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Concomitants of Dual Generalized Order Statistics from Farlie Gumbel Morgenstern Type Bivariate Gumbel Distribution

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Abstract

Burkschat *et al.* (2003) introduced the concept of dual generalized order statistics to enable a common approach to descendingly ordered random variables like reverse order statistics and lower record values. In this paper probability density function of single and the joint probability density function of two concomitants of dual generalized order statistics from Farlie Gumbel Morgenstern type bivariate Gumbel distribution have been obtained and expression for marginal and joint moments generating functions are derived. Further, the results are deduced for reverse order statistics and lower record values.

Keywords and Phrases: Dual Generalized Order Statistics, Farlie Gumbel Morgenstern Type Bivariate Gumbel Distribution, Concomitants, Marginal and Joint Moments Generating Functions.

AMS Classification: Primary 62E15; Secondary 62G30.

1 Introduction

The Farlie Gumbel Morgenstern (FGM) family of bivariate distributions has found extensive use in practice. This family is characterized by the specified marginal distribution functions $F_X(x)$ and $F_Y(y)$ of random variables X and Y respectively and a parameter α, resulting in the bivariate distribution function (*df*) is given by

$$F_{X,Y}(x,y) = F_X(x)F_Y(y)[1 + \alpha(1 - F_X(x))(1 - F_Y(y))]. \tag{1}$$

with the corresponding probability density function (*pdf*)

$$f_{X,Y}(x,y) = f_X(x)f_Y(y)[1 + \alpha(2F_X(x) - 1)(1 - 2F_Y(y))]. \tag{2}$$

Here, $f_X(x)$ and $f_Y(y)$ are the marginals of $f_{X,Y}(x,y)$. The parameter α is known as the association parameter. The two random variables X and Y are independent when α is zero. Such a model was originally introduced by Morgenstern (1956) and investigated by Gumbel (1960) for exponential marginals. The general form in (1) is due to Farlie (1960) and Johnson and Kotz (1975). The admissible range of association parameter α is $-1 \leq \alpha \leq 1$ and the Pearson correlation coefficient ρ between X and Y can never exceed 1/3.

The conditional *df* and *pdf* of Y given X, are given by

$$F_{Y|X}(y|x) = F_Y(y)[1 + \alpha(1 - F_X(x))(1 - F_Y(y))] \tag{3}$$

and

$$f_{Y|X}(y|x) = f_Y(y)[1 + \alpha(1 - 2F_X(x))(1 - 2F_Y(y))] \tag{4}$$

[*c.f.* Beg and Ahsanullah, 2007]

In this paper, we have considered FGM type bivariate Gumbel distribution with *pdf*

$$f(x,y) = e^{-x}e^{-y}e^{-e^{-x}}e^{-e^{-y}}[1 + \alpha(1 - 2e^{-e^{-x}})(1 - 2e^{-e^{-y}})], \\ -\infty < x, y < \infty, -1 \leq \alpha \leq 1 \tag{5}$$

and corresponding *df*

$$F(x,y) = e^{-e^{-x}}e^{-e^{-y}}[1 + \alpha(1 - 2e^{-e^{-x}})(1 - 2e^{-e^{-y}})], \\ -\infty < x, y < \infty, -1 \leq \alpha \leq 1 \tag{6}$$

Thus, the conditional *pdf* of Y given X is

$$f(y|x) = e^{-y}e^{-e^{-y}}[1 + \alpha(1 - 2e^{-e^{-x}})(1 - 2e^{-e^{-y}})], \\ -\infty < x, y < \infty, -1 \leq \alpha \leq 1 \tag{7}$$

and the marginal *pdf* and *df* of X are

$$f(x) = e^{-x}e^{-e^{-x}}, \quad -\infty < x < \infty. \tag{8}$$

$$F(x) = e^{-e^{-x}}, \quad -\infty < x < \infty. \tag{9}$$

respectively.

Kamps (1995) introduced the concept of generalized order statistics (*gos*). Using the concept of *gos*, Burkschat *et al.* (2003) introduced the concept of dual generalized order statistics (*dgos*) as follows:

Let X be a continuous random variables with *df* $F(x)$ and *pdf* $f(x)$, $x \in (\alpha, \beta)$. Further, Let $n \in N$, $k \geq 1$, $\tilde{m} = (m_1, m_2, ..., m_) \in \Re^{n-1}$, $M_r = \sum_{j=r}^{n-1} m_j$ such that $\gamma_r = k + (n-r) + M_r \geq 1$ for all $r \in 1, 2, ..., n-1$. Then $X_d(r, n, \tilde{m}, k)$ $r = 1, 2, ..., n$ are called *dgos* if their *pdf* is given by

$$k\Big(\prod_{j=1}^{n-1}\gamma_j\Big)\Big(\prod_{i=1}^{n-1}\left[F(x_i)\right]^{m_i}f(x_i)\Big)\left[F(x_n)\right]^{k-1}f(x_n) \tag{10}$$

on the cone $F^{-1}(1) > x_1 \geq x_2 \geq \ldots \geq x_n > F^{-1}(0)$.

If $m_i = m = 0$, $i = 1, 2, ..., n-1$, $k = 1$, then $X_d(r, n, m, k)$ reduces to the $(n - r + 1) - th$ reverse order statistics, $X_{n-r+1:n}$ from the sample $X_1, X_2, ..., X_n$ and when $m = -1$, then $X_d(r, n, m, k)$ reduces to $k - th$ lower record values.

The *pdf* of $X_d(r, n, m, k)$ is

$$f_{X_d(r,n,m,k)} = \frac{C_{r-1}}{(r-1)!}[F(x)]^{\gamma_r - 1}f(x)g_m^{r-1}(F(x)) \tag{11}$$

and joint *pdf* of $X_d(r, n, m, k)$ and $X_d(s, n, m, k)$, is $1 \leq r < s \leq n$.

$$\begin{aligned} f_{X_d(r,s,n,m,k)}(x,y) &= \frac{C_{s-1}}{(r-1)!(s-r-1)!}\,[F(x)]^m f(x)\; g_m^{r-1}\big(F(x)\big) \\ &\times[h_m\big(F(y)\big) - h_m\big(F(x)\big)]^{s-r-1}[F(y)]^{\gamma_s-1}f(y),\ \alpha \leq x < y \leq \beta \end{aligned} \tag{12}$$

where

$$C_{r-1} = \prod_{i=1}^{r}\gamma_i, \quad \gamma_i = k + (n-i)(m+1)$$

$$h_m(x) = \begin{cases} -\dfrac{1}{m+1}\,x^{m+1} & , \quad m \neq -1 \\ -\log x & , \quad m = -1 \end{cases}$$

and $g_m(x) = h_m(x) - h_m(1)$, $x \in (0, 1)$.

Let $(X_i, Y_i), i = 1, 2, ..., n$, be n pairs of independent random variables from some bivariate population with *df* $F(x, y)$. If we arrange the $X-$ variates in descending order as $X_d(1,n,m,k) \geq X_d(2,n,m,k) \geq ... \geq X_d(n,n,m,k)$, then $Y-$ variates paired (not necessarily in descending order) with these *dgos* are called the concomitants of *dgos* and are denoted by $Y_{[1,n,m,k]}, Y_{[2,n,m,k]}, ..., Y_{[n,n,m,k]}$. The *pdf* of $Y_{[r,n,m,k]}$, the $r-th$ concomitant *dgos*, is given as

$$g_{[r,n,m,k]}(y) = \int_{-\infty}^{\infty} f_{Y|X}(y|x)\ f_{X_d(r,n,m,k)}(x)\ dx \tag{13}$$

and the joint *pdf* of $Y_{[r,n,m,k]}$ and $Y_{[s,n,m,k]}$ $1 \leq r < s \leq n$ is

$$g_{[r,s,n,m,k]}(y_1, y_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{x_1} f_{Y|X}(y_1|x_1)\ f_{Y|X}(y_2|x_2)\ f_{X_d(r,s,n,m,k)}(x_1, x_2)\ dx_1\ dx_2 \tag{14}$$

where $f_{X_d(r,s,n,m,k)}(x)$ is the joint *pdf* of $X_d(r,n,m,k)$ and $X_d(s,n,m,k)$, $1 \leq r < s \leq n$.

The most important use of concomitants arises in selection procedures when $k < n$ individuals are chosen on the basis of their $X-$ values. Then the corresponding $Y-$ values represent performance on an associated characteristic. For example, X might be the score of a candidate on a screening test and Y the score on a later test.

There are vast literature which deals with concomitants. An excellent review on concomitants of order statistics is given Bhattacharya (1984) and David and Nagaraja (1998). Balasubramanian and Beg (1996, 1997, 1998) studied the concomitants for bivariate exponential distribution of Marshall- Olkin, Morgesnstern type bivariate exponential distribution and Gumbel's bivariate exponential distribution and gave the recurrence relation between single and product moment of order statistics. Begum and Khan (1997, 1998, 2000) studied the concomitants for Gumbel's bivariate Weibull distribution, bivariate Burr distribution, Marshall and Olkin bivariate Weibull distribution and gave expression for single and product moment of order statistics. Ahsanullah and Beg (2006) studied the concomitants for Gumbel's bivariate exponential distribution and derived the recurrence relation between single and product moment of generalized order statistics.

Here in this paper we have considered FGM type bivariate Gumbel distribution and obtained *pdf* for $r-th$, $1 \leq r \leq n$ and the joint *pdf* of $r-th$ and $s-th$, concomitants of *dgos*. Also, moment generating function (mgf) and cumulant generating function(cgf) are obtained and expressions for means, variances and covariances are derived.

2 Probability Density Function of $Y_{[r,n,m,k]}$

For the FGM type bivariate Gumbel distribution as given in (5), using (7) and (11) in (13), the *pdf* of $r-th$ concomitants of *dgos* $Y_{[r,n,m,k]}$ for $m \neq -1$ is given as

$$g_{[r,n,m,k]}(y) = \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} e^{-y} e^{-e^{-y}}$$

$$\times \int_{-\infty}^{\infty} e^{-x} \left[e^{-e^{-x}}\right]^{\gamma_{r-i}} [1 + \alpha(1 - 2e^{-e^{-x}})(1 - 2e^{-e^{-y}})] dx. \tag{15}$$

Setting $e^{-x} = t$ in (15), we get

$$= \frac{C_{r-1} e^{-y} e^{-e^{-y}}}{(r-1)!(m+1)^{r-1}} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \Big[\frac{1}{\gamma_{r-i}} + \alpha\Big(\frac{1}{\gamma_{r-i}} - \frac{2}{\gamma_{r-i}+1}\Big)(1 - 2e^{-e^{-y}})\Big]. \tag{16}$$

$$= \frac{C_{r-1}}{(r-1)!(m+1)^r} e^{-y} e^{-e^{-y}} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i}$$

$$\times \Big[\frac{1}{\frac{k}{m+1} + n - r + i} + \alpha\Big(\frac{1}{\frac{k}{m+1} + n - r + i} - \frac{2}{\frac{k+1}{m+1} + n - r + i}\Big)(1 - 2e^{-e^{-y}})\Big]. \tag{17}$$

$$= \frac{C_{r-1}}{(r-1)!(m+1)^r} e^{-y} e^{-e^{-y}} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \Big[B\Big(\frac{k}{m+1} + n - r + i, 1\Big)$$

$$+\alpha\Big\{B\Big(\frac{k}{m+1} + n - r + i, 1\Big) - 2B\Big(\frac{k+1}{m+1} + n - r + i, 1\Big)\Big\}(1 - 2e^{-e^{-y}})\Big]. \tag{18}$$

For real positive p, c and a positive integer b, we have

$$\sum_{a=0}^{b} (-1)^a \binom{b}{a} B(a + p, c) = B(p, c + b). \tag{19}$$

Thus, using (19) in (18), we get

$$g_{[r,n,m,k]}(y) = \frac{C_{r-1}}{(r-1)!(m+1)^r} e^{-y} e^{-e^{-y}} \Big[B\Big(\frac{k}{m+1} + n - r, r\Big)$$

$$+\alpha\Big\{B\Big(\frac{k}{m+1}+n-r,r\Big)-2B\Big(\frac{k+1}{m+1}+n-r,r\Big)\Big\}(1-2e^{-e^{-y}})\Big]. \quad (20)$$

After simplification, we get

$$g_{[r,n,m,k]}(y)=e^{-y}e^{e^{-y}}\Big[1+\alpha(1-2e^{-e^{-y}})\Big\{1-2\prod_{i=1}^{r}\Big(1+\frac{1}{\gamma_i}\Big)^{-1}\Big\}\Big]. \quad (21)$$

Remark 2.1: Set $m=0,k=1$ in (21), to get the *pdf* of $r-th$ concomitants of reverse order statistics from FGM type bivariate Gumbel distribution as

$$g_{[n-r+1:n]}(y)=e^{-y}e^{e^{-y}}\Big[1+\alpha(1-2e^{-e^{-y}})\Big\{1-\frac{2(n-r+1)}{n+1}\Big\}\Big].$$

By replacing $(n-r+1)$ by r, we get the *pdf* of ordinary order statistics.

$$g_{[r:n]}(y)=e^{-y}e^{e^{-y}}\Big[1+\alpha\Big\{1-\frac{2r}{n+1}\Big\}(1-2e^{-e^{-y}})\Big].$$

Remark 2.2: At $m=-1$ in (21), we get the *pdf* of $r-th$ concomitants of $k-th$ lower record values from Farlie FGM type bivariate Gumbel distribution as

$$g_{[r,n,-1,k]}(y)=e^{-y}e^{e^{-y}}\Big[1+\alpha(1-2e^{-e^{-y}})\Big\{1-2\Big(\frac{k}{k+1}\Big)^{r}\Big\}\Big].$$

3 Moment Generating Function of $Y_{[r,n,m,k]}$

In this section,we derive the moment generating function (mgf) of $Y_{[r,n,m,k]}$ for FGM type bivariate Gumbel distribution by using the results of the previous section. In view of (21), the mgf of $Y_{[r,n,m,k]}$ is given as

$$M_{[r,n,m,k]}(t)=\int_{-\infty}^{\infty}e^{ty}e^{-y}e^{e^{-y}}\Big[1+\alpha(1-2e^{-e^{-y}})\Big\{1-2\prod_{i=1}^{r}\Big(1+\frac{1}{\gamma_i}\Big)^{-1}\Big\}\Big]\,dy. \quad (22)$$

Let $e^{-y}=z$ in (22), then we have

$$=\int_{0}^{\infty}z^{(1-t)-1}e^{-z}\Big[1+\alpha(1-2e^{-z})\Big\{1-2\prod_{i=1}^{r}\Big(1+\frac{1}{\gamma_i}\Big)^{-1}\Big\}\Big]\,dz \quad (23)$$

$$= \Gamma(1-t)\Big[1+\alpha(1-2^t)\Big\{1-2\prod_{i=1}^{r}\Big(1+\frac{1}{\gamma_i}\Big)^{-1}\Big\}\Big]. \tag{24}$$

Remark 3.1: Set $m = 0, k = 1$ in (24), to get the mgf of concomitants of reverse order statistics from FGM type bivariate Gumbel distribution as

$$M_{[n-r+1:n]}(t) = \Gamma(1-t)\Big[1+\alpha(1-2^t)\Big\{1-\frac{2(n-r+1)}{n+1}\Big\}\Big].$$

Further, by replacing $(n-r+1)$ by r, we get the mgf of concomitants of ordinary order statistics

$$M_{[r:n]}(t) = \Gamma(1-t)\Big[1-\alpha(1-2^t)\Big\{1-\frac{2r}{n+1}\Big\}\Big].$$

Remark 3.2: Setting $m = -1$ in (24), we get the mgf of concomitants of $k-th$ lower record values from FGM type bivariate Gumbel distribution as

$$M_{[r,n,-1,k]}(t) = \Gamma(1-t)\Big[1+\alpha(1-2^t)\Big\{1-2\Big(\frac{k}{k+1}\Big)^r\Big\}\Big].$$

Remark 3.3: The cumulative generating function (cgf) of $r-th$ $dgos$ $Y_{[r,n,m,k]}$ for FGM type bivariate Gumbel distribution is

$$K_{[r,n,m,k]}(t) = \ln\ \Gamma(1-t) + \ln\ \Big[1+\alpha(1-2^t)\Big\{1-2\prod_{i=1}^{r}\Big(1+\frac{1}{\gamma_i}\Big)^{-1}\Big\}\Big].$$

Since, mean= $E\Big(Y_{[r,n,m,k]}\Big) = \mu_1[r,n,m,k] = \frac{d}{dt}K_{[r,n,m,k]}(t)$ at $t = 0$

and variance= $\mu_2[r,n,m,k] = \frac{d^2}{dt^2}K_{[r,n,m,k]}(t)$ at $t = 0$.

Thus,

$$\mu_{1\ [r,n,m,k]} = \alpha\Big[2\prod_{i=1}^{r}\Big(1+\frac{1}{\gamma_i}\Big)^{-1}-1\Big]\ln 2 - \psi(1). \tag{25}$$

$$\mu_{2\ [r,n,m,k]} = \frac{\Pi^2}{6} - \alpha(\ln 2)^2\Big[1-2\prod_{i=1}^{r}\Big(1+\frac{1}{\gamma_i}\Big)^{-1}\Big].\Big[1+\alpha\Big\{1-2\prod_{i=1}^{r}\Big(1+\frac{1}{\gamma_i}\Big)^{-1}\Big\}\Big]. \tag{26}$$

where, $\psi(x) = \frac{d}{dx}\ln\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}$, $\psi(1) = -\gamma \simeq 0.5772156649...$ is known as Eulers constant.

4 Joint Probability Density Function of $Y_{[r,n,m,k]}$ and $Y_{[s,n,m,k]}$

For the FGM type bivariate Gumbel distribution as given in (5), using (7) and (12) in (14), the joint *pdf* of $r-th$ and $s-th$ concomitants of *dgos* $Y_{[r,n,m,k]}$ and $Y_{[s,n,m,k]}$ for $m \neq -1$ is given as

$$g_{[r,s,n,m,k]}(y_1,y_2) = \frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}} \; e^{-y_1}e^{-y_2}e^{-e^{-y_1}}e^{-e^{-y_2}}$$
$$\times \sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}(-1)^{i+j}\binom{r-1}{i}\binom{s-r-1}{j}\int_{-\infty}^{\infty} e^{-x_1}\left(e^{-e^{-x_1}}\right)^{(s-r+i-j)(m+1)}$$
$$\times\ [1+\alpha(1-2e^{-e^{-x_1}})(1-2e^{-e^{-y_1}})]I(x_1,y_2)dx_1, \tag{27}$$

where,

$$I(x_1,y_2) = \int_{-\infty}^{x_1} e^{-x_2}\left(e^{-e^{-x_2}}\right)^{\gamma_{s-j}}[1+\alpha(1-2e^{-e^{-x_2}})(1-2e^{-e^{-y_2}})]dx_2. \tag{28}$$

By setting $e^{-x_2}=t$ in (28), we get

$$I(x_1,y_2) = \Big[\frac{e^{-e^{-(\gamma_{s-j})x_1}}}{\gamma_{s-j}} + \alpha\Big(\frac{e^{-e^{-(\gamma_{s-j})x_1}}}{\gamma_{s-j}} - \frac{e^{-e^{-(\gamma_{s-j}+1)x_1}}}{\gamma_{s-j}+1}\Big)(1-2e^{-e^{-y_2}})\Big]. \tag{29}$$

Now putting the value of $I(x_1,y_2)$ from (29) in (27), we obtain

$$g_{[r,s,n,m,k]}(y_1,y_2) = e^{-y_1}e^{-y_2}e^{-e^{-y_1}}e^{-e^{-y_2}}\Big[1+\alpha^2\left(1-2e^{-e^{-y_1}}\right)\left(1-2e^{-e^{-y_2}}\right)$$
$$\times\Big\{1-2\prod_{i=1}^{r}\Big(1+\frac{1}{\gamma_i}\Big)^{-1}-2\prod_{i=1}^{s}\Big(1+\frac{1}{\gamma_i}\Big)^{-1}+\ 4\prod_{i=1}^{r}\Big(1+\frac{2}{\gamma_i}\Big)^{-1}\Big\}$$
$$+\alpha\Big\{\Big(1-2e^{-e^{-y_1}}\Big)+\Big(1-2e^{-e^{-y_2}}\Big)\Big\}\Big\{1-2\prod_{i=1}^{s}\Big(1+\frac{1}{\gamma_i}\Big)^{-1}\Big\}\Big]. \tag{30}$$

5 Joint Moment Generating Function of two concomitants $Y_{[r,n,m,k]}$ and $Y_{[s,n,m,k]}$

Joint moment generating function of two concomitant $Y[r,n,m,k]$ and $Y[s,n,m,k]$ is given by

$$M_{[r,s,n,m,k]}(t_1,t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t_1y_1}e^{t_2y_2}\;\; g_{[r,s,n,m,k]}(y_1,y_2)\; dy_1\; dy_2. \tag{31}$$

In view of (30) and (31), we have

$$M_{[r,s,n,m,k]}(t_1,t_2)$$
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{t_1 y_1} e^{t_2 y_2} e^{-y_1} e^{-y_2} e^{-e^{-y_1}} e^{-e^{-y_2}} \Big[1+\alpha\Big\{\Big(1-2e^{-e^{-y_1}}\Big)+\Big(1-2e^{-e^{-y_2}}\Big)\Big\}$$
$$\times\Big\{1-2\prod_{i=1}^{s}\Big(1+\frac{1}{\gamma_i}\Big)^{-1}\Big\} + \alpha^2 \Big(1-2e^{-e^{-y_1}}\Big)\Big(1-2e^{-e^{-y_2}}\Big)$$
$$\times\Big\{1-2\prod_{i=1}^{r}\Big(1+\frac{1}{\gamma_i}\Big)^{-1} - 2\prod_{i=1}^{s}\Big(1+\frac{1}{\gamma_i}\Big)^{-1} + 4\prod_{i=1}^{r}\Big(1+\frac{2}{\gamma_i}\Big)^{-1}\Big\}\Big] dy_1\, dy_2. \qquad (32)$$

Let $z_1 = e^{-y_1}$, then

$$M_{[r,s,n,m,k]}(t_1,t_2)$$
$$= \Gamma(1-t_1)\int_{-\infty}^{\infty} e^{t_2 y_2} e^{-y_2} e^{-e^{-y_2}} \Big[1+\alpha\Big\{\Big(1-2^{t_1}\Big)+\Big(1-2e^{-e^{-y_2}}\Big)\Big\}$$
$$\times\Big\{1-2\prod_{i=1}^{s}\Big(1+\frac{1}{\gamma_i}\Big)^{-1}\Big\} + \alpha^2 \Big(1-2^{t_1}\Big)\Big(1-2e^{-e^{-y_2}}\Big)$$
$$\times\Big\{1-2\prod_{i=1}^{r}\Big(1+\frac{1}{\gamma_i}\Big)^{-1} - 2\prod_{i=1}^{s}\Big(1+\frac{1}{\gamma_i}\Big)^{-1} + 4\prod_{i=1}^{r}\Big(1+\frac{2}{\gamma_i}\Big)^{-1}\Big\}\Big] dy_2. \qquad (33)$$

Setting $z_2 = e^{-y_2}$ in (33) and simplifying, we have

$$M_{[r,s,n,m,k]}(t_1,t_2) = \Gamma(1-t_1)\Gamma(1-t_2)\Big[1+\alpha\Big\{\Big(1-2^{t_1}\Big)+\Big(1-2^{t_2}\Big)\Big\}$$
$$\times\Big\{1-2\prod_{i=1}^{s}\Big(1+\frac{1}{\gamma_i}\Big)^{-1}\Big\} + \alpha^2 \Big(1-2^{t_1}\Big)\Big(1-2^{t_2}\Big)$$
$$\times\Big\{1-2\prod_{i=1}^{r}\Big(1+\frac{1}{\gamma_i}\Big)^{-1} - 2\prod_{i=1}^{s}\Big(1+\frac{1}{\gamma_i}\Big)^{-1} + 4\prod_{i=1}^{r}\Big(1+\frac{2}{\gamma_i}\Big)^{-1}\Big\}\Big]. \qquad (34)$$

Remark 5.1: Set $m = 0, k = 1$ in (34), to get the joint mgf of two concomitants of reverse order statistics from FGM type bivariate Gumbel distribution as

$$M_{[n-s+1,n-r+1:n]}(t_1,t_2) = \Gamma(1-t_1)\Gamma(1-t_2)\Big[1+\alpha\Big\{\Big(1-2^{t_1}\Big)+\Big(1-2^{t_2}\Big)\Big\}\Big\{1-\frac{2(n-s+1)}{n+1}\Big\}$$
$$+\alpha^2 \Big(1-2^{t_1}\Big)\Big(1-2^{t_2}\Big)\Big\{1-\frac{2(n-r+1)}{n+1}-\frac{2(n-s+1)}{n+1}+\frac{4(n-r+1)(n-r+2)}{(n+1)(n+2)}\Big\}\Big].$$

Now replacing $(n-r+1)$ by s and $(n-s+1)$ by r , we get the joint mgf of two concomitants of ordinary order statistics

$$M_{[r,s:n]}(t_1,t_2)=\Gamma(1-t_1)\Gamma(1-t_2)\Big[1-\alpha\Big\{\Big(1-2^{t_1}\Big)+\Big(1-2^{t_2}\Big)\Big\}\Big\{1-\frac{2r}{n+1}\Big\}$$
$$+\alpha^2\ \Big(1-2^{t_1}\Big)\Big(1-2^{t_2}\Big)\Big\{1-\frac{2s}{n+1}-\frac{2r}{n+1}+\frac{4s(s+1)}{(n+1)(n+2)}\Big\}\Big].$$

Remark 5.2: Setting $m=-1$ in (34), we get the joint mgf of two concomitants of $k-th$ lower record values from FGM type bivariate Gumbel distribution as

$$M_{[r,s,n,-1,k]}(t_1,t_2)=\Gamma(1-t_1)\Gamma(1-t_2)\Big[1+\alpha\Big\{\Big(1-2^{t_1}\Big)+\Big(1-2^{t_2}\Big)\Big\}\Big\{1-2\Big(\frac{k}{k+1}\Big)^s\Big\}$$
$$+\alpha^2\ \Big(1-2^{t_1}\Big)\Big(1-2^{t_2}\Big)\Big\{1-2\Big(\frac{k}{k+1}\Big)^r-2\Big(\frac{k}{k+1}\Big)^s+\ 4\Big(\frac{k}{k+2}\Big)^r\Big\}\Big].$$

Remark 5.3: Joint cgf of two concomitant $Y[r,n,m,k]$ and $Y[s,n,m,k]$ for FGM type bivariate Gumbel distribution is given as

$$K_{[r,s,n,m,k]}(t_1,t_2)=\ \ln\ \Gamma(1-t_1)+\ \ln\ \Gamma(1-t_2)$$
$$+\ \ln\ \Big[1+\alpha\Big\{\Big(1-2^{t_1}\Big)+\Big(1-2^{t_2}\Big)\Big\}\Big\{1-2\prod_{i=1}^{s}\Big(1+\frac{1}{\gamma_i}\Big)^{-1}\Big\}+\alpha^2\ \Big(1-2^{t_1}\Big)\Big(1-2^{t_2}\Big)$$
$$\times\Big\{1-2\prod_{i=1}^{r}\Big(1+\frac{1}{\gamma_i}\Big)^{-1}-2\prod_{i=1}^{s}\Big(1+\frac{1}{\gamma_i}\Big)^{-1}+\ 4\prod_{i=1}^{r}\Big(1+\frac{2}{\gamma_i}\Big)^{-1}\Big\}\Big]. \tag{35}$$

Noting that,

$$Cov\Big[Y_{[r,n,m,k]},Y_{[s,n,m,k]}\Big]=\frac{d^2}{dt_1\ st_2}K_{[r,s,n,m,k]}(t_1,t_2)\ \ at\ t_=0,\ t_2=0$$

Then we get

$$Cov\Big[Y_{[r,n,m,k]},Y_{[s,n,m,k]}\Big]=\Big(\alpha\ln 2\Big)^2\Big[\Big\{1-2\prod_{i=1}^{r}\Big(1+\frac{1}{\gamma_i}\Big)^{-1}-2\prod_{i=1}^{s}\Big(1+\frac{1}{\gamma_i}\Big)^{-1}$$
$$+\ 4\prod_{i=1}^{r}\Big(1+\frac{2}{\gamma_i}\Big)^{-1}\Big\}-\Big\{1-2\prod_{i=1}^{s}\Big(1+\frac{1}{\gamma_i}\Big)^{-1}\Big\}^2\Big]. \tag{36}$$

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