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Constructions of Nested Group Divisible Designs

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Abstract

In this paper, some approaches are developed for constructing nested group divisible (NGD) designs. One approach is developed based on the patterned methods of constructions of NGD designs, starting from known GD designs with two associate classes. The approach is described based on the process of augmentation of incidence matrices mentioned in additional remarks (ii). Some series of NGD designs are also obtained.

Keywords and Phrases: GD design, NGD design.

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1 Introduction

The use of block designs is widely recognized in many fields of experimentation. With some types of experimental material however, there may be more sources of variation than can be eliminated by ordinary block designs. For such situations Preece (1967) introduced nested balanced incomplete block designs.

Nested group divisible designs, which are three associate partially balanced incomplete block (PBIB) designs were introduced by Roy (1953) as hierarchical group divisible incomplete block designs with m- associate classes, and subsequently studied by Raghavarao (1960), Roy (1962), Hinkelmann and Kempthorne (1963). But many methods of constructing NGD designs are not available in literature. These design are useful as three factor experiments having balance as well as orthogonal factorial structure. The purpose of this paper is to give two kinds of general methods of constructing NGD designs. It seems that methods of constructing NGD designs are not rich in available literature.

A nested group divisible (NGD) design is an arrangement of v treatment into b blocks such that:

(i) each treatment is replicated exactly in r blocks;

(ii) each block contains excatly k $(< v)$ treatments;

(iii) there exists an association scheme between v = pmn treatments, which are partitioned into p sets of m groups of n treatments each, such that:

(a) any two treatments from the same set and the same group are first associates;

(b) any two treatments from the same set but from different groups are second associates;

(c) any two treatments from different sets are third associates,

(iv) any two treatments which are ith associate occur together in $\lambda_i (i = 1, 2, 3)$ blocks.

The nested group divisible association scheme can be displayed in an $pm \times n$ array as

$$\begin{bmatrix} A^{(1)} \\ A^{(2)} \\ . \\ . \\ A^{(p)} \end{bmatrix} \text{ with } A^{(i)} = \begin{bmatrix} a_{11}^{(i)} & a_{12}^{(i)} & . & a_{1n}^{(i)} \\ a_{21}^{(i)} & a_{22}^{(i)} & . & a_{2n}^{(i)} \\ . & . & . & . \\ a_{m1}^{(i)} & a_{m2}^{(i)} & . & a_{mn}^{(i)} \end{bmatrix}, \text{ i = 1,2, \ldots, p.}$$

where $a_{st}^{(i)}$ denotes the tth treatment of the sth group in the ith set.

If n_i denotes the number of treatments which are ith associates of any treatment θ (Say), then $n_1 = n - 1,\ n_2 = n(m - 1),\ n_3 = mn(p - 1)$

We shall denote the usual group divisible (GD) design by GD $(v, b, r, k, \lambda_1, \lambda_2; m, n)$ and the parameters of a NGD design by $(v^*, b^*, r^*, k^*, \lambda_1^*, \lambda_2^*, \lambda_3^*; p, m, n)$.

In case if $\lambda_2^* = \lambda_3^*$, then a NGD design reduces to a GD design by combining the second and third associate classes.

Note that when $\lambda_2^* = 0$, the design is disconnected and hence we put the restriction that $\lambda_2^* > 0$.

Here patterned method of constructing NGD designs, from known GD designs mostly are self complementary is provided first. Further more, other method is presented in the additional remarks (ii).

The definitions of other terms discussed here are from Raghavrao (1971).

The following notations are used throughout the paper: I$_s$, denotes the identity matrix of order s, $J_{s\times t}\,(O_{s\times t})$ denotes the matrix of order $s \times t$ whose all elements are unity (zero). N is the incidence matrix of a GD design $(v, b, r, k, \lambda_1, \lambda_2; m, n)$.

2 Constructions using GD designs

In this section, further methods of construction of NGD designs, different from those of Duan and Kageyama (1993), Miao, Kageyama and Duan (1996) are presented.

In what follows, N denotes $v \times b$ incidence matrix of GD design and $\bar{N} = J - N$ be the $v \times b$ incidence matrix of complement of GD design.

Theorem 2.1 When $m \geq 2$, the existence of a GD design with parameters $v = 2k,\ b = 2r,\ r,\ k,\ \lambda_1,\ \lambda_2;\ m, n$ implies the existence of a NGD design with parameters

$$v^* = 2v,\ b^* = 4b,\ r^* = 4r,\ k^* = 2k,\ \lambda_1^* = 4\lambda_1,\ \ \lambda_2^* = 4\lambda_2,\ \ \lambda_3^* = 2r;\ p = 2, m, n.$$

Proof: Suppose N is the incidence matrix of order $v \times b$ of GD design and J-N is the incidence matrix of its complementary design, then we have, incidence structure

$$M = \begin{bmatrix} N & N & J-N & J-N \\ N & J-N & J-N & N \end{bmatrix}$$

is the incidence matrix of the required NGD design.
Let M' denotes the usual transpose of M, then the matrix structure $M\,M'$ is written as

$$MM' = I_2 \otimes A + (J_2 - I_2) \otimes B.$$

Where $\otimes$ denotes the Kronecker product of matrices.
Now

$$\begin{aligned} A &= 2NN' + 2\,(J-N)\,(J'-N')\,, \\ &= 2NN' + 2JJ' - 2NJ' - 2JN' + 2NN' \\ &= 4NN' \qquad (\because b = 2r,\ JN' = NJ' = rJ) \end{aligned} \tag{1}$$

and

$$\begin{aligned} B &= NN' + N\,(J'-N')\, + \,(J-N)\,(J'-N')\, + \,(J-N)\,N', \\ &= JJ' \\ &= bJ \,=\, 2rJ \end{aligned} \tag{2}$$

From (1)., we have $\lambda_1^* = 4\lambda_1$, $\lambda_2^* = 4\lambda_2$ and from (2), we get $\lambda_3^* = 2r$.
The parameters $v^*,\ b^*,\ r^*,\ k^*$ are obvious. Hence the theorem.

Example 2.1 Consider a GD design R_{46} in Clatworthy (1973), with parameters $v = 6,\ b = 14, r = 7,\ k = 3, \lambda_1 = 2, \lambda_2 = 3;\ m = 3,\ n = 2$ using theorem 2.1, we get a NGD design with parameters $v^* = 12,\ b^* = 56,\ r^* = 28,\ k^* = 6,\ \lambda_1^* = 8,\ \lambda_2^* = 12,\ \lambda_3^* = 14;\ m = 3,\ n = 2,\ p = 2$.

Further, this NGD design is resolvable.

Remark 2.1:- m and n will remain the same in NGD association scheme as they were in GD association scheme. It is also to be noted that the groups in NGD association' scheme are always formed considering the solution (plan) of the design as per the number of occurrences of $\lambda_1^*,\ \lambda_2^*\ and\ \lambda_3^*$.

Theorem 2.2 The existence of a GD design with parameters $v = 2k,\ b = 2r,\ r,\ k,$ $\lambda_1, \lambda_2\,;\ m, n$ implies the existence of a NGD design with parameters

$v^* = 2v,\ b^* = 4b + 2,\ r^* = 4r + 1,\ k^* = 2k,\ \lambda_1^* = 4\lambda_1 + 1,\ \lambda_2^* = 4\lambda_2 + 1,\ \lambda_3^* = 2r;$ $m = 2,\ n = k,\ p = 2.$

Proof: It follows from the pattern in which N be the incidence matrix of original GD design, J is all one matrix of appropriate order and J* be the column vector of all one. Then it can be shown that

$$\begin{bmatrix} J^* & 0 & N & N & J-N & J-N \\ O & J^* & N & J-N & J-N & N \end{bmatrix}$$

is the incidence matrix of the required NGD design.

Theorem 2.3 The existence of a GD design with parameters $v = 2k,\ b = 2r,\ r,\ k,$ $\lambda_1, \lambda_2\,;\ m, n$ implies the existence of a NGD design with parameters

$v^* = 4v,\ b^* = 8b,\ r^* = 6r,\ k^* = 3k,\ \lambda_1^* = 6\lambda_1,\ \lambda_2^* = 6\lambda_2,\ \lambda_3^* = 2r;\ m^* = m,\ n^* = n,\ p = 4.$

Proof: Let N be the incidence matrix of original GD design, and J-N be the incidence matrix of complementary GD design of the original design. Then the incidence structure is the incidence matrix of the required NGD design,

$$M = \begin{bmatrix} 0 & 0 & N & N & N & J-N & N & N \\ N & N & 0 & 0 & N & N & N & J-N \\ N & J-N & N & N & 0 & 0 & J-N & J-N \\ J-N & J-N & J-N & N & N & N & 0 & 0 \end{bmatrix} \tag{3}$$

Is the incidence matrix of the required NGD design.
Suppose M' is the transpose of M given in (3), then

$$MM' = I_4 \otimes A + (J_4 - I_4) \otimes B \tag{4}$$

Where

$$\begin{aligned} A &= 5NN' + (J-N)\ (J'-N') \\ &= 5NN' + JJ' - NJ' - JN' + NN' \\ &= 6NN' + bJ - rJ - rJ \\ &= 6NN' \qquad (\because\ b = 2r) \end{aligned} \tag{5}$$

and
$$\begin{aligned}
B &= NN' + N\,(J' - N') + N\,(J' - N') + (J - N)\;(J' - N')\\
&= NN' + NJ' - NN' + NJ' - NN' + JJ' - JN' - NJ' + NN'\\
&= NJ' + NJ' + JJ' - JN' - NJ'\\
&= JJ' = bJ \qquad (\because\; JN' = NJ' = rJ)\\
&= 2rJ \qquad (\because\; b = 2r)
\end{aligned} \tag{6}$$

From (5) we have $\lambda_1^* = 6\lambda_1$ *and* $\lambda_2^* = 6\lambda_2,$ *also from* (6) *we get* $\lambda_3^* = 2r$. The proof is completed.

Theorem 2.4 : The existence of a GD designs with parameters $v = 2k,\; b = 2r,$ $r,\; k,\; \lambda_1, \lambda_2\,;\, m, n$ implies the existence of a NGD design with parameters

$$v^* = 3v,\; b^* = 4b,\; r^* = 4r,\; k^* = 3k,\; \lambda_1^* = 4\lambda_1,\; \lambda_2^* = 4\lambda_2,\; \lambda_3^* = 2r;\; m,\; n,\; p = 3$$

Proof: Suppose N be the incidence matrix of order $v \times b$ of the original GD design and J-N be the incidence matrix of the complementary design of the original GD design. Then the incidence structure

$$M = \begin{bmatrix} N & N & N & N \\ N & J-N & N & J-N \\ N & N & J-N & J-N \end{bmatrix} \tag{7}$$

is the incidence matrix of order $v^* \times b^*$ of the NGD design.
We construct the matrix MM' as follows

$$MM' = \begin{bmatrix} A & B & B \\ B & A & B \\ B & B & A \end{bmatrix} \tag{8}$$

Where $A = 4NN'$
also $A = NN' + (J - N)\;(J' - N') + NN' + (J - N)\;(J' - N')$

$$\begin{aligned}
&= 2NN' + JJ' - NJ' - JN' + NN' + JJ' - NJ' - JN' + NN'\\
&= 4NN' + 2bJ' - 2NJ' - 2JN'\\
&= 4NN' \qquad (\because NJ' = JN' = rJ)
\end{aligned} \tag{9}$$

now

$$\begin{aligned}
B &= NN' + (J - N)\,(J' - N') + N\,(J' - N') + (J - N)\;(J' - N')\\
&= NN' + JN' - NN' + NJ' - NN' + JJ' - NJ' - JN' + NN'\\
&= JJ'\\
&= bJ\\
&= 2rJ
\end{aligned} \tag{10}$$

From (9) we have $\lambda_1^* = 4\lambda_1$, $\lambda_2^* = 4\lambda_2$ and from (10), we get $\lambda_3^* = 4r$. The proof is completed.

Theorem 2.5 The existence of a self complementary GD design with parameters $v,\ b,\ r,\ k,\ \lambda_1, \lambda_2\,;\ m, n$ implies the existence of a NGD design with parameters

$$v^* = 9v,\ b^* = 9b,\ r^* = 4b{+}r,\ k^* = 4v{+}k,\ \lambda_1^* = 4b{+}\lambda_1,\ \lambda_2^* = 4b{+}\lambda_2,\ \lambda_3^* = 4r;\ m,\ n,\ p = 9.$$

Proof: let N be the incidence matrix of GD design with parameters $v,\ b,\ r,\ k,\ \lambda_1, \lambda_2\,;\ m, n$ and J is flat matrix of order $v \times b$ whose all elements are one, 0 is a null matrix of appropriate order. Then the incidence structure

$$M = \begin{bmatrix} N & 0 & J & 0 & J & 0 & J & 0 & J \\ 0 & N & 0 & J & 0 & 0 & 0 & J & J \\ J & 0 & N & J & 0 & J & J & 0 & 0 \\ 0 & J & J & N & 0 & J & 0 & 0 & J \\ J & 0 & 0 & 0 & N & J & 0 & J & J \\ 0 & 0 & J & J & J & N & 0 & J & 0 \\ J & J & J & 0 & 0 & 0 & N & J & 0 \\ 0 & J & 0 & 0 & J & J & J & N & 0 \\ J & J & 0 & J & J & 0 & 0 & 0 & N \end{bmatrix} \tag{11}$$

is the incidence matrix of required NGD design. Let M' be the transpose of M, then MM' is written as

$$MM' = I_9 \otimes A + (J_9 - I_9) \otimes B \tag{12}$$

Where

$$\begin{aligned} A &= NN' + 4JJ' \\ &= NN' + 4bJ, \end{aligned} \tag{13}$$

$$\begin{aligned} B &= JJ' + JJ' \\ \text{and} \quad &= 2bJ \\ &= 4rJ \end{aligned}$$

$$\begin{aligned} B &= NJ' + JN' + JJ' \qquad (\because b = 2r) \\ &= rJ + rJ + bJ \\ &= 2rJ + bJ \\ &= 4rJ \qquad (\because b = 2r) \end{aligned} \tag{14}$$

From (13), we have $\lambda_1^* = 4b + r_1$, $\lambda_2^* = 4v + k$, and from (14), we get $\lambda_3^* = 4r$. the proof is completed.

Several series of self complementary GD designs can be found in Kageyama and Tanaka (1981).

3 Additional remarks

(i) Bhagwandas and Parihar (1982), Bhagwandas, Banerjee and Kageyama (1985) and Banerjee, Kageyama and Bhagwandas (1987) gave several methods of patterned construction of GD designs starting from a BIB design. These GD designs may be extended to the NGD designs by the application of theorems described here.
(ii) If N_1 and N_2 are incidence matrices of NGD designs both having the same number of treatments and the same association structure, then $N = [N_1 \colon N_2]$ is a NGD design. Also, if either of $N_i (i = 1, 2)$, is a NGD and another is GD design or BIB design with the same number of treatments and GD design having the same association scheme within the sets, and then N is again a NGD design.

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