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Simulated Tests for Normality: A Comparative Study

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Abstract

The subject of assessing whether a data set is from a specific distribution has received a good deal of attention. This topic is critically important for the normal distribution. Often the distributions of the test statistics are intractable. Here we consider simulation based distributions for several commonly used normality test statistics, such as, Anderson-Darling A^2 test, Chi-square test, Shapiro-Wilk W test, Shapiro-Francia W' test, D'Agostino-Pearson test, and Jarque-Bera test. Practitioners are used to with the Chi-square test because all other tests are dependent on specialized tables and/or software. Here, we give algorithms, how those specialized tables can be generated and then the respective tests can be implemented without much difficulty. A power comparison is also performed using simulation.

Keywords and Phrases: Central moments; Kurtosis; Legendre polynomials; Monte-Carlo simulation; Normal score; Skewness.

AMS Classification: Primary 62J02; Secondary 62J20.

1 Introduction

The subject of assessing whether a data set is from a specific distribution has received a good deal of attention. This topic is critically important for normal distributions.

Recently, Wah (2011), Rahman and Wu (2013a), and Rahman and Wu (2013b) compared several normality tests. Here we compare six different commonly used parametric goodness of fit tests for normality through simulation. In this study, we consider simulation based distributions for several commonly used normality test statistics, such as, Anderson-Darling A^2 test, Chi-square test, Shapiro-Wilk W test, Shapiro-Francia W' test, D'Agostino-Pearson test, and Jarque-Bera test. A brief motivation for this study is given in Section 2. Descriptions of all the seven (previously mentioned six plus traditional Chi-square test) tests considered are given in Section 3. In Section 4, a power comparison is presented. A brief conclusion based on the simulation results is given in Section 5. In Appendix, the simulation results are provided.

2 Motivation

The most commonly used method to check for normality is the Normal Probability Plot. The most commonly used method to test for normality is the Chi-square goodness-of-fit test, which is very simple to comprehend and very easy to implement. Some softwares are developed to implement other specialized tests but the accessibility of these softwares is limited to the general practitioners. Most tests rely on tables which are also not easily accessible. Here we give brief descriptions of the tests under consideration, implementation of algorithms, and finally a comparison using simulation. In addition, we also consider traditional Chi-square goodness-of-fit test along with simulation based Chi-square test.

3 Tests for Normality

3.1 Anderson-Darling Test

A distribution function test is suggested by Anderson and Darling (1952). The Anderson-Darling A^2 statistic is computed as

$$A^2 \quad = \quad -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left\{\ln\Phi_i + \ln(1-\Phi_i)\right\}, \tag{1}$$

where Φ_i's are the normal cumulative distribution function (CDF) value for the i^{th} ordered data point. Large sample approximations of the percentiles were given by Anderson and Darling (1952) based on simulation and then through polynomial fittings with respect to the sample size. Due to computational developments, now one can use simulation to generate the percentiles of the A^2 statistic for the sample size under consideration and the null hypothesis that the data is from the standard normal distribution. Then a p-value can be obtained for the observed A^2 statistic for the data at hand.

3.2 Chi-square Test

Pearson's chi-squared test is used to assess two types of comparison: tests of goodness of fit and tests of independence. In this paper, we use it to establish whether or not an observed frequency distribution differs from a hypothetical distribution. The test-statistic is defined as

$$\chi^2 = \sum_{i=1}^{g} \frac{(O_i - E_i)^2}{E_i}, \tag{2}$$

where g is the number of groups, O_i is an observed frequency while E_i is the expected frequency under the null hypothesis. Then χ^2 follows approximate Chi-square distribution with $g - k - 1$ degrees of freedom, where k is the number of parameters estimated for the distribution under consideration. A rule can be maintained that the expected frequencies are at least 5. For moderate to small samples, the highest number of groups possible should be considered. For large samples, the highest number of groups might over smooth in complying with Chi-square approximation. To examine such a behavior, different number of groups can be considered while comparisons are made. In literature, number of classes in the Chi-square test is analyzed by Dahiya and Gurland (1971), Hamdan (1963), Mann and Wald (1942), and Williams (1950). Here we consider expected frequencies as 5 throughout our simulations.

Due to computational developments, now one can use simulation to generate the percentiles of the χ^2 statistic for the sample size under consideration and the null hypothesis that the data is from the standard normal distribution. Then a p-value can be obtained for the observed χ^2 statistic for the data at hand. Here we denote such statistic as χ^2_s, where s stands for simulation.

3.3 Shapiro-Wilk W Test

Let $(X_1, X_2, \cdots, X_n)$ be a random sample to be tested for departure from normality, based on the ordered sample $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$, and let $\mathbf{m}_{n\times 1}$ denote the vector of expected values $\mathbf{V}_{n\times n}$ variance-covariance matrix of the standard normal order statistics. Shapiro and Wilk (1965) suggested the following test.
Define

$$W = \frac{\left(\sum_{i=1}^{n} a_i X_{(i)}\right)^2}{\sum_{i=1}^{n} (X_i - \bar{X})^2}, \tag{3}$$

where $\bar{X}$ is the sample mean and $\mathbf{a}_{n\times 1} = \frac{\mathbf{m}'\mathbf{V}^{-1}}{(\mathbf{m}'\mathbf{V}^{-1}\mathbf{V}^{-1}\mathbf{m})^{1/2}}$. Note that W equals the square of the standard product-moment correlation coefficient between the $X_{(i)}$ and

a_i, and therefore measures the straightness of the normal probability plot of the $X_{(i)}$; small values of W indicate non-normality. In literature, a_i's are tabulated for limited sample sizes and some approximations are provided for larger samples. Here, we suggest obtaining a_i's using simulation and then creating a simulation of the W statistic distribution under the null hypothesis for the sample size needed prior to implementing the test.

3.4 Shapiro-Francia W' Test

$$W' = \frac{\left(\sum_{i=1}^{n} m_i X_{(i)}\right)^2}{\sum_{i=1}^{n} m_i^2 \times \sum_{i=1}^{n} \left(X_i - \bar{X}\right)^2}, \tag{4}$$

a modified W statistic proposed by Shapiro and Francia (1972).Sarkadi (1975) discussed about consistency of this test. Implementation of W' is easier and gained popularity as it requires only the means of the order statistics of the standard normal variates unlike W which also requires covariance matrix of the order statistics of the standard normal variates. Harter (1961) provided the approximate means of the order statistics of the standard normal variates. Parrish (1992) provided the more accurate means of the order statistics of the standard normal variates using Legendre Polynomials, which are accurate up to thirty two decimal places. Following Parrish (1992), Rahman and Pearson (2000) showed that by simulation the means of the order statistics of the standard normal variates can be approximated pretty accurately up to about eight decimal places which serves the purpose in most cases.

Rahman and Pearson (2000) showed that through exclusive Monte-Carlo simulation, that is, incorporating approximate expected values, the W' percentiles can be computed pretty accurately, while Rahman and Ali (1999) provided revised W' percentiles using Parrish (1992) expected values.

Hence, before implementing W' test one can easily simulate the means of the order statistics of the standard normal variates and W' percentiles and then compute the p-value for the data at hand. Similar can be done for the W statistic.

3.5 D'Agostino-Pearson Test

Pearson (1895) suggested that the following sample estimates could be used to describe nonnormal distributions and used as the bases for tests of normality. Let $(X_1, \cdots, X_n)$

denote a sample of n observations. M_j's are the j-th sample central moments, and $\overline{X}$ is the sample mean. $\sqrt{b_1}$ and b_2 are defined as

$$\sqrt{b_1} = \frac{M_3}{\sqrt{M_2^3}} = \frac{\frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^3}{\left(\frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2\right)^{3/2}}, \tag{5}$$

and

$$b_2 = \frac{M_4}{M_2^2} = \frac{\frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^4}{\left(\frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2\right)^{2}}, \tag{6}$$

where $M_j = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^j$.

D'Agostino et al. (1990) have done separate tests based on $\sqrt{b_1}$ and b_2. They indicate how these two can be used in conjunction with normal probability plotting. They have also provided the procedures on how to calculate the normal approximations of the test statistics $X(\sqrt{b_1})$ and $X(b_2)$ when the sample sizes are large enough ($n > 8$ and $n \geq 20$, respectively), where $X(\sqrt{b_1})$ is the standard normal score for the respective percentile position for $\sqrt{b_1}$ value, and $X(b_2)$ is similarly defined as $X(\sqrt{b_1})$. In this paper, we will obtain the estimates of these two statistics $X(\sqrt{b_1})$ and $X(b_2)$ by simulations. The test proposed by D'Agostino and Pearson (1973) combines $\sqrt{b_1}$ and b_2 for an omnibus test. The test statistic is

$$DPC = X^2(\sqrt{b_1}) + X^2(b_2). \tag{7}$$

Under the normal null hypothesis, DPC follows a Chi-square distribution with 2 degrees of freedom. In literature, simulated $\sqrt{b_1}$ and b_2 values, and their tables are given for a wide range of sample sizes.

At this computational age, one can easily simulate percentiles for $\sqrt{b_1}$ and b_2 under normality assumption prior to implementing the DPC test. Computational steps:

- Step 1: Simulate samples of size n from $N(0,1)$, compute $\sqrt{b_1}$ and b_2, store the values.

- Step 2: Compute $\sqrt{b_1}$ and b_2 for the data, obtain percentile positions of $\sqrt{b_1}$ and b_2 in the stored respective empirical distributions in Step 1, compute DPC in (7) and then obtain the upper tail p-value by using the Chi-square distribution with 2 degrees of freedom.

Alternatively, instead of using the Chi-square table, one can use the simulated empirical distribution to determine the critical values or the p-values in making the decisions. Let us call such test DPS. Computational steps:

- Step 1: Simulate samples of size n from $N(0,1)$, compute $\sqrt{b_1}$ and b_2, store the values.
- Step 2: Separately, take samples of size n from $N(0,1)$, compute $\sqrt{b_1}$ and b_2, obtain the percentile positions of $\sqrt{b_1}$ and b_2 in the stored respective empirical distributions in Step 1, compute DPC in (7) and store.
- Step 3: Compute $\sqrt{b_1}$ and b_2 for the data, obtain percentile positions of $\sqrt{b_1}$ and b_2 in the stored respective empirical distributions in Step 1, compute DPC in (7) and then obtain the upper tail p-value by using the empirical distribution in Step 2.

Rahman and Wu (2013b) showed that performance of DPS is better than DPC, hence we will consider DPS in this study.

3.6 Jarque-Bera Test

Jarque and Bera (1987) suggested a moment based statistic

$$JBC = \frac{n}{6}\left(b_1 + \frac{1}{4}(b_2 - 3)^2\right), \tag{8}$$

where b_1 and b_2 are defined above.

For large samples, JBC follows a Chi-square distribution with 2 degrees of freedom. Computational steps:

- Compute b_1 and b_2 for the data, compute JBC in (8) and then obtain the upper tail p-value by using the Chi-square distribution with 2 degrees of freedom.

Alternatively, instead of using the Chi-square table one can use the simulated empirical distribution to determine the critical values or the p-values in making the decisions. Let us call such test JBS. Computational steps:

- Step 1: Simulate samples of size n from $N(0,1)$, compute b_1 and b_2, compute JBC in (8) and store.
- Step 2: Compute b_1 and b_2 for the data, compute JBC in (8) and then obtain the upper tail p-value by using the empirical distribution in Step 1.

Here, we will consider JBS in our comparisons.

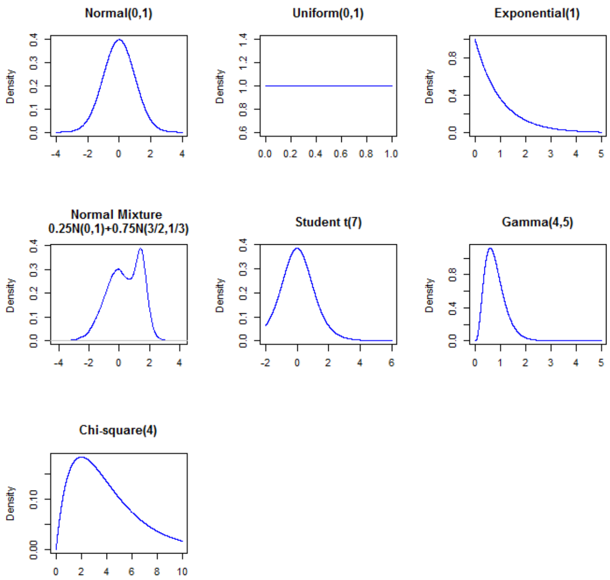

Figure 1: Different Distributions

4 Power Comparison

All seven tests mentioned in Section 3 are compared using simulation. Data were generated from $N(0,1)$ to investigate the null distributions. Then data were generated from some non-normal distributions, such as, $Uniform(0,1)$, $Exponential(1)$, a mixture $\frac{1}{4}N(0,1)+\frac{3}{4}N\left(\frac{3}{2},\frac{1}{3}\right)$, t_7, $Gamma(4,5)$, and χ^2_4 to compare the powers of the tests. All the alternative distributions are shown in Figure 1. Samples are considered of sizes $n=20,30,50$, and 100. In all simulations 10,000 replications were considered. Proportions of rejections (p-values when the null distribution is considered) were computed and their percentiles are given in the tables in the Appendix. Levels of significances are considered 1%, 5% and 10%.

Test	Abbreviation
D'Agostino-Pearson Test	DPS
Jarque-Bera Test	JBS
Anderson-Darling Test	ADS
Shapiro-Wilk Test	SWS
Shapiro-Francia Test	SFS
χ^2 Simulation Test	C2S
χ^2 Table Test	C2T

We notice that when data are generated from the standard normal distribution, that is, the null hypothesis is true, powers are the level of significances, all powers are close to the respective level of significances, except for $C2T$ for smaller sample sizes. For $Uniform(0,1)$ alternatives, JBS has poor performance compared to all others. DPS has the best performance for larger samples.

5 Concluding Remarks

In general, the further the alternative distributions are away from the normal distribution, the more powerful the tests are. The Chi-square test is the least powerful test for the sample sizes considered in this study. For very large samples, the Chi-square test might outperform others but might not be noticeable. All other tests are more powerful but more computational difficulty. At this computational age, the researchers should overcome the hesitance and use one of the other tests other than the Chi-square test. All other tests are equally computational burdensome but in this study as we experienced that they could be performed with some computational background and without much difficulty. If one has to use the Chi-square test, should use the number of groups as high as possible at least for the samples of sizes below one hundred.

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Appendix

Rejection Proportions

	$\alpha = 0.01$				$\alpha = 0.05$				$\alpha = 0.10$			
	$n = 20$	$n = 30$	$n = 50$	$n = 100$	$n = 20$	$n = 30$	$n = 50$	$n = 100$	$n = 20$	$n = 30$	$n = 50$	$n = 100$
TS	$Normal(0,1),\ \sqrt{\beta_1} = 0, \beta_2 = 3$											
DPS	0.0194	0.0181	0.0192	0.0155	0.0575	0.0592	0.0534	0.0513	0.1038	0.0948	0.1011	0.0955
JBS	0.0095	0.0094	0.0103	0.0116	0.0504	0.0536	0.0519	0.0502	0.1072	0.0962	0.1011	0.1017
ADS	0.0128	0.0094	0.0096	0.0082	0.0516	0.0540	0.0442	0.0502	0.1014	0.1009	0.0967	0.0931
SWS	0.0107	0.0107	0.0075	0.0092	0.0516	0.0560	0.0467	0.0520	0.1042	0.0999	0.1025	0.0961
SFS	0.0102	0.0118	0.0102	0.0103	0.0501	0.0538	0.0470	0.0541	0.1021	0.1015	0.0975	0.1042
$C2S$	0.0116	0.0099	0.0074	0.0091	0.0541	0.0524	0.0489	0.0519	0.1130	0.1103	0.1041	0.0944
$C2T$	0.0199	0.0121	0.0100	0.0091	0.1085	0.0659	0.0524	0.0536	0.2475	0.1346	0.1041	0.1057
	$Uniform(0,1),\ \sqrt{\beta_1} = 0, \beta_2 = 1$											
DPS	0.0508	0.1752	0.4211	0.9712	0.1905	0.3896	0.7634	0.9953	0.2995	0.5171	0.8698	0.9983
JBS	0.0001	0.0000	0.0000	0.0000	0.0214	0.0172	0.0806	0.2888	0.1442	0.1800	0.2760	0.5390
ADS	0.0432	0.0888	0.2549	0.8041	0.1742	0.2989	0.5566	0.9521	0.2898	0.4406	0.7227	0.9785
SWS	0.0162	0.0453	0.9300	0.9347	0.0424	0.4627	0.1572	0.4579	0.1858	0.3706	0.2976	1.0000
SFS	0.0062	0.0273	0.0638	0.7368	0.0462	0.3490	0.2766	0.9261	0.0828	0.3789	0.7115	0.9978
$C2S$	0.0243	0.0204	0.0810	0.6093	0.1079	0.1135	0.1968	0.7801	0.1791	0.1943	0.2815	0.8481
$C2T$	0.0380	0.0211	0.0818	0.5989	0.1759	0.1135	0.1968	0.7807	0.3925	0.2372	0.2896	0.8501
	$Exponential(1),\ \sqrt{\beta_1} = 5, \beta_2 = 3$											
DPS	0.4505	0.6375	0.8560	0.9990	0.6004	0.7914	0.9573	0.9999	0.6970	0.8551	0.9898	1.0000
JBS	0.3923	0.5393	0.7941	0.9989	0.8114	0.9455	0.9973	1.0000	0.9299	0.9870	0.9998	1.0000
ADS	0.5598	0.8221	0.9800	1.0000	0.7796	0.9373	0.9971	1.0000	0.8609	0.9632	0.9983	1.0000
SWS	0.6820	0.9801	0.9583	0.9971	0.6356	0.9578	0.9974	1.0000	0.9593	0.9422	1.0000	1.0000
SFS	0.5548	0.8089	0.9841	1.0000	0.7560	0.9540	0.9993	1.0000	0.8907	0.9888	0.9997	1.0000
$C2S$	0.1324	0.5705	0.8707	0.9978	0.2566	0.6965	0.9847	0.9996	0.3551	0.7451	0.9905	0.9999
$C2T$	0.1818	0.5746	0.8707	0.9987	0.3508	0.6965	0.9848	0.9996	0.5501	0.7659	0.9905	0.9999
	$NormalMixture : \frac{1}{4}N(0,1) + \frac{3}{4}N\left(\frac{3}{2}, \frac{1}{3}\right),\ \sqrt{\beta_1} = 5.4819, \beta_2 = 5.5438$											
DPS	0.0289	0.0396	0.0681	0.1748	0.0886	0.1391	0.1910	0.4642	0.1523	0.2162	0.3559	0.6303
JBS	0.0006	0.0005	0.0003	0.0002	0.0062	0.0045	0.0025	0.0016	0.0398	0.0238	0.0138	0.0072
ADS	0.0385	0.0713	0.1519	0.4491	0.1409	0.2209	0.3490	0.7173	0.2400	0.3250	0.5050	0.8323
SWS	0.0209	0.0996	0.1182	0.1497	0.0481	0.1165	0.3795	0.2392	0.1968	0.2928	0.3938	0.6517
SFS	0.0189	0.0255	0.0791	0.2590	0.0914	0.1173	0.2548	0.6068	0.1566	0.2364	0.4226	0.8166
$C2S$	0.0182	0.0408	0.1209	0.2548	0.0955	0.1334	0.2657	0.4772	0.1696	0.2086	0.3764	0.6224
$C2T$	0.0296	0.0422	0.1215	0.2418	0.1614	0.1334	0.2753	0.4817	0.3477	0.2400	0.3845	0.6262
	t-distribution with 7 degrees of freedom, $\sqrt{\beta_1} = 0, \beta_2 = 5$											
DPS	0.0955	0.1319	0.1722	0.2812	0.1652	0.2117	0.2790	0.4106	0.2179	0.2720	0.3388	0.5022
JBS	0.0620	0.0890	0.1365	0.2400	0.1379	0.1925	0.2475	0.3518	0.1965	0.2421	0.2969	0.4237
ADS	0.0422	0.0562	0.0763	0.1312	0.1194	0.1434	0.1914	0.2767	0.1860	0.2224	0.2595	0.3827
SWS	0.0506	0.0709	0.1050	0.1117	0.1194	0.1665	0.1669	0.2034	0.1637	0.1967	0.2683	0.3391
SFS	0.0613	0.0849	0.1517	0.2757	0.1482	0.2089	0.2744	0.4309	0.2346	0.2861	0.3579	0.5043
$C2S$	0.0269	0.0246	0.0244	0.0233	0.0796	0.0844	0.0872	0.0829	0.1492	0.1595	0.1535	0.1655
$C2T$	0.0433	0.0229	0.0222	0.0236	0.1411	0.1045	0.0872	0.0912	0.3058	0.1896	0.1535	0.1692
	$Gamma(4,5),\ \sqrt{\beta_1} = 1, \beta_2 = 4.5$											
DPS	0.1362	0.2244	0.3825	0.6708	0.2601	0.3597	0.5531	0.8786	0.3358	0.4441	0.6385	0.9476
JBS	0.1129	0.1796	0.3486	0.6692	0.3659	0.5069	0.7306	0.9704	0.5419	0.6855	0.8706	0.9930
ADS	0.1045	0.1884	0.3430	0.7352	0.2435	0.3638	0.5804	0.8904	0.3581	0.4877	0.6977	0.9384
SWS	0.2400	0.2550	0.2203	0.3538	0.2441	0.3758	0.3985	0.5166	0.5165	0.1592	0.3077	0.5052
SFS	0.1574	0.1649	0.3516	0.8820	0.2939	0.4109	0.7264	0.9046	0.2992	0.4230	0.8125	0.9937
$C2S$	0.0288	0.0611	0.0927	0.2039	0.0897	0.1749	0.2756	0.4324	0.1533	0.2460	0.3826	0.6021
$C2T$	0.0500	0.0625	0.1079	0.2077	0.1546	0.1749	0.2759	0.4324	0.3260	0.2824	0.3828	0.6080
	$\chi^2_4,\ \sqrt{\beta_1} = 1.41, \beta_2 = 6$											
DPS	0.2772	0.3619	0.6194	0.9591	0.3935	0.5616	0.7863	0.9949	0.4767	0.6701	0.8892	0.9987
JBS	0.2271	0.2874	0.5409	0.9394	0.5277	0.7369	0.9387	0.9995	0.7483	0.9037	0.9851	1.0000
ADS	0.2655	0.4429	0.7469	0.9850	0.4685	0.6541	0.8886	0.9986	0.5872	0.7733	0.9394	0.9992
SWS	0.2197	0.6109	0.4545	0.9000	0.6791	0.5273	1.0000	0.9873	0.7467	0.7043	0.9720	0.9970
SFS	0.2776	0.4453	0.8280	0.9938	0.5167	0.6487	0.9391	0.9999	0.7205	0.8329	0.9936	0.9996
$C2S$	0.0521	0.1768	0.3730	0.7147	0.1381	0.3076	0.6413	0.8707	0.2300	0.4302	0.7544	0.9339
$C2T$	0.0814	0.1895	0.3994	0.7051	0.2158	0.3523	0.6519	0.8801	0.4161	0.4676	0.7546	0.9361