International Journal of Statistical Sciences *ISSN 1683–5603*
Vol. 11(Special Issue), 2011, pp 135-144

An Extended Inverse Gaussian Model

Serge B. Provost and Iman Mabrouk
Department of Statistical and Actuarial Sciences
University of Western Ontario
London, Ontario, Canada, N6A 5B7

[Received October 13, 2010; Revised February 19, 2011; Accepted March 8, 2011]

Abstract

The generalized inverse Gaussian distribution which was popularized in the late seventies by Ole Barndorff-Neilsen is extended in this paper by incorporating an additional parameter in its density function. A reduced version of the resulting model is also being considered. The effects of the parameters are described for both distributions. Additionally, several of their statistical functions are provided. Data sets pertaining to maximum flood levels and snowfall precipitations were fitted with several statistical models for comparison purposes.

Keywords and Phrases: Inverse Gaussian distribution, Generalized exponential models, Moments, Density estimation, Goodness-of-fit.

AMS Classification: Primary 62E15; Secondary 62G07.

1 Introduction

The inverse Gaussian distribution with positive parameters μ and λ (also known as Wald's distribution) has density function

$$f(x) = \frac{\sqrt{\lambda}}{\sqrt{2\pi\, x^3}}\; e^{-\lambda(x-\mu)^2/(2x\,\mu^2)}\; \mathcal{I}_{\Re^+}(x)\,, \tag{1}$$

where $\mathcal{I}_{\mathcal{B}}(x)$ denotes the indicator function of the set $\mathcal{B}$, $\Re^+$ being the set of positive real numbers. This distribution has numerous applications in various fields of scientific investigation. For instance, Seshadri (1999) points out applications in connection with actuarial models, electrical networks, life testing, hydrology, demography, physiology, meteorology, small area estimation, traffic noise intensity, remote sensing, and market research.

As explained by Dugué (1941) and Seshadri (1997), a generalization called the Generalized Inverse Gaussian ($\mathcal{GIG}$) distribution was originally proposed by Etienne Halphen. This distribution was later popularized by Barndorff-Neilsen (1977) and Jørgensen (1982). Its density function is given by

$$f(x) = \frac{(\phi/\theta)^{\frac{\lambda}{2}}}{2\,\mathcal{K}_\lambda(\sqrt{\theta\,\phi})}\, x^{\lambda-1}\, e^{-\frac{1}{2}(\theta\, x^{-1}+\phi\, x)}\, \mathcal{I}_{\Re^+}(x), \quad \phi>0,\, \theta>0,\, \lambda\in\Re, \tag{2}$$

where $\mathcal{K}_\lambda(\cdot)$ is a modified Bessel function of the second type that has the following integral representation:

$$\mathcal{K}_\lambda(\eta) = \frac{1}{2}\int_0^\infty x^{\lambda-1} e^{\frac{1}{2}\,\eta\,(x+x^{-1})}\mathrm{d}x. \tag{3}$$

Incidentally, $\mathcal{K}_\lambda(\cdot)$ is a built-in function in the symbolic computing package *Mathematica*. As explained in Abramowitz and Stegun (1972), the modified Bessel functions of the first and second types, namely $I_\lambda(w)$ and $\mathcal{K}_\lambda(w)$, are the two linearly independent solutions of the differential equation $w^2\,\frac{\mathrm{d}^2 y}{\mathrm{d}w^2} + w\,\frac{\mathrm{d}y}{\mathrm{d}w} - (w^2+\lambda^2)\,y = 0$.

We are proposing an extension of the $\mathcal{GIG}$ distribution that will be referred as to the Extended Inverse Gaussian ($\mathcal{EIG}$) distribution. Its density function is given by

$$f_E(x) = \frac{\delta\ (\nu/\tau)^{\frac{\delta+\xi+1}{2\delta}}\, x^{\delta+\xi}\, e^{-\tau x^{-\delta}-\nu x^{\delta}}}{2\,\mathcal{K}_{\frac{\delta+\xi+1}{\delta}}\,(2\sqrt{\nu\tau})}\, \mathcal{I}_{\Re^+}(x) \tag{4}$$

where $\xi\in\Re$, $\nu>0$, $\tau>0$ and $\delta>0$. By introducing a single additional parameter, we aim to obtain a more flexible modeling distribution while keeping the resulting model relatively parsimonious. A location parameter could also be introduced in (4) for modeling purposes. Note that the $\mathcal{GIG}$ density function can be obtained from (4) by making the following substitutions: $\delta=1,\ \tau=\theta/2,\ \nu=\phi/2$ and $\xi=\lambda-2$.

A reduced model called the Reduced Extended Inverse Gaussian ($\mathcal{REIG}$) distribution, is obtained by omitting $e^{-\nu\, x^{\delta}}$ (or equivalently letting $\nu=0$) in the density function (4), which gives

$$f_R(x) = \frac{\delta\,\tau^{-\frac{\xi+\delta+1}{\delta}}}{\Gamma\big(-\frac{\xi+\delta+1}{\delta}\big)}\, x^{\xi+\delta}\, e^{-\tau\, x^{-\delta}}\, \mathcal{I}_{\Re^+}(x)\,, \quad \xi\in\Re,\, \nu>0,\, \tau>0, \quad \delta>0, \tag{5}$$

provided that $1+\delta+\xi<0$.

Another reduced version of the $\mathcal{EIG}$ model is obtained by omitting $e^{-\tau\, x^{-\delta}}$ (or equivalently letting $\tau=0$) in the density function (4), which yields

$$g(x) = \frac{\delta\,\nu^{\frac{\delta+\xi+1}{\delta}}}{\Gamma\big(\frac{\delta+\xi+1}{\delta}\big)}\, x^{\xi+\delta} e^{-\nu\, x^{\delta}}\, \mathcal{I}_{\Re^+}(x), \tag{6}$$

where $\delta + \xi > -1$. This density function is in fact a Reparameterized Generalized Gamma ($\mathcal{RGG}$) density function, which is obtained by letting $\beta = \delta,\ \theta = \nu^{-1/\beta}$ and $k = \frac{\delta+\xi+1}{\delta}$ in the generalized gamma density,

$$g_1(x) = \frac{\beta}{\theta^{k\,\beta}\,\Gamma(k)}\,x^{k\,\beta\,-\,1}e^{-(\frac{x}{\theta})^{\beta}}\,\mathcal{I}_{\Re^+}(x)\,. \tag{7}$$

For specific distributional results in connection with the generalized gamma distribution, the reader is referred to Johnson *et al.* (1994).

The parameter effects on the $\mathcal{EIG}$ and $\mathcal{REIG}$ distributions are described in Section 2. Moment expressions as well as other statistical functions are included in Section 3 for both the $\mathcal{EIG}$ and $\mathcal{REIG}$ distributions. Two data sets are fitted with the proposed models in Section 4. The Anderson-Darling and the Cramér-von Mises statistics are employed as measures of discrepancy. Even though the extended inverse Gaussian distribution relies on one or two additional parameters when compared to other models, it ought to be given due consideration when the objective is to obtain the best possible fit with respect to an empirical cumulative distribution function.

2 Parameter Effects

This section illustrates graphically how the extended generalized inverse Gaussian model and its reduced version are affected by their parameters.

2.1 The Extended Inverse Gaussian ($\mathcal{EIG}$) Model

Figures 1-3 indicate that the parameters ξ, δ and ν somewhat affect the shape of the $\mathcal{EIG}$ model while ξ and τ have a noticeable shifting effect on the distribution. Moreover, the parameters ν and τ in the density expression (4) are clearly scale parameters.

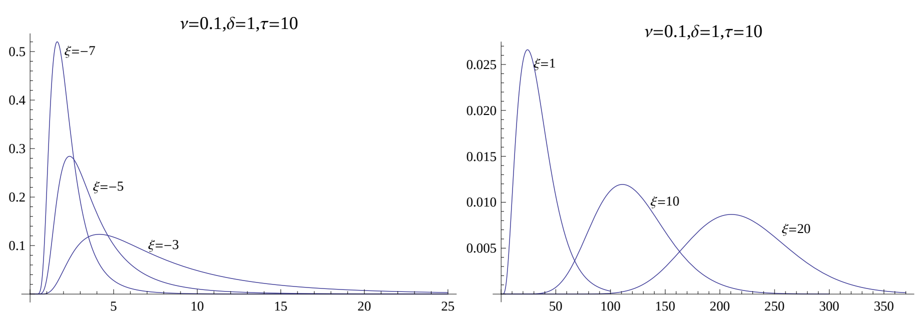

Figure 1: Effect of ξ on the $\mathcal{EIG}$ distribution.

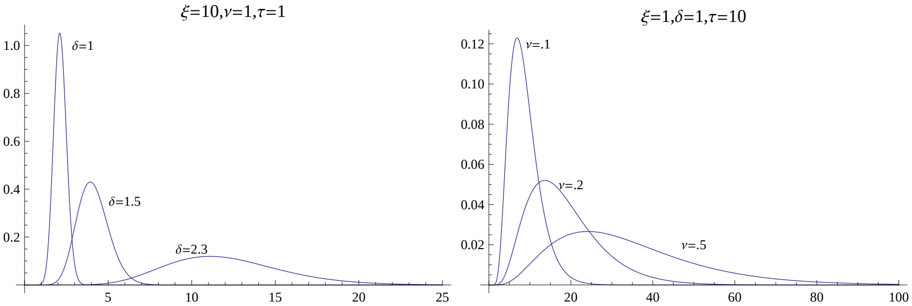

Figure 2: Effects of δ (left panel) and ν (right panel) on the $\mathcal{EIG}$ model.

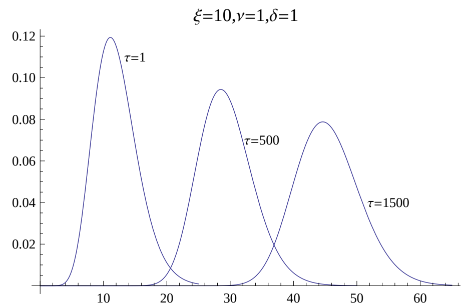

Figure 3: Effect of τ on the $\mathcal{EIG}$ distribution.

2.2 The Reduced Extended Inverse Gaussian ($\mathcal{REIG}$) Model

Figure 4 suggests that the parameter δ acts somewhat as a shifting parameter while ξ affects the shape of the $\mathcal{REIG}$ distribution. The scale parameter τ acts as a shifting parameter as it did for the $\mathcal{EIG}$ model.

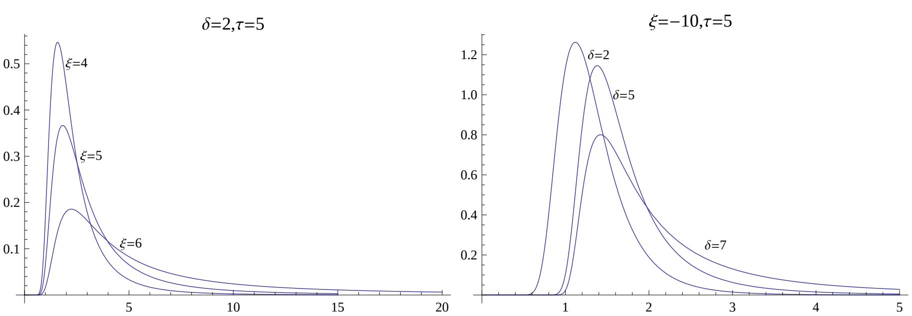

Figure 4: Effects of ξ (left panel) and δ (right panel) on the $\mathcal{REIG}$ model.

3 Certain Statistical Functions

Some statistical functions are provided in the next two subsections in connection with the $\mathcal{EIG}$ and $\mathcal{REIG}$ models.

3.1 The Extended Inverse Gaussian ($\mathcal{EIG}$) Model

Let X be an $\mathcal{EIG}$ random variable. Then,
(i) its h^{th} moment is

$$E(X^h) = \frac{\nu^{-\frac{h}{\delta}}(\nu\tau)^{\frac{h}{2\delta}}\mathcal{K}_{\frac{h+\delta+\xi+1}{\delta}}\,(2\sqrt{\nu\tau})}{\mathcal{K}_{\frac{\delta+\xi+1}{\delta}}\,(2\sqrt{\nu\tau})}\ ; \tag{8}$$

(ii) its expectation, $E(X)$, is as given above for $h = 1$.
(iii) its variance, $E(X^2) - (E(X))^2$, can be directly obtained from (8).
(iv) its skewness is given by $(E(X^3) - 3E(X^2)\,\mu + 2\,\mu^3)/\sigma^3$;
(v) its kurtosis is given by $(E(X^4) - 4E(X^3)\,\mu + 6E(X^2)\,\mu^2 - 3\,\mu^4)/\sigma^4 - 3$;
(vi) its mode is

$$2^{-1/\delta}\left(\frac{\delta+\xi+\sqrt{4\delta^2\nu\tau+\delta^2+2\delta\xi+\xi^2}}{\delta\,\nu}\right)^{\frac{1}{\delta}}.$$

3.2 The Reduced Extended Inverse Gaussian ($\mathcal{REIG}$) Model

Let X be a $\mathcal{REIG}$ random variable. Then,
(i) its h^{th} moment is

$$E(X^h) = \frac{\tau^{k/\delta}\,\Gamma\big(-\frac{k+\xi+\delta+1}{\delta}\big)}{\Gamma\big(-\frac{\xi+\delta+1}{\delta}\big)}\ ; \tag{9}$$

(ii) its expectation, $E(X)$, is as given above for $h = 1$.
(iii) its variance, $E(X^2) - (E(X))^2$, can be directly obtained from (9).
(iv) its skewness is given by $(E(X^3) - 3E(X^2)\,\mu + 2\,\mu^3)/\sigma^3$;
(v) its kurtosis is given by $(E(X^4) - 4E(X^3)\,\mu + 6E(X^2)\,\mu^2 - 3\,\mu^4)/\sigma^4 - 3$;
(vi) its mode is

$$e^{-\frac{i\pi}{\delta}}\,\delta^{\frac{1}{\delta}}\,(\xi+\delta)^{-1/\delta}\,\tau^{\frac{1}{\delta}}\ ;$$

(vii) its cumulative distribution function (CDF) is

$$F_R(y) = \frac{y^{\xi+1}\,(y^{-\rho})^{\frac{\xi+1}{\rho}}\,\Gamma\left(-\frac{\xi+\rho+1}{\rho}, y^{-\rho}\tau\right)}{\Gamma\left(-\frac{\xi+\rho+1}{\rho}\right)},$$

where $\Gamma(\alpha, \beta)$ denotes the incomplete gamma function.

4 Numerical Examples

In order to assess the adequacy of a statistical model with respect to a given data set, we make use of the following goodness-of-fit statistics:

(i) The Anderson-Darling statistic given by

$$A_0^2 = \Big(1+\frac{0.2}{\sqrt{n}}\Big)\Big(-n-\frac{1}{n}\sum_{i=1}^{n}(2\,i-1)\log(z_i(1-z_{n+1-i}))\Big), \tag{10}$$

where z_i is equal to the cumulative distribution function of the distribution under consideration evaluated at the point $x_i,\ i=1,\dots,n$, with the x_i's being the *ordered* observations from a sample of size n and the model parameters having been replaced by their respective maximum likelihood estimates;

(ii) The Cramér-von Mises statistic, that is,

$$W_0^2 = \Big(1+\frac{0.2}{\sqrt{n}}\Big)\Big(\frac{1}{12\,n}+\sum_{i=1}^{n}\Big(z_i-\frac{2\,i-1}{2\,n}\Big)^2\Big), \tag{11}$$

where the z_i's are as defined above. The smaller these statistics are, the better the fit. Related considerations are discussed for instance in D'Agostino and Stephens (1986).

The following density functions, all related to the $\mathcal{EIG}$ model, will be considered. The gamma density function which is given by

$$f(x) = \frac{x^{\theta-1}e^{-x/\phi}}{\phi^{\theta}\,\Gamma(\theta)}\,\mathcal{I}_{\Re^+}(x), \quad \theta>0,\,\phi>0\,, \tag{12}$$

is clearly a particular case of the $\mathcal{EIG}$ density as specified by (4) with $\xi=\theta-2,\ \delta=1$, $\nu=1/\phi$ and $\tau=0$. On letting $\xi=-5/2,\ \delta=1,\ \nu=\lambda/(2\mu^2)$ and $\tau=\lambda/2$ in (4), the inverse Gaussian distribution with parameters $\mu\in\Re$ and $\lambda>0$ whose density is given by (1), is also seen to be a special case of the $\mathcal{EIG}$ distribution. The reparameterized generalized gamma ($\mathcal{RGG}$) density as given in (6) can be obtained from the $\mathcal{EIG}$ model by letting $\tau=0$ in (4). The $\mathcal{EIG}$ density reduces to the Weibull density function,

$$f(x) = \theta\,\phi\,x^{\phi-1}e^{-\theta\,x^{\phi}}\,\mathcal{I}_{\Re^+}(x)\,, \quad \theta>0,\,\phi>0\,, \tag{13}$$

with the substitutions, $\delta=\phi,\ \tau=0\ \nu=\theta$ and $\xi=-1$ in (4). The relationship between the $\mathcal{GIG}$ density, as given in (2), and the $\mathcal{EIG}$ density function is specified in Section 1. Finally, the $\mathcal{REIG}$ model as defined by the density (5) is obtained by letting $\nu=0$ in (4). Two data sets were fitted with each one of these models as well as the lognormal distribution, and the resulting parameter estimates and goodness-of-fit statistics were tabulated. Several of the fitted cumulative distribution functions are graphically displayed along the empirical cumulative distribution functions for comparison purposes.

4.1 Maximum Flood Level Data

Consider the data set presented in Table 1. This data which was studied by Dumonceaux and Antle (1973), consists of maximum flood levels (in millions cubic of feet per second) of the Susquehanna River at Harrisburg, Pennsylvania, observed over 20 four-year periods.

Table 1: Maximum Flood Level Data

.654	.613	.402	.379	.269	.740	.416	.338	.315	.449
.297	.423	.379	.3235	.418	.412	.494	.392	.484	.265

This data was fitted to several distributions including those specified by (4) and (5). We made use of the symbolic computing package *Mathematica* in conjunction with the command *NMaximize* applied to the loglikelihoods to estimate the parameters. This command always attempts to find a global maximum subject to certain constraints. In this case, such constraints are specified by inequalities that certain functions of the parameters should satisfy and intervals within which the parameters can vary. The determination of such intervals was guided by the parameter estimates obtained for the reduced models. The results are presented in Table 4.2. For comparison purposes, the lognormal model whose parameters estimates were found to be $\hat{\mu} = -0.8978$ and $\sigma = 0.2692$, was also considered. It can be seen that the proposed $\mathcal{EIG}$ model and its reduced version provide a better fit than that resulting from the other models. Figures 5 and 6 show the cumulative distribution functions of the lognormal, $\mathcal{RGG}$, $\mathcal{REIG}$ and $\mathcal{EIG}$ models superimposed on the empirical cumulative distribution function. Admittedly, the $\mathcal{EIG}$ and $\mathcal{REIG}$ models fit the data nearly equally well in this case. However, it should be noted that the sample size is minute and that only scant data is available in the tails of the distribution, which apparently precludes taking full advantage of the additional parameter in this instance.

Table 2: Parameter Estimates and A_0^2 & W_0^2 for the Flood Data

	$\hat{\xi}$	$\hat{\delta}$	$\hat{\nu}$	$\hat{\tau}$	A_0^2	W_0^2
Weibull	-1	3.5260	14.450	0	.8213	0.1400
Gamma	11	1	30.769	0	0.4433	0.0712
Inverse Gaussian	-2.5	1	15.745	2.8195	0.3514	0.0558
Lognormal($-0.8978, .2692$)					0.3470	0.0540
$\mathcal{RGG}$	339.113	0.0364	9600	0	0.3390	0.0560
$\mathcal{GIG}$	-16.567	1	0.005	5.7343	0.2861	0.0449
$\mathcal{REIG}$	-10	2.3	0	0.3108	0.2567	0.0436
$\mathcal{EIG}$	-9.95	2.24	0.09	0.34	0.2551	0.0437

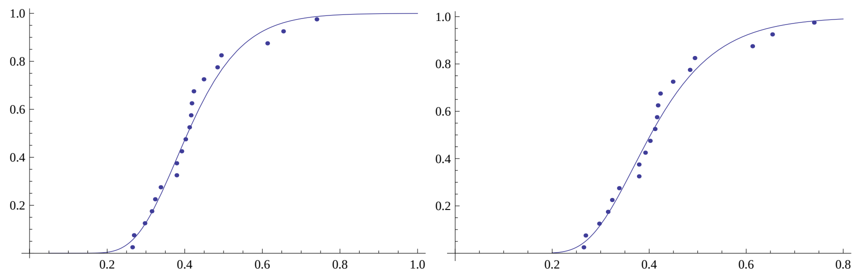

Figure 5: CDF (solid line) and empirical CDF (dots) for the flood data set. Left panel: Lognormal; Right panel: $\mathcal{GIG}$.

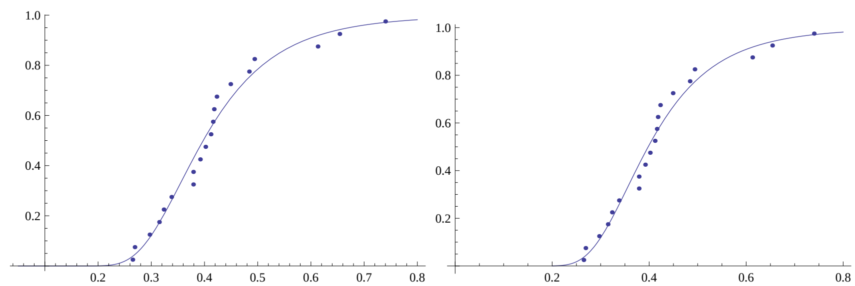

Figure 6: CDF (solid line) and empirical CDF (dots) for the flood data set. Left panel: $\mathcal{EIG}$; Right panel: $\mathcal{REIG}$.

4.2 Snowfall Precipitations in Buffalo

The same models are now fitted to the Buffalo snowfall data set, as given in Table 3 (and available for instance from the S-PLUS data library). This set comprises a record of the annual snowfall precipitations in centimeters over 63 consecutive years in the city of Buffalo. It can be seen from Table 4 that the $\mathcal{EIG}$ distribution provides the best fit. In this case, the goodness-of-fit measures indicate that a close fit can also be obtained by making use of the $\mathcal{RGG}$ distribution. This is corroborated by the graphs of the cumulative distribution functions superimposed on the empirical cumulative distribution function (Figures 7 and 8). Again, the lognormal was considered as an alternative model. In this case, the $\mathcal{EIG}$ model clearly produces a superior fit as compared to the $\mathcal{REIG}$ model.

Table 3: The Snowfall Precipitation Data

25	39.8	39.9	40.1	46.7	49.1	49.6	51.2	51.6	53.5	54.7
55.5	55.9	58	60.3	63.6	65.4	66.1	69.3	70.9	71.4	71.5
71.8	72.9	74.4	76.2	77.8	78.2	78.4	79	79.3	79.6	80.7
82.4	82.4	83	83.6	83.6	84.8	85.5	87.4	88.7	89.6	89.8
89.9	90.9	97.	98.3	101.4	102.4	103.9	104.5	105.2	110	110.5
110.5	113.7	114.5	115.6	120.5	120.7	124.7	126.4			

Table 4: Parameter Estimates and A_0^2 & W_0^2 for the Snowfall Data

	$\hat{\xi}$	$\hat{\delta}$	$\hat{\nu}$	$\hat{\tau}$	A_0^2	W_0^2
Inverse Gaussian	-2.5	1	0.0548	353.261	0.8676	0.1504
Lognormal(-4.3368, .3270)					0.7752	0.1284
$\mathcal{REIG}$	-53.66	0.1671	0	650.2	0.7417	0.0886
Gamma	8	1	0.124536	0	0.4840	0.0792
$\mathcal{GIG}$	7.97	1	0.1219	0.0025	0.4291	0.0532
Weibull	-1	3.8338	3.37×10^{-8}	0	0.2964	0.0454
$\mathcal{RGG}$	1.2889	3.629	9.31×10^{-8}	0	0.2817	0.0428
$\mathcal{EIG}$	0.3268	3	2.22×10^{-6}	0.029	0.2698	0.0407

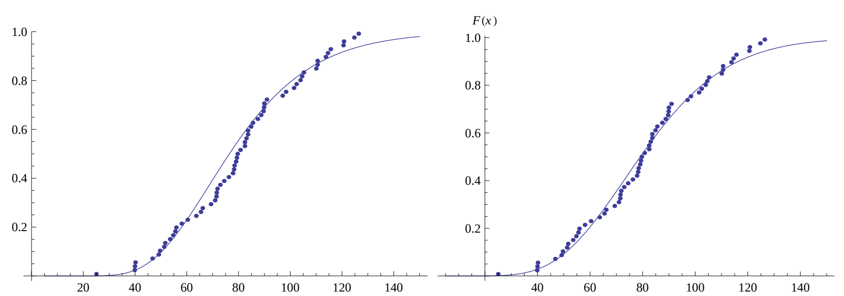

Figure 7: CDF (solid line) and empirical CDF (dots) for the snowfall data set. Left panel: Lognormal; Right panel: $\mathcal{GIG}$.

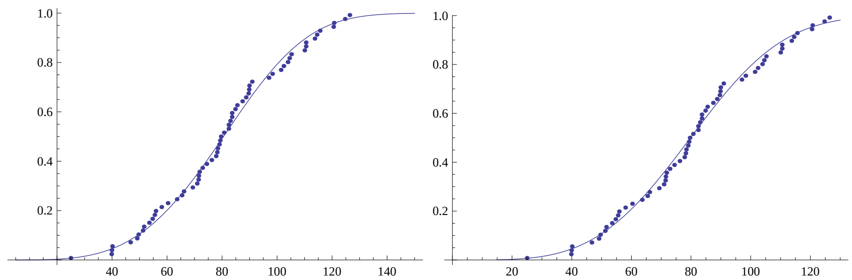

Figure 8: CDF (solid line) and empirical CDF (dots) for the snowfall data set. Left panel: $\mathcal{RGG}$; Right panel: $\mathcal{EIG}$.

References

[1] Abramowitz, M. and Stegun, I. A. (1972). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. Ed. 9. Dover, New York.

[2] Barndorff-Nielsen O. E. (1977). Exponentially decreasing distributions for the logarithm of the particle size. *Proceedings of the Royal Society, London, Series A, Mathematical and Physical Sciences*, **353**, 401–419.

[3] D'Agostino, R. B. and Stephens, M. A. (1986). *Goodness-of-fit Techniques*. CRC Press, Boca Raton.

[4] Dugué, D. (1941). Sur un nouveau type the courbe de fréquence. *Comptes Rendus de l'Académie des Sciences*, **213**, 634–635.

[5] Dumonceaux, R. and Antle, C. (1973). *Discrimination between the Lognormal and the Weibull distributions*. Technometrics, **15**(4), 923–926.

[6] Johnson, N. L., Kotz, S. and Balakrishnan, N. (1994). *Distributions in Statistics – Continuous Univariate Distributions*. Volume 1. Wiley-Interscience, New York.

[7] Jørgensen, B. (1982). *Statistical Properties of the Inverse Gaussian Distribution*, Lecture Notes in Statistics. Eds: D. Brillinger, S. Fienberg, J. Gani, J. Hartigan, J. Kiefer, and K. Krickeberg. Springer-Verlag, New York.

[8] Seshadri, V. (1993). *The Inverse Gaussian Distribution: A Case Study in Exponential Families*. Oxford Science Publication, Oxford.

[9] Seshadri, V. (1997). *Halphen's Laws*. pp. 302–306 in Encyclopedia of Statistical Sciences, Update Volume 1. Eds: S. Kotz, C. B. Read, D. L. Banks. Wiley, New York.

[10] Seshadri, V. (1999). *The Inverse Gaussian Distribution: Statistical Theory and Applications*. Springer-Verlag, New York.