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Improved Exponential Chain Ratio and Product-Type Estimators for Finite Population Mean in Double Sampling

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Abstract

An exponential chain ratio and product-type estimators in double sampling are proposed for estimating finite population mean of the study variate, when the information on another additional auxiliary character is available along with the main auxiliary character. The expressions for the bias and mean square error (MSE) of the proposed estimators have been obtained in two different cases. An analytical and numerical comparison of the proposed estimators with other existing members of estimators shows that the proposed estimators are more efficient under certain realistic conditions.

Keywords and Phrases: Auxiliary information, Study variate, Exponential chain ratio and product-type estimators in double sampling, MSE and Efficiency.

AMS Classification: 62D05.

1 Introduction

In sample surveys, supplementary information is used at either selection or estimation stage or both, to improve the precision of the estimate of the population parameter. The literature on survey sampling describes several methods of using the auxiliary variable at the estimation stage. This includes among others; linear regression estimator, ratio estimator, product estimator and difference estimator. When the auxiliary variable is used at the estimation stage and the relation between the study variable (Y) and auxiliary variable(X) is highly positive, such that the regression line passes through the origin, the classical ratio method of estimation proposed by Cochran (1940) is most preferred. On the other hand, when the relation between the variables is highly negative, the classical product method of estimation by Robson (1957) and Murthy (1964) is most preferred.

The use of ratio and product strategies in survey sampling solely depend upon the knowledge of population mean $\bar{X}$ of the auxiliary character X. However, there are situations of practical importance; where the population mean $\bar{X}$ is not known before the start of the survey. In such a situation, a sample of size n_1 is selected initially by using a suitable sampling design and its sample mean $\bar{x_1}$ is used to estimate population mean $\bar{X}$, then a subsample of size $n(n < n_1)$ is selected to estimate the population mean of the study and auxiliary variables. However, if the population mean $\bar{Z}$ of another auxiliary variable Z, closely related to X but compared to X remotely related to Y is known (i.e. $\rho_{yx} > \rho_{yz}$), it is preferable to estimate $\bar{X}$ by $\bar{X} = \frac{\bar{x_1}\bar{Z}}{\bar{z_1}}$, which would provide better estimate of $\bar{X}$ than $\bar{x_1}$ to the terms of order $o(n^{-1})$ if $\frac{\rho_{xz}C_x}{C_z} > \frac{1}{2}$, where C_x, C_z and ρ_{yx}, ρ_{yz} and ρ_{xz} are coefficient of variation of x, z and correlation coefficient between y and x; y and z; x and z respectively. This technique is known as chaining. The chain regression estimator was first introduced by Swain (1970). Chand (1975), Sukhatme and Chand (1977), Kiregyera (1980, 1984) proposed some chain ratio and regression type estimators based on two auxiliary variates. Isaki (1983), Singh and Singh (2001), Singh et al. (2001), Prasad et al. (2002), Pradhan (2005), Singh and Choudhury (2012) and many authors have suggested some improved chain ratio, product, regression type estimators in double sampling.

Let us consider a finite population $U = (U_1, U_2, U_3, ..., U_N)$ of size N units and the value of the variables on the i^{th} unit be (y_i, x_i), where $i = 1, 2, 3, ..., N$. Let $\bar{Y} = \frac{1}{N}\sum_{i=1}^{N} y_i$ and $\bar{X} = \frac{1}{N}\sum_{i=1}^{N} x_i$ are the population means of the study variable y and the auxiliary variable x respectively. For estimating the population mean $\bar{Y}$ of y, a simple random sample of size n is drawn without replacement from the population U. Then the classical ratio and product-type estimators are respectively as

$\bar{Y_R} = \bar{y}\frac{\bar{X}}{\bar{x}}$ if $\bar{x} \neq 0$ and $\bar{Y_P} = \bar{y}\frac{\bar{x}}{\bar{X}}$,

where $\bar{y}$ and $\bar{x}$ are the sample means of y and x respectively based on a sample of size n out of the population of size N units and $\bar{X}$ is the known as population mean of x. With known population mean $\bar{X}$, Bahl and Tuteja (1991) suggested the exponential ratio and product type estimators as

$\bar{Y}_{Re} = \bar{y}\ \exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right)$ and $\bar{Y}_{Pe} = \bar{y}\ \exp\left(\frac{\bar{x}-\bar{X}}{\bar{x}+\bar{X}}\right)$

respectively for the population mean $\bar{Y}$.

If the population mean $\bar{X}$ of the auxiliary variable x is not known before start of the survey, a first-phase sample of size n_1 is drawn from the population, on which only the auxiliary variable x is observed. Then a second-phase sample of size n is drawn, on which both study variable y and auxiliary variable x are observed. Let $\bar{x}_1 = \frac{1}{n_1}\sum_{i=1}^{n_1} x_i$ denotes the sample mean of size n_1 based on the first phase sample and $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$ and $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ denote the sample means of variables Y and X respectively, obtained from the second phase sample of size n .

Singh and Vishwakarma (2007) suggested the exponential ratio and product-type estimators for $\bar{Y}$ in double sampling respectively as

$\bar{Y}^{d}_{Re} = \bar{y}\ \exp\left(\frac{\bar{x}_1-\bar{x}}{\bar{x}_1+\bar{x}}\right)$ and $\bar{Y}^{d}_{Pe} = \bar{y}\ \exp\left(\frac{\bar{x}-\bar{x}_1}{\bar{x}+\bar{x}_1}\right)$.

If the population mean $\bar{Z}$ of another auxiliary variate Z, closely related to X but compared to X remotely related to Y is available and $\bar{z}_1 = \frac{1}{n_1}\sum_{i=1}^{n_1} z_i$ be the sample mean of Z. The chain ratio and product estimators in double sampling suggested by Chand (1975) are respectively given as

$\bar{Y}^{dc}_{R} = \bar{y}\frac{\bar{x}_1}{\bar{x}}\frac{\bar{Z}}{\bar{z}_1}$ and $\bar{Y}^{dc}_{P} = \bar{y}\frac{\bar{x}}{\bar{x}_1}\frac{\bar{z}_1}{\bar{Z}}$.

Singh and Choudhury (2012) suggested the exponential chain ratio and product-type estimators for $\bar{Y}$ in double sampling respectively as

$\bar{Y}^{dc}_{Re} = \bar{y}\ exp\left(\frac{\bar{x}_1\frac{\bar{Z}}{\bar{z}_1}-\bar{x}}{\bar{x}_1\frac{\bar{Z}}{\bar{z}_1}+\bar{x}}\right)$ and $\bar{Y}^{dc}_{Pe} = \bar{y}\ exp\left(\frac{\bar{x}-\bar{x}_1\frac{\bar{Z}}{\bar{z}_1}}{\bar{x}+\bar{x}_1\frac{\bar{Z}}{\bar{z}_1}}\right)$.

In this paper, under SRSWOR, we present a class of exponential chain ratio and product-type estimators in double sampling based on Singh and Choudhury (2012) and obtain the bias and the MSE of class of estimators to the first order of approximation. Numerical illustrations are given to show the performance of the proposed estimator over other estimators.

2 Proposed estimators based on the estimators $\bar{Y}^{dc}_{Re}$ and $\bar{Y}^{dc}_{Pe}$

Motivated by Singh and Choudhury (2012), we have proposed the following modified exponential chain ratio and product-type estimators in double sampling respectively as

$$t_1^* = \bar{y}exp\left\{\frac{\bar{x}_1\left(\frac{a\bar{Z}+b}{a\bar{z}_1+b}\right)-\bar{x}}{\bar{x}_1\left(\frac{a\bar{Z}+b}{a\bar{z}_1+b}\right)+\bar{x}}\right\} = \bar{y}exp\left\{\frac{\bar{x}_1\left(\frac{\bar{U}}{\bar{u}_1}\right)-\bar{x}}{\bar{x}_1\left(\frac{\bar{U}}{\bar{u}_1}\right)+\bar{x}}\right\} \tag{1}$$

and

$$t_2^* = \bar{y}exp\left\{\frac{\bar{x}-\bar{x}_1\left(\frac{a\bar{Z}+b}{a\bar{z}_1+b}\right)}{\bar{x}+\bar{x}_1\left(\frac{a\bar{Z}+b}{a\bar{z}_1+b}\right)}\right\} = \bar{y}exp\left\{\frac{\bar{x}-\bar{x}_1\left(\frac{\bar{U}}{\bar{u}_1}\right)}{\bar{x}+\bar{x}_1\left(\frac{\bar{U}}{\bar{u}_1}\right)}\right\}, \tag{2}$$

where $a\,(\neq 0)$ and b are scalar constants, $\bar{u}_1 = a\bar{z}_1 + b$ and $\bar{U} = a\bar{Z} + b$.

Remarks

(i) For $(a, b) = (1, 0)$, the estimator t_1^* of equation (1) reduces to the 'exponential chain ratio-type estimator'$\left(\bar{Y}_{Re}^{dc}\right)$ in double sampling.
(ii) For $(a, b) = (1, 0)$, the estimator t_2^* of equation (2) reduces to the 'exponential chain product-type estimator'$\left(\bar{Y}_{Pe}^{dc}\right)$ in double sampling.

The bias and the MSE of the proposed estimators are obtained for the following two cases.
Case I: When the second phase sample is a *subsample* of the first phase sample.
Case II: When the second phase sample is drawn *independently* of the first phase sample.

3 Bias and MSE of t_1^* and t_2^* for Case I

To obtain the bias (B) and mean square error (M) of estimators t_1^* and t_2^*, we write $e_0 = \frac{\bar{y}-\bar{Y}}{\bar{Y}}$, $e_1 = \frac{\bar{x}-\bar{X}}{\bar{X}}$, ${e'}_1 = \frac{\bar{x}_1-\bar{X}}{\bar{X}}$ and $e_2 = \frac{\bar{z}_1-\bar{Z}}{\bar{Z}}$ such that

$$\begin{cases} E(e_0) = E(e_1) = E(e_1') = E(e_2) = 0,\ E(e_0^2) = \frac{1-f}{n}C_y^2, \\ E(e_1^2) = \frac{1-f}{n}C_x^2,\ E(e_1'^2) = \frac{1-f_1}{n_1}C_x^2,\ E(e_2^2) = \frac{1-f_1}{n_1}C_z^2, \\ E(e_0e_1) = \frac{1-f}{n}C_{yx}C_x^2,\ E(e_0e_1') = \frac{1-f_1}{n_1}C_{yx}C_x^2,\ E(e_0e_2) = \frac{1-f_1}{n_1}C_{yz}C_z^2, \\ E(e_1e_1') = \frac{1-f_1}{n_1}C_x^2,\ E(e_1e_2) = \frac{1-f_1}{n_1}C_{zx}C_z^2,\ E(e_1'e_2) = \frac{1-f_1}{n_1}C_{zx}C_z^2. \end{cases} \tag{3}$$

where $f = \frac{n}{N}$, $f_1 = \frac{n_1}{N}$, $C_y = \frac{S_y}{\bar{Y}}$, $C_x = \frac{S_x}{\bar{X}}$, $C_z = \frac{S_z}{\bar{Z}}$, $C_{yx} = \frac{\rho_{yx}C_y}{C_x}$, $C_{yz} = \frac{\rho_{yz}C_y}{C_z}$, $C_{zx} = \frac{\rho_{zx}C_x}{C_z}$, $\rho_{yx} = \frac{S_{yx}}{S_yS_x}$, $\rho_{yz} = \frac{S_{yz}}{S_yS_z}$, $\rho_{zx} = \frac{S_{zx}}{S_zS_x}$, $S_y^2 = \frac{1}{N-1}\sum_{i=1}^N\left(y_i - \overline{Y}\right)^2$, $S_x^2 = \frac{1}{N-1}\sum_{i=1}^N\left(x_i - \bar{X}\right)^2$, $S_z^2 = \frac{1}{N-1}\sum_{i=1}^N\left(z_i - \overline{Z}\right)^2$, $S_{xy} = \frac{1}{N-1}\sum_{i=1}^N\left(y_i - \overline{Y}\right)\left(x_i - \bar{X}\right)$, $S_{yz} = \frac{1}{N-1}\sum_{i=1}^N\left(y_i - \overline{Y}\right)\left(z_i - \bar{Z}\right)$ and $S_{zx} = \frac{1}{N-1}\sum_{i=1}^N\left(z_i - \overline{Z}\right)\left(x_i - \bar{X}\right)$.

Expanding the right hand side of (1) and (2) in terms of e's, multiplying out and neglecting the terms of e's of power greater than two, we have

$$
\begin{aligned}
t_1^* - \bar{Y} \quad \cong \quad & \bar{Y}\Big\{e_0 + \frac{1}{2}\left(e_1' - e_1 - \phi e_2 + e_0 e_1' - e_0 e_1 - \phi e_0 e_2\right) \\
& -\frac{1}{4}\left(e_1'^2 - e_1^2 - \phi^2 e_2^2 + e_1 e_1' + \phi e_1' e_2 - \phi e_1 e_2\right) + \frac{1}{8}\left(e_1'^2 + e_1^2 + \phi^2 e_2^2\right)\Big\} \quad (4)
\end{aligned}
$$

and

$$
\begin{aligned}
t_2^* - \bar{Y} \quad \cong \quad & \bar{Y}\Big\{e_0 + \frac{1}{2}\left(e_1 - e_1' + \phi e_2 - e_0 e_1' + e_0 e_1 + \phi e_0 e_2\right) \\
& -\frac{1}{4}\left(e_1^2 + e_1'^2 + \phi^2 e_2^2 + e_1 e_1' + \phi e_1' e_2 - \phi e_1 e_2\right) + \frac{1}{8}\left(e_1'^2 + e_1^2 + \phi^2 e_2^2\right)\Big\} \quad (5)
\end{aligned}
$$

where $\phi = \frac{a\bar{Z}}{a\bar{Z}+b}$.
Therefore, the bias of the estimators t_1^* and t_2^* can be obtained by using the results of equation (3) in equations (4) and (5) as
$B\left(t_1^*\right)_I = \bar{Y}\left\{\frac{3}{8}\left(\frac{1-f^*}{n}C_x^2 + \phi^2\frac{1-f_1}{n_1}C_z^2\right) - \frac{1}{2}\left(\frac{1-f^*}{n}C_{yx}C_x^2 - \phi\frac{1-f_1}{n_1}C_{yz}C_z^2\right)\right\}$
and
$B\left(t_2^*\right)_I = \bar{Y}\left\{-\frac{1}{8}\left(\frac{1-f^*}{n}C_x^2 + \phi^2\frac{1-f_1}{n_1}C_z^2\right) + \frac{1}{2}\left(\frac{1-f^*}{n}C_{yx}C_x^2 + \phi\frac{1-f_1}{n_1}C_{yz}C_z^2\right)\right\}$
where $f^* = \frac{n}{n_1}$.
From equations (4) and (5), we have

$$
t_1^* - \bar{Y} \cong \left\{e_0 + \frac{1}{2}\left(e_1' - e_1 - \phi e_2\right)\right\} \tag{6}
$$

and

$$
t_2^* - \bar{Y} \cong \left\{e_0 + \frac{1}{2}\left(e_1 - e_1' + \phi e_2\right)\right\} \tag{7}
$$

Squaring both sides of equations (6) and (7), taking expectations and using the results of equation (3), we get the MSE of the estimators t_1^* and t_2^* to the first degree approximation as

$$
\begin{aligned}
M\left(t_1^*\right)_I \quad = \quad & \bar{Y}^2\left\{\frac{1-f}{n}C_y^2 + \frac{1}{4}\left(\frac{1-f^*}{n}C_x^2 + \phi^2\frac{1-f_1}{n_1}C_z^2\right)\right. \\
& \left. -\left(\frac{1-f^*}{n}C_{yx}C_x^2 + \phi\frac{1-f_1}{n_1}C_{yz}C_z^2\right)\right\} \qquad (8)
\end{aligned}
$$

and

$$
\begin{aligned}
M\left(t_2^*\right)_I \quad = \quad & \bar{Y}^2\left\{\frac{1-f}{n}C_y^2 + \frac{1}{4}\left(\frac{1-f^*}{n}C_x^2 + \phi^2\frac{1-f_1}{n_1}C_z^2\right)\right. \\
& \left. +\left(\frac{1-f^*}{n}C_{yx}C_x^2 + \phi\frac{1-f_1}{n_1}C_{yz}C_z^2\right)\right\} \qquad (9)
\end{aligned}
$$

Differentiating equation (8) with respect to ϕ yields its optimum value as

$$\phi_{opt.} = 2C_{yz}. \tag{10}$$

Substituting the value of $\phi_{opt.}$ from equation (10) in equation (8), we get the optimum MSE of t_1^* as

$$opt.M\left(t_1^*\right)_I = \bar{Y}^2\left\{\frac{1-f}{n}C_y^2 + \frac{1-f^*}{n}C_x^2\left(\frac{1}{4} - C_{yx}\right) - \frac{1-f_1}{n_1}C_{yz}^2C_z^2\right\} \tag{11}$$

Differentiating equation (9 with respect to ϕ yields its optimum value as

$$\phi_{opt.} = -2C_{yz}. \tag{12}$$

Substituting the value of $\phi_{opt.}$ from equation (12) in equation (9), we get the optimum MSE of t_2^* as

$$opt.M\left(t_2^*\right)_I = \bar{Y}^2\left\{\frac{1-f}{n}C_y^2 + \frac{1-f^*}{n}C_x^2\left(\frac{1}{4} + C_{yx}\right) - \frac{1-f_1}{n_1}C_{yz}^2C_z^2\right\} \tag{13}$$

The MSE of usual unbiased estimator $\bar{y}$ under SRSWOR scheme is

$$M\left(\bar{y}\right) = \bar{Y}^2\frac{1-f}{n}C_y^2 \tag{14}$$

To the first degree approximation, the MSE of estimators $\bar{Y}_R^{dc}, \bar{Y}_P^{dc}, \bar{Y}_{Re}^{dc}$ and $\bar{Y}_{Re}^{dc}$ are

$$M\left(\bar{Y}_R^{dc}\right)_I = \bar{Y}^2\left\{\frac{1-f}{n}C_y^2 + \frac{1-f^*}{n}C_x^2\left(1-2C_{yx}\right) + \frac{1-f_1}{n_1}C_z^2\left(1-2C_{yz}\right)\right\} \tag{15}$$

$$M\left(\bar{Y}_P^{dc}\right)_I = \bar{Y}^2\left\{\frac{1-f}{n}C_y^2 + \frac{1-f^*}{n}C_x^2\left(1+2C_{yx}\right) + \frac{1-f_1}{n_1}C_z^2\left(1+2C_{yz}\right)\right\} \tag{16}$$

$$M\left(\bar{Y}_{Re}^{dc}\right)_I = \bar{Y}^2\left\{\frac{1-f}{n}C_y^2 + \frac{1}{4}\left(\frac{1-f^*}{n}C_x^2 + \frac{1-f_1}{n_1}C_z^2\right) - \left(\frac{1-f^*}{n}C_{yx}C_x^2 + \frac{1-f_1}{n_1}C_{yz}C_z^2\right)\right\} \tag{17}$$

and

$$M\left(\bar{Y}_{Pe}^{dc}\right)_I = \bar{Y}^2\left\{\frac{1-f}{n}C_y^2 + \frac{1}{4}\left(\frac{1-f^*}{n}C_x^2 + \frac{1-f_1}{n_1}C_z^2\right) + \left(\frac{1-f^*}{n}C_{yx}C_x^2 + \frac{1-f_1}{n_1}C_{yz}C_z^2\right)\right\} \tag{18}$$

respectively.

3.1 Efficiency Comparisons of t_1^* and t_2^*

From the equations (11) and (17), we have
$M\left(\bar{Y}_{Re}^{dc}\right)_I - opt.M\left(t_1^*\right)_I = \bar{Y}^2\frac{1-f_1}{n_1}C_z^2\left(\frac{1}{2}-C_{yz}\right)^2 > 0.$

From the above comparison, it is clear that the proposed exponential chain ratio-type estimator (t_1^*) is more efficient than the exponential chain ratio estimator $\left(\bar{Y}_{Re}^{dc}\right)$ in double sampling.
From the equations (13) and (18), we have
$M\left(\bar{Y}_{Pe}^{dc}\right)_I - opt.M\left(t_2^*\right)_I = \bar{Y}^2\frac{1-f_1}{n_1}C_z^2\left(\frac{1}{2}+C_{yz}\right)^2 > 0.$

From the above comparison, we observe that the proposed exponential chain product-type estimator (t_2^*) is more efficient than exponential chain product estimator $\left(\bar{Y}_{Pe}^{dc}\right)$ in double sampling.

4 Bias and MSE of t_1^* and t_2^* for Case II

To obtain Bias and MSE of estimators t_1^* and t_2^*, we have

$$\begin{cases} E(e_0) = E(e_1) = E(e_1') = E(e_2) = 0,\ E(e_0^2) = \frac{1-f}{n}C_y^2, \\ E(e_1^2) = \frac{1-f}{n}C_x^2,\ E(e_1'^2) = \frac{1-f_1}{n_1}C_x^2,\ E(e_2^2) = \frac{1-f_1}{n_1}C_z^2,\ E(e_0e_1) = \frac{1-f}{n}C_{yx}C_x^2, \\ E(e_1'e_2) = \frac{1-f}{n}C_{xz}C_z^2,\ E(e_0e_1') = E(e_0e_2) = E(e_1e_1') = E(e_1e_2) = 0. \end{cases} \tag{19}$$

Taking expectations in equations (4) and (5) and using the results of equation (19), we get the bias of the estimators t_1^* and t_2^* to the first degree approximation as
$B\left(t_1^*\right)_{II} = \bar{Y}\left\{\frac{1}{8}\left(f^{**}C_x^2 + 3\phi^2\frac{1-f_1}{n_1}C_z^2\right) + \frac{1}{4}\left(\frac{1-f^*}{n}C_x^2 - \phi\frac{1-f_1}{n_1}C_{xz}C_z^2\right) - \frac{1}{2}\frac{1-f}{n}C_{yx}C_x^2\right\}$
and
$B\left(t_2^*\right)_{II} = \bar{Y}\left\{\frac{1}{8}\left(f^{**}C_x^2 - \phi^2\frac{1-f_1}{n_1}C_z^2\right) - \frac{1}{4}\left(\frac{1-f^*}{n}C_x^2 + \phi\frac{1-f_1}{n_1}C_{xz}C_z^2\right) + \frac{1}{2}\frac{1-f}{n}C_{yx}C_x^2\right\}$
Squaring both the sides of equations (6) and (7), taking expectations and using the results of equation (19), we get the MSE of t_1^* and t_2^* as

$$M\left(t_1^*\right)_{II} = \bar{Y}^2\left\{\frac{1-f}{n}C_y^2 + \frac{1}{4}\left(f^{**}C_x^2 + \phi^2\frac{1-f_1}{n_1}C_z^2\right) - \frac{1}{2}\left(\frac{1-f}{n}C_{yx}C_x^2 + \phi\frac{1-f_1}{n_1}C_{xz}C_z^2\right)\right\} \tag{20}$$

and

$$M\left(t_2^*\right)_{II} = \bar{Y}^2\left\{\frac{1-f}{n}C_y^2 + \frac{1}{4}\left(f^{**}C_x^2 + \phi^2\frac{1-f_1}{n_1}C_z^2\right) - \frac{1}{2}\phi\frac{1-f_1}{n_1}C_{xz}C_z^2 + \frac{1-f}{n}C_{yx}C_x^2\right\} \tag{21}$$

where $f^{**} = \frac{1-f}{n} + \frac{1-f_1}{n_1}$.

Differentiating equation (20) with respect to ϕ yields its optimum value as

$$\phi_{opt.} = C_{xz} \tag{22}$$

Substituting the value of $\phi_{opt.}$ from equation (22) in equation (20), we get the optimum MSE of t_1^* as

$$opt.M\left(t_1^*\right)_{II} = \bar{Y}^2\left\{\frac{1-f}{n}C_y^2 + \frac{1}{4}\left(f^{**}C_x^2 - \frac{1-f_1}{n_1}C_{xz}^2C_z^2\right) - \frac{1-f}{n}C_{yx}C_x^2\right\} \quad (23)$$

Differentiating equation (21) with respect to ϕ yields its optimum value as

$$\phi_{opt.} = C_{xz} \quad (24)$$

Substituting the value of $\phi_{opt.}$ from equation (24) in equation (21), we get the optimum MSE of t_2^* as

$$opt.M\left(t_2^*\right)_{II} = \bar{Y}^2\left\{\frac{1-f}{n}C_y^2 + \frac{1}{4}\left(f^{**}C_x^2 - \frac{1-f_1}{n_1}C_{xz}^2C_z^2\right) + \frac{1-f}{n}C_{yx}C_x^2\right\} \quad (25)$$

To the first degree approximation, the MSE of estimators $\bar{Y}_R^{dc}, \bar{Y}_P^{dc}, \bar{Y}_{Re}^{dc}$ and $\bar{Y}_{Re}^{dc}$ are

$$M\left(\bar{Y}_R^{dc}\right)_{II} = \bar{Y}^2\left\{\frac{1-f}{n}C_y^2 + \frac{1-f}{n}C_x^2\left(1-2C_{yx}\right) + \frac{1-f_1}{n_1}C_x^2 + \frac{1-f_1}{n_1}C_z^2\left(1-2C_{xz}\right)\right\} \quad (26)$$

$$M\left(\bar{Y}_P^{dc}\right)_{II} = \bar{Y}^2\left\{\frac{1-f}{n}C_y^2 + \frac{1-f}{n}C_x^2\left(1+2C_{yx}\right) + \frac{1-f_1}{n_1}C_x^2 + \frac{1-f_1}{n_1}C_z^2\left(1-2C_{xz}\right)\right\} \quad (27)$$

$$M\left(\bar{Y}_{Re}^{dc}\right)_{II} = \bar{Y}^2\left\{\frac{1-f}{n}C_y^2 + \frac{1}{4}\left(f^{**}C_x^2 + \frac{1-f_1}{n_1}C_z^2\right) - \frac{1-f}{n}C_{yx}C_x^2 - \frac{1}{2}\frac{1-f_1}{n_1}C_{xz}C_z^2\right\} \quad (28)$$

and

$$M\left(\bar{Y}_{Pe}^{dc}\right)_{II} = \bar{Y}^2\left\{\frac{1-f}{n}C_y^2 + \frac{1}{4}\left(f^{**}C_x^2 + \frac{1-f_1}{n_1}C_z^2\right) + \frac{1-f}{n}C_{yx}C_x^2 - \frac{1}{2}\frac{1-f_1}{n_1}C_{xz}C_z^2\right\} \quad (29)$$

respectively.

4.1 Efficiency Comparisons of t_1^* and t_2^* with the estimators $\bar{Y}_{Re}^{dc}$ and $\bar{Y}_{Pe}^{dc}$

From the equations (23) and (28), we have
$M\left(\bar{Y}_{Re}^{dc}\right)_{II} - opt.M\left(t_1^*\right)_{II} = \bar{Y}^2\frac{1}{4}\frac{1-f_1}{n_1}C_z^2\left(1-C_{xz}\right)^2 > 0.$

From the above comparison, we observed that the proposed estimator t_1^* is more efficient than the double sampling exponential chain ratio-type estimator $\left(\bar{Y}_{Re}^{dc}\right)$.
From the equations (25) and (29), we have
$M\left(\bar{Y}_{Pe}^{dc}\right)_{II} - opt.M\left(t_2^*\right)_{II} = \bar{Y}^2\frac{1-f_1}{n_1}C_z^2\left(1-C_{xz}\right)^2 > 0.$

This shows that the proposed estimator t_2^* is more efficient than the double sampling exponential chain product-type estimator $\left(\bar{Y}_{Pe}^{dc}\right)$.

5 Empirical Study

To examine the merits of the proposed estimators, we have considered four natural population data sets. The sources of populations, nature of the variates y, x and z; and the values of the various parameters are given as follows.

Population I -Source: Cochran (1977)

Y: Number of 'Placebo' children, X: Number of paralytic polio cases in the placebo group, Z: Number of paralytic polio cases in the 'not inoculated' group.

N=34, n=10, n_1=15, $\bar{Y}$=4.92, $\bar{X}$=2.59, $\bar{Z}$=2.91, ρ_{yx}=0.7326, ρ_{yz}=0.6430, ρ_{zx}=0.6837, C_y^2=1.0248, C_x^2=1.5175, C_z^2=1.1492.

Population II -Source: Sukhatme and Chand (1977)

Y: Apple trees of bearing age in 1964, X: Bushels of apples harvested in 1964, Z: Bushels of apples harvested in 1959.

N=200, n=20, n_1=30, $\bar{Y} = 0.103182\times10^4$, $\bar{X} = 0.293458\times10^4$, $\bar{Z} = 0.365149\times10^4$, ρ_{yx}=0.93, ρ_{yz}=0.77, ρ_{zx}=0.84, C_y^2=2.55280, C_x^2=4.02504, C_z^2=2.09379.

Population III -Source: Srivastava *et al.*(1989)

Y: The measurement of weight of children , X: Mid arm circumference of children, Z: Skull circumference of children.

N=82, n=25, n_1=43, $\bar{Y}$=5.60 kg, $\bar{X}$=11.90 cm, $\bar{Z}$=39.80 cm, ρ_{yx}=0.09, ρ_{yz}=0.12, ρ_{zx}=0.86, C_y^2=0.0107, C_x^2=0.0052, C_z^2=0.0008.

Population IV-Source: Srivastava *et al.*(1989)

Y: The measurement of weight of children , X: Mid arm circumference of children, Z: Skull circumference of children.

N=55, n=18, n_1=30, $\bar{Y}$=17.08 kg, $\bar{X}$=16.92 cm, $\bar{Z}$=50.44 cm, ρ_{yx}=0.54, ρ_{yz}=0.51, ρ_{zx}=–0.08, C_y^2=0.0161, C_x^2=0.0049, C_z^2=0.0007.

To observe the relative performance of different estimators of $\bar{Y}$, we have computed the percentage relative efficiencies of the proposed estimators (t_1^* *and* t_2^*), exponential chain ratio-type$\left(\bar{Y}_{Re}^{dc}\right)$, exponential chain product-type $\left(\bar{Y}_{Pe}^{dc}\right)$, chain ratio $\left(\bar{Y}_{R}^{dc}\right)$ and chain product $\left(\bar{Y}_{P}^{dc}\right)$ estimators in double sampling and sample mean per unit estimator $\bar{y}$ with respect to usual unbiased estimator $\bar{y}$ for Case I and Case II. The findings are presented in Table 1 and 2.

Table 1: Percentage relative efficiencies of $\bar{Y}_R^{dc}, \bar{Y}_P^{dc}, \bar{Y}_{Re}^{dc}, \bar{Y}_{Pe}^{dc}, t_1^*$ and t_2^* w.r.t. $\bar{y}$ for Case I

Estimators→	$\bar{y}$	$\bar{Y}_R^{dc}$	$\bar{Y}_P^{dc}$	$\bar{Y}_{Re}^{dc}$	$\bar{Y}_{Pe}^{dc}$	t_1^*	t_2^*
Population I	100.00	136.91	25.96	184.36	47.55	186.71	72.45
Population II	100.00	279.93	26.02	247.82	46.58	293.97	74.35
Population III	100.00	81.92	70.22	97.11	88.38	97.12	97.74
Population IV	100.00	131.91	61.01	120.57	78.75	131.12	255.53

Table 2: Percentage relative efficiencies of $\bar{Y}_R^{dc}, \bar{Y}_P^{dc}, \bar{Y}_{Re}^{dc}, \bar{Y}_{Pe}^{dc}, t_1^*$ and t_2^* w.r.t. $\bar{y}$ for Case II

Estimators→	$\bar{y}$	$\bar{Y}_R^{dc}$	$\bar{Y}_P^{dc}$	$\bar{Y}_{Re}^{dc}$	$\bar{Y}_{Pe}^{dc}$	t_1^*	t_2^*
Population I	100.00	87.63	21.24	141.68	42.15	158.73	44.31
Population II	100.00	182.67	19.16	220.59	37.90	230.35	37.95
Population III	100.00	68.82	58.68	91.06	82.82	204.74	83.55
Population IV	100.00	116.68	48.81	122.79	70.87	127.02	71.19

From the Table 1 and Table 2, it is clear that the proposed exponential chain ratio-type estimator t_1^* is more efficient than the usual unbiased estimator $\bar{y}$, chain ratio and product estimators $\left(\bar{Y}_R^{dc}\ and\ \bar{Y}_P^{dc}\right)$ in double sampling, exponential chain ratio and product-type estimators $\left(\bar{Y}_{Re}^{dc}\ and\ \bar{Y}_{Pe}^{dc}\right)$ in double sampling for both the cases except population III in Case I.

6 Proposed estimators based on the estimators t_1^* and t_2^*

Based on the estimators t_1^* and t_2^*, we propose the following exponential chain ratio and product-type estimators in double sampling respectively as

$$t_1^{**} = \bar{y}exp\left\{\frac{\left(\frac{\bar{x}_1}{\bar{x}}\right)^{\alpha}\left(\frac{\bar{U}}{\bar{u}_1}\right)-1}{\left(\frac{\bar{x}_1}{\bar{x}}\right)^{\alpha}\left(\frac{\bar{U}}{\bar{u}_1}\right)+1}\right\} \tag{30}$$

and

$$t_2^{**} = \bar{y}exp\left\{\frac{1-\left(\frac{\bar{x}_1}{\bar{x}}\right)^{\alpha}\left(\frac{\bar{U}}{\bar{u}_1}\right)}{1+\left(\frac{\bar{x}_1}{\bar{x}}\right)^{\alpha}\left(\frac{\bar{U}}{\bar{u}_1}\right)}\right\} \tag{31}$$

where α is a suitably chosen constant. Some members of these proposed estimators are given in Table 3.

7 Bias and MSE of t_1^{**} and t_2^{**} for Case I

Expanding the right hand side of equations (30) and (31) in terms of e's, multiplying out and neglecting the terms of e's of power greater than two, we have

$$\begin{aligned} t_1^{**}-\bar{Y} \cong\ & \bar{Y}\left[e_0+\frac{1}{2}\left\{\alpha e_1'-\alpha e_1-\phi e_2-\alpha e_0 e_1+\alpha e_0 e_1'-\phi e_0 e_2+\frac{\alpha(\alpha-1)}{2}e_1'^2\right\}\right. \\ & +\frac{\alpha(\alpha+1)}{4}e_1^2-\frac{1}{4}\left(\alpha^2 e_1'^2-2\alpha^2 e_1' e_1+\alpha^2 e_1^2-\phi^2 e_2^2\right) \\ & \left.+\frac{1}{8}\left(\alpha^2 e_1'^2+\alpha^2 e_1^2+\phi^2 e_2^2+2\alpha\phi e_1' e_2\right)\right] \end{aligned} \tag{32}$$

Table 3: Some existing members of proposed class of estimators t_1^{**} and t_2^{**}

Value of constants in estimator t_1^{**}			
α	a	b	Estimator
1	1	0	$\bar{Y}_{Re}^{dc}=\bar{y}exp\left(\frac{\bar{x_1}\frac{\bar{Z}}{\bar{z_1}}-\bar{x}}{\bar{x_1}\frac{\bar{Z}}{\bar{z_1}}+\bar{x}}\right)$ Singh and Choudhury (2012)
1	–	–	$t_1^{*}=\bar{y}exp\left\{\frac{\bar{x}_1\left(\frac{a\bar{Z}+b}{a\bar{z}_1+b}\right)-\bar{x}}{\bar{x}_1\left(\frac{a\bar{Z}+b}{a\bar{z}_1+b}\right)+\bar{x}}\right\}$
Value of constants in estimator t_2^{**}			
α	a	b	Estimator
1	1	0	$\bar{Y}_{Pe}^{dc}=\bar{y}\ exp\left(\frac{\bar{x}-\bar{x}_1\frac{\bar{Z}}{\bar{z}_1}}{\bar{x}+\bar{x}_1\frac{\bar{Z}}{\bar{z}_1}}\right)$ Singh and Choudhury (2012)
1	–	–	$t_2^{*}=\bar{y}exp\left\{\frac{\bar{x}-\bar{x}_1\left(\frac{a\bar{Z}+b}{a\bar{z}_1+b}\right)}{\bar{x}+\bar{x}_1\left(\frac{a\bar{Z}+b}{a\bar{z}_1+b}\right)}\right\}$

and

$$\begin{aligned} t_2^{**}-\bar{Y} \quad &\cong \quad \bar{Y}\left[e_0+\frac{1}{2}\left\{\alpha e_1-\alpha e_1'+\phi e_2+\alpha e_0e_1-\alpha e_0e_1'+\phi e_0e_2-\frac{\alpha\left(\alpha-1\right)}{2}e_1'^2\right\}\right. \\ &-\frac{\alpha\left(\alpha+1\right)}{4}e_1^2-\frac{1}{4}\left(\alpha^2e_1^2-\alpha^2e_1'^2+\phi^2e_2^2\right) \\ &\left.+\frac{1}{8}\left(\alpha^2e_1'^2+\alpha^2e_1^2+\phi^2e_2^2-2\alpha^2e_1'e_1-2\alpha\phi e_1'e_2+2\alpha\phi e_1e_2\right)\right] \end{aligned} \tag{33}$$

Therefore, the bias of the estimators t_1^{**} and t_2^{**} can be obtained by using the results of equation (3) in equations (32) and (33) as

$$B\left(t_1^{**}\right)_I=\bar{Y}\left[\frac{1}{8}\left\{\alpha\left(\alpha+1\right)\frac{1-f^*}{n}C_x^2+3\phi^2\frac{1-f_1}{n_1}C_z^2\right\}-\frac{1}{2}\left(\alpha\frac{1-f^*}{n}C_{yx}C_x^2+\phi\frac{1-f_1}{n_1}C_{yz}C_z^2\right)\right]$$

and

$$B\left(t_2^{**}\right)_I=\bar{Y}\left[\frac{1}{8}\left\{\alpha\left(\alpha-1\right)\frac{1-f^*}{n}C_x^2-\phi^2\frac{1-f_1}{n_1}C_z^2\right\}+\frac{1}{2}\left(\alpha\frac{1-f^*}{n}C_{yx}C_x^2+\phi\frac{1-f_1}{n_1}C_{yz}C_z^2\right)\right]$$

From equations (32) and (33), we have

$$t_1^{**}-\bar{Y}\cong\left\{e_0+\frac{1}{2}\left(e_1'-\alpha e_1-\phi e_2\right)\right\} \tag{34}$$

and

$$t_2^{**}-\bar{Y}\cong\left\{e_0+\frac{1}{2}\left(e_1-\alpha e_1'+\phi e_2\right)\right\} \tag{35}$$

Squaring both the sides of equations (34) and (37), taking expectations and using the results of equation (3), we get the MSE of t_1^{**} and t_2^{**} as

$$\begin{aligned} M\left(t_1^{**}\right)_I &= \bar{Y}^2\left\{\frac{1-f}{n}C_y^2+\frac{1}{4}\left(\alpha^2\frac{1-f^*}{n}C_x^2+\phi^2\frac{1-f_1}{n_1}C_z^2\right)\right. \\ &\left.-\left(\alpha\frac{1-f^*}{n}C_{yx}C_x^2+\phi\frac{1-f_1}{n_1}C_{yz}C_z^2\right)\right\} \end{aligned} \tag{36}$$

and

$$\begin{aligned} M\left(t_2^{**}\right)_I &= \bar{Y}^2\left\{\frac{1-f}{n}C_y^2+\frac{1}{4}\left(\alpha^2\frac{1-f^*}{n}C_x^2+\phi^2\frac{1-f_1}{n_1}C_z^2\right)\right. \\ &\left.+\left(\alpha\frac{1-f^*}{n}C_{yx}C_x^2+\phi\frac{1-f_1}{n_1}C_{yz}C_z^2\right)\right\} \end{aligned} \tag{37}$$

Differentiating in equation (36) with respect to α and ϕ separately, yields optimum values of α and ϕ as
$\alpha_{opt.} = C_{yx}$ and $\phi_{opt.} = C_{yz}$.

Substituting the above optimum values of $\alpha_{opt.}$ and $\phi_{opt.}$ in equation (36), we obtain the optimum MSE of the estimator t_1^{**} as

$$opt.M\left(t_1^{**}\right)_I = \bar{Y}^2\left(\frac{1-f}{n}C_y^2-\frac{1-f^*}{n}C_{yx}^2C_x^2-\frac{1-f_1}{n_1}C_{yz}^2C_z^2\right) \tag{38}$$

Differentiating in equation (37) with respect to α and ϕ separately, yields optimum values of α and ϕ as
$\alpha_{opt.} = -2C_{yx}$ and $\phi_{opt.} = -2C_{yz}$.

Substituting the above optimum values of $\alpha_{opt.}$ and $\phi_{opt.}$ in equation (37), we obtain the optimum MSE of the estimator t_2^{**} as

$$opt.M\left(t_2^{**}\right)_I = \bar{Y}^2\left(\frac{1-f}{n}C_y^2-\frac{1-f^*}{n}C_{yx}^2C_x^2-\frac{1-f_1}{n_1}C_{yz}^2C_z^2\right) \tag{39}$$

From equations (38) and (39), we have observed that the optimum MSE of the estimators t_1^{**} and t_2^{**} are same in case of their optimality.

7.1 Efficiency Comparison of the estimator t_1^{**} (or t_2^{**}) with the estimators t_1^* and t_2^*

From the equations (11) and (38) or (39), we have
$opt.M\left(t_1^{*}\right)_I - \left\{opt.M\left(t_1^{**}\right)_I \ or \ opt.M\left(t_2^{**}\right)_I\right\} = \bar{Y}^2\frac{1-f^*}{n}C_x^2\left(\frac{1}{2}-C_{yx}\right)^2 > 0.$

From the equations (13) and (38) or (39), we have

$opt.M\left(t_2^*\right)_I - \left\{opt.M\left(t_1^{**}\right)_I \text{ or } opt.M\left(t_2^{**}\right)_I\right\} = \bar{Y}^2 \frac{1-f^*}{n} C_x^2 \left(\frac{1}{2} + C_{yx}\right)^2 > 0.$

From the above expressions, it is clear that the estimator t_1^{**} (or t_2^{**}) is more efficient than the estimators t_1^* and t_2^* in case of its optimality.

8 Bias and MSE of t_1^{**} and t_2^{**} for Case II

Taking expectations in equations (32) and (33) and using the results from equation (19), we get the bias of the estimators t_1^{**} and t_2^{**} respectively as

$$\begin{aligned} B\left(t_1^{**}\right)_{II} &= \bar{Y}\left[\frac{1}{8}\alpha\left\{(\alpha+2)\frac{1-f}{n} + (\alpha-2)\frac{1-f_1}{n_1}\right\}C_x^2 + \frac{3}{8}\phi^2\frac{1-f_1}{n_1}C_z^2 \right. \\ &\quad \left. -\alpha\left(\frac{1}{2}\frac{1-f}{n}C_{yx}C_x^2 + \frac{1}{4}\phi C_{xz}C_z^2\right)\right] \end{aligned}$$

and

$$\begin{aligned} B\left(t_2^{**}\right)_{II} &= \bar{Y}\left[\frac{1}{8}\alpha\left\{(\alpha-2)\frac{1-f}{n} + (\alpha+2)\frac{1-f_1}{n_1}\right\}C_x^2 - \frac{1}{8}\phi^2\frac{1-f_1}{n_1}C_z^2 \right. \\ &\quad \left. +\alpha\left(\frac{1}{2}\frac{1-f}{n}C_{yx}C_x^2 - \frac{1}{4}\phi C_{xz}C_z^2\right)\right] \end{aligned}$$

Squaring both the sides of equations (34) and (35), taking expectations and using the results from equation (19), we get the MSE of t_1^{**} and t_2^{**} respectively as

$$\begin{aligned} M\left(t_1^{**}\right)_{II} &= \bar{Y}^2\left\{\frac{1-f}{n}C_y^2 + \frac{1}{4}\left(\alpha^2 f^{**}C_x^2 + \phi^2\frac{1-f_1}{n_1}C_z^2\right) \right. \\ &\quad \left. -\alpha\left(\frac{1-f}{n}C_{yx}C_x^2 + \phi\frac{1}{2}\frac{1-f_1}{n_1}C_{xz}C_z^2\right)\right\} \end{aligned} \tag{40}$$

and

$$\begin{aligned} M\left(t_2^{**}\right)_{II} &= \bar{Y}^2\left\{\frac{1-f}{n}C_y^2 + \frac{1}{4}\left(\alpha^2 f^{**}C_x^2 + \phi^2\frac{1-f_1}{n_1}C_z^2\right) \right. \\ &\quad \left. +\alpha\left(\frac{1-f}{n}C_{yx}C_x^2 - \phi\frac{1}{2}\frac{1-f_1}{n_1}C_{xz}C_z^2\right)\right\} \end{aligned} \tag{41}$$

Differentiation in equation (40) with respect to α and ϕ separately, yields optimum values of α and ϕ as
$\alpha_{opt.} = \frac{C}{A+BC_{xz}}$ and $\phi_{opt.} = \frac{CC_{xz}}{A+BC_{xz}}$,

where $A = f^{**}C_x$, $B = -\frac{1-f_1}{n_1}\rho_{xz}C_z$ and $C = 2\frac{1-f}{n}\rho_{yx}C_y$.

Substituting the above optimum values of $\alpha_{opt.}$ and $\phi_{opt.}$ in equation (40), we obtain the optimum MSE of the estimator t_1^{**} as

$$opt.M\left(t_1^{**}\right)_{II} = \bar{Y}^2 \frac{1-f}{n} C_y^2 \left\{ 1 - \frac{\frac{1-f}{n} C_{yx}^2 C_x^2}{\frac{1-f}{n} + \frac{1-f_1}{n_1} \left(\frac{C_{xz} C_z}{C_x} \right)^2} \right\} \tag{42}$$

Differentiation in equation (41) with respect to α and ϕ separately, yields optimum values of α and ϕ as
$\alpha_{opt.} = \frac{C}{A+BC_{xz}}$ and $\phi_{opt.} = \frac{CC_{xz}}{A+BC_{xz}}$.

Substituting the above optimum values of $\alpha_{opt.}$ and $\phi_{opt.}$ in equation (41), we obtain the optimum MSE of the estimator t_2^{**} as

$$opt.M\left(t_2^{**}\right)_{II} = \bar{Y}^2 \frac{1-f}{n} C_y^2 \left\{ 1 - \frac{\frac{1-f}{n} C_{yx}^2 C_x^2}{\frac{1-f}{n} + \frac{1-f_1}{n_1} \left(\frac{C_{xz} C_z}{C_x} \right)^2} \right\} \tag{43}$$

From equations (42) and (43), we observe that the optimum MSE of the estimators t_1^{**} and t_2^{**} are same in case of their optimality.

8.1 Efficiency Comparison of the estimator t_1^{**} (or t_2^{**}) with the estimators t_1^* and t_2^*

From the equations (23) and (42) or (43), we have
$opt.M\left(t_1^*\right)_{II} - \left\{opt.M\left(t_1^{**}\right)_{II} \ or \ opt.M\left(t_2^{**}\right)_{II}\right\} =$

$$\bar{Y}^2 \left\{ \frac{1}{4} f^{**} C_x^2 + \frac{\frac{1-f}{n} \frac{C_{yx}^2 C_x^2}{C_y^2}}{\frac{1-f}{n} + \frac{1-f_1}{n_1} \frac{C_{xz}^2 C_z^2}{C_x^2}} - \frac{1}{4} \frac{1-f_1}{n_1} C_{xz}^2 C_z^2 - \frac{1-f}{n} C_{yx} C_x^2 \right\} > 0 \text{ if}$$

$$\frac{1}{4} f^{**} C_x^2 + \frac{\frac{1-f}{n} \frac{C_{yx}^2 C_x^2}{C_y^2}}{\frac{1-f}{n} + \frac{1-f_1}{n_1} \frac{C_{xz}^2 C_z^2}{C_x^2}} > \frac{1}{4} \frac{1-f_1}{n_1} C_{xz}^2 C_z^2 + \frac{1-f}{n} C_{yx} C_x^2.$$

From the equations (25) and (42) or (43), we have
$opt.M\left(t_2^*\right)_{II} - \left\{opt.M\left(t_1^{**}\right)_{II} \ or \ opt.M\left(t_2^{**}\right)_{II}\right\} =$

$$\bar{Y}^2 \left\{ \frac{1}{4} f^{**} C_x^2 + \frac{\frac{1-f}{n} \frac{C_{yx}^2 C_x^2}{C_y^2}}{\frac{1-f}{n} + \frac{1-f_1}{n_1} \frac{C_{xz}^2 C_z^2}{C_x^2}} - \frac{1}{4} \frac{1-f_1}{n_1} C_{xz}^2 C_z^2 + \frac{1-f}{n} C_{yx} C_x^2 \right\} > 0 \text{ if}$$

$$\frac{1}{4} f^{**} C_x^2 + \frac{\frac{1-f}{n} \frac{C_{yx}^2 C_x^2}{C_y^2}}{\frac{1-f}{n} + \frac{1-f_1}{n_1} \frac{C_{xz}^2 C_z^2}{C_x^2}} > \frac{1}{4} \frac{1-f_1}{n_1} C_{xz}^2 C_z^2 - \frac{1-f}{n} C_{yx} C_x^2.$$

9 Empirical Study

To observe the relative performances of different estimators of $\bar{Y}$, we have computed the percentage relative efficiencies of the proposed estimators t_1^{**} and t_2^{**} with respect to $\bar{y}$ by using the population data sets given in Section 5.

Table 4: PREs of estimators t_1^*, t_2^*, t_1^{**} and t_2^{**} w.r.t. $\bar{y}$

Estimators→	t_1^*	t_1^{**}	t_2^*	t_2^{**}
Case I				
Population I	186.71	189.27	72.45	189.27
Population II	293.97	326.41	74.35	326.41
Population III	97.12	101.07	97.74	101.07
Population IV	131.12	138.66	255.53	138.66
Case II				
Population I	158.73	172.10	44.31	172.10
Population II	230.35	369.89	37.95	369.89
Population III	204.74	100.73	83.55	100.73
Population IV	127.02	126.24	71.19	126.24

10 Conclusions

From Table 1 and Table 2, it is evident that the proposed exponential chain ratio and product-type estimators (t_1^* and t_2^*) have shown their gain in efficiencies over the estimators proposed by Singh and Choudhury (2012). From Table 4, it is also clear that the further proposed estimators t_1^{**} and t_2^{**} are more efficient than the estimators t_1^* and t_2^* in about all population data sets. So the use of the proposed estimators t_1^*, t_2^* and hence t_1^{**}, t_2^{**} are preferable in practice over other estimators taken into considerations.

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References

[1] Bahl, S. and Tuteja, R. K. (1991). Ratio and Product type exponential estimator, *Information and Optimization Sciences*, **12**, 159–163.

[2] Chand, L. (1975). *Some ratio type estimator based on two or more auxiliary variables*, Unpublished Ph. D. dissertation, Lowa State University, Ames, Lowa.

[3] Cochran, W. G. (1940). The estimation of the yields of the cereal experiments by sampling for the ratio of grain to total produce, *The Journal of Agricultural Science*, **30(2)**, 262–275.

[4] Cochran, W. G. (1977). *Sampling techniques*, New-York: John Wiley and Sons.

[5] Isaki, C. T. (1983). Variance estimation using auxiliary information, *Journal of American Statistical Association*, **78**, 117–123.

[6] Kiregyera, B. (1980). A chain ratio-type estimator in finite population in double sampling using two auxiliary variables, *Metrika*, **27**, 217–223.

[7] Kiregyera, B. (1984). Regression-type estimators using two auxiliary variables and the model of double sampling from finite populations, *Metrika*, **31(3–4)**, 215–226.

[8] Murthy, M. N. (1964). Product method of estimation, *Sankhyā*, **26**, 69-74.

[9] Pradhan, B. K. (2005). A Chain Regression Estimator in Two Phase Sampling Using Multiauxiliary Information, *Bulletin of the Malaysian Mathematical Sciences Society (2)*, **28(1)**, 81–86.

[10] Prasad, B., Singh, R.S. and Singh, H.P. (2002). Modified chain ratio estimators for finite population mean using two auxiliary variables in double sampling, *Statistics in Transition*, **5(6)**, 1051–1066.

[11] Robson, D. S. (1957). Applications of multivariate polykays to the theory of unbiased ratio-type estimation, *Journal of the American Statistical Association*, **52**, 511–522.

[12] Singh A.K. and Singh, H.P. (2001). Dual to chain ratio type estimator in double sampling using two auxiliary variables. *Journal of Ravishankar University*, **14B (Science)**, 99 -106.

[13] Singh, A.K., Singh, H.P. and Upadhyaya, L.N. (2001). A generalized chain estimator for finite population mean in two-phase sampling, *Journal of the Indian Society of Agricultural Statistics*, **54(3)**, 370–375.

[14] Singh, B.K. and Choudhury, S. (2012). Exponential chain ratio and product type estimators for finite population mean under double sampling scheme, *Global Journal of Science Frontier Research Mathematics and Decision Sciences*, **12(6)**, 13–24.

[15] Singh, H.P. (1986). On the estimation of mean using supplementary information on two auxiliary characters in double sampling, *Current Science*, **55(6)**, 302–304.

[16] Singh, H.P. (1993). A chain ratio-cum-difference estimator using two auxiliary variates in double sampling, *Journal of Ravishankar University*, **6, B**, 79–83.

[17] Singh, H.P. and Ruiz-Espejo, M. (2000). An improved class of chain regression estimators in two-phase sampling, *Statistics and Decisions*, **18**, 205–218.

[18] Singh, H.P. and Vishwakarma, G.K. (2007). Modified exponential ratio and product estimators for finite population mean in double sampling, *Austrian Journal of Statistics*, **36(3)**, 217–225.

[19] Singh, V. P. and Singh, H. P. (1997-1998). Chain estimators for population ratio in double sampling, *Aligarh Journal of Statistics*, **17**, 85–100.

[20] Singh, V.K., Singh, Hari P., Singh, H. P. and Shukla, D.K. (1994). A general class of chain estimators for ratio and product of two means of a finite population, *Communications in Statistics-Theory and Methods*, **23(5)**, 1341–1355.

[21] Srivnstava, Rani S., Srivastava, S.P., and Khare, B.B. (1989). Chain ratio type estimator for ratio of two population means using auxiliary characters, *Communications in Statistics Theory and Methods*, **18(10)**, 3917–3926.

[22] Sukhatme, B.V. and Chand, L. (1977). Multivariate ratio-type estimators, *Proceedings of American Statistical Association, Social Statistics Section*, 927–931.

[23] Swain A.K.P.C. (1970). A note on the use of multiple auxiliary variables in sample surveys, *Trabajos de Estadistica y de Investigacion Operative*, **21(3)**, 135–141.

[24] Tracy, D.S. and Singh, H.P. (1999). A general class of chain regression estimators in two-phase sampling, *Journal of Applied Statistical Science*, **8(4)**, 205–216.

[25] Upadhyaya, L.N., Dubey, S.P. and Singh, H.P. (1992). A class of ratio-in-regression estimators using two auxiliary variables in double sampling, *The Journal of Scientific Research*, **42**, 127–134.

[26] Upadhyaya, L.N., Kushwaha, K.S. and Singh, H.P. (1990). A modified chain ratio-type estimator in two-phase sampling using multi auxiliary information, *Metron*, **48(1–4)**, 381-393.