (1967) considered the mixture of two generalized gamma distributions. Gamma mixtures of gamma distributions were also studied by Bhattacharya (1966) and by Roy et al. (1999). We define gamma mixture of normal moment distributions following Roy and Sinha (1995). The general form of normal moment distribution is defined as

$$g(x;k) = \frac{x^{2k}e^{-\frac{1}{2}x^2}}{2^{k+\frac{1}{2}}\Gamma\left(k+\frac{1}{2}\right)} \qquad -\infty < x < \infty \tag{1}$$

where, k is an integer. The form of the distribution is a density function proportional to the factor $f(x;k)$, where,

$$f(x;k) = x^{2k}e^{-\frac{1}{2}x^2}$$

hence the name '**normal moment distribution**'. The main properties of the distribution (1) are obtained as follows:
Mean $= 0$, Variance $= 2k+1$, Skewness $= 0$ and Kurtosis $= (2k+3)/(2k+1)$.

It is clear that the distribution (1) turns to a standard normal distribution for $k = 0$. It is observed that the distribution is symmetrical and bimodal. The mode of the distribution shifts to both sides of the mean as the value of k increases.

2 Preliminaries

When the parameter θ of a family of distributions, given by the density function $f(x;\theta)$ is subject to chance variation, then mixture of distributions occur very often. The mixing distribution, say, $g(\theta)$ is then a probability distribution on the parameter of the distribution. The general formula for the finite mixture is

$$\sum_{i=1}^{k} f(x;\theta_i)g(\theta_i) \tag{2}$$

The infinite analogue in which g is a density function is

$$\int f(x;\theta)g(\theta)d\theta \tag{3}$$

The mixtures of gamma distributions defined by Roy et al. (1999). Now we shall define a class of distributions that we call gamma mixture of distributions.

3 Main Results

At first gamma mixture of distributions is defined. In the light of this definition, we then define gamma mixture of normal moment distributions. The main results of the paper are presented in the forms of some definitions and theorems.

Definition 3.1. *A random variable X is said to have a gamma mixture distributions if its probability density function is given by*

$$f(x,\theta,v) = \int_0^\infty \frac{\theta^v e^{-r\theta} r^{v-1}}{\Gamma(v)} g(x;r,p)dr \tag{4}$$

where $g(x;r,p)$ is a probability function or probability density function.

The name gamma mixture distributions comes from the fact that the distribution (4) is the weighted average of $g(x;r,p)$ with weights equal to the ordinates of gamma distribution.

Definition 3.2. *A random variable X having density function*

$$f(x,\theta,v) = \int_0^\infty \frac{\theta^v e^{-r\theta} r^{v-1}}{\Gamma(v)} \frac{e^{-\frac{1}{2}x^2} x^{2r}}{2^{r+\frac{1}{2}}\Gamma\left(r+\frac{1}{2}\right)} dr \tag{5}$$

is said to have a gamma mixture of normal moment distributions with parameter θ and v.

We now state some characteristics of the distribution in the following theorem.

Theorem 3.1. *If X follows a gamma mixture of normal moment distribution with parameters θ and v then its characteristic function is obtained as*

$$\phi_x(t) = \int_0^\infty \sum_{x=0}^{\infty} \frac{\theta^v e^{-x\theta} r^{v-1} e^{-\frac{t^2}{2}} \binom{2r}{2m} (it)^{2m} \Gamma\left(r+\frac{1}{2}-m\right)}{\Gamma(v) 2^m \Gamma\left(r+\frac{1}{2}\right)} dr \tag{6}$$

Proof. The characteristic function of X is given by

$$\begin{aligned}
\phi_x(t) = E(e^{itx}) &= \int_0^\infty \frac{\theta^v e^{-r\theta} r^{v-1}}{\Gamma(v)} \int_{-\infty}^{\infty} \frac{e^{itx} x^{2r} e^{-\frac{1}{2}x^2}}{2^{r+\frac{1}{2}} \Gamma\left(r+\frac{1}{2}\right)} dx dr \\
&= \int_0^\infty c_r \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} e^{-\frac{1}{2}(x-it)^2} x^{2r} dx dr \\
&= \int_0^\infty c_r e^{-\frac{t^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2} (u+it)^2 du dr, \text{ putting } u = x - it \\
&= \int_0^\infty c_r e^{-\frac{t^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2} \sum_{m=0}^{2r} \binom{2r}{m} (it)^m u^{2r-m} du dr \qquad (7)
\end{aligned}$$

where, $c_r = \frac{\theta^v e^{-r\theta} r^{v-1}}{\Gamma(v) 2^{r+\frac{1}{2}} \Gamma\left(r+\frac{1}{2}\right)}$.

Now if m is odd, the integral (7) is an odd function of u and the value of the integral becomes zero. In order to make the function even, let us replace m by $2m$, then (7) takes the form

$$\begin{aligned}
\phi_x(t) &= \int_0^\infty c_r e^{-\frac{t^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2} \sum_{m=0}^{r} \binom{2r}{2m} (it)^m u^{2r-2m} du dr \\
&= \int_0^\infty c_r e^{-\frac{t^2}{2}} \sum_{m=0}^{r} \binom{2r}{2m} (it)^{2m} 2^{r+\frac{1}{2}-m} \Gamma\left(r+\frac{1}{2}-m\right) dr \qquad (8)
\end{aligned}$$

Now, putting the value of c_r in (8) we get

$$\phi_x(t) = \int_0^\infty \sum_{m=0}^{r} \frac{\theta^v e^{-r\theta} r^{v-1} e^{-\frac{t^2}{2}} \binom{2r}{2m} (it)^{2m} \Gamma\left(r+\frac{1}{2}-m\right)}{\Gamma(v) 2^m \Gamma\left(r+\frac{1}{2}\right)} dr$$

This completes the proof.

Theorem 3.2. *If X follows a gamma mixture of normal moment distribution with parameters θ and v then its sth moments about mean is given by*

$$\mu_{2s} = \int_0^\infty \frac{\theta^v e^{-r\theta} r^{v-1}}{\Gamma(v)} \frac{2^s \Gamma\left(r+s+\frac{1}{2}\right)}{\Gamma\left(r+\frac{1}{2}\right)} dr \tag{9}$$

and mean $= \mu_1' = 0$; *variance* $= \mu_2 = \frac{2v+\theta}{\theta}$; $\beta_1 = 0$; $\beta_2 = \frac{4v(v+1)+8v\theta+3\theta^2}{(2v+\theta)^2}$, *where* β_1 *and* β_2 *are the measures of the skewness and kurtosis respectively.*

Proof. Now by definition,

$$\mu_{2s+1}' = E(X^{2s+1}) = \int_0^\infty \frac{\theta^v e^{-r\theta} r^{v-1}}{\Gamma(v)} \int_{-\infty}^\infty \frac{x^{2s+2r+1} e^{-\frac{x^2}{2}}}{2^{r+\frac{1}{2}}\Gamma\left(r+\frac{1}{2}\right)} dxdr;\ s = 0, 1, \cdots \tag{10}$$

since, in the above expression, the integral is an odd function of X, we get $\mu_{2s+1}' = 0$ and and hence $\mu_{2s+1} = 0$. Now,

$$\begin{aligned} \mu_{2s}' = \mu_{2s} = E(X^{2s}) &= \int_0^\infty \frac{\theta^v e^{-r\theta} r^{v-1}}{\Gamma(v)} \int_{-\infty}^\infty \frac{(x^2)^{r+s} e^{-\frac{x^2}{2}}}{2^{r+\frac{1}{2}}\Gamma\left(r+\frac{1}{2}\right)} \\ &= \int_0^\infty \frac{\theta^v e^{-r\theta} r^{v-1}}{\Gamma(v)} \frac{2^s\Gamma\left(r+s+\frac{1}{2}\right)}{\Gamma\left(r+\frac{1}{2}\right)} dr;\ s = 1, 2, 3 \end{aligned} \tag{11}$$

In particular, mean $= \mu_1' = 0$; variance $= \mu_2 = \frac{2v+\theta}{\theta}$; $\mu_3 = 0$; $\mu_4 = \frac{4v(v+1)+8v\theta+3\theta^2}{\theta^2}$. Hence, $\beta_1 = 0$ and $\beta_2 = \frac{4v(v+1)+8v\theta+3\theta^2}{(2v+\theta)^2}$. This completes the proof.

Remarks. If $v = 0$, then all the values of $\phi_x(t)$, μ_1', μ_2, μ_3, μ_4, β_1 and β_2 are true for normal distribution with zero mean and unit variance.

Again, if $v = 1$, the distribution (5) reduces to an exponential mixture of normal moment distributions.

4 Estimation of the Parameters

The problem of estimation of the parameters in a parametric mixture model is rather complicated. Pearson (1894) and Rider (1961) are the investigator who dealt with the estimation problem of the parameters of mixture distributions. Both of them used the method of moments. Sometimes maximum likelihood estimation was used to estimate the parameters of the mixture distributions.

Generally, for mixture distributions, solution of the maximum likelihood equations become unduely tedious which is also true for gamma mixture of normal moment distributions. Here we estimate the parameters of the distribution by the method of moments.

Case I. We assume that the parameter v in the distribution (5) is known. Then the distribution contains only one parameter, viz., θ.

Consider a random sample $X_1, X_2, \cdots, X_n$ from the distribution having density function (5). Then the second raw sample moment is obtained as

$$m_2' = \frac{\sum_{i=1}^{n} x_i^2}{n} = s^2 \tag{12}$$

We have already found

$$\mu_2' = \mu_2 = \frac{2v + \theta}{\theta} \tag{13}$$

By the method of moments, equating (12) and (13), we obtain

$$s^2 = \frac{2v + \theta}{\theta} \tag{14}$$

Hence,

$$\hat{\theta} = \frac{2v}{s^2 - 1}. \tag{15}$$

Case II. When θ is known, then the moment estimate of v is given by

$$\hat{v} = \frac{\theta(s^2 - 1)}{2}. \tag{16}$$

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