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Effect Size Estimation with Vague Prior Information

A. K. Md. Ehsanes Saleh

School of Mathematics and Statistics
Carleton University
Ottawa, Canada

Amal F. Ghania

School of Mathematics and Statistics
Carleton University
Ottawa, Canada

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Abstract

It is well-known that tests of hypothesis reveals whether the experimental results are significantly different than the control results. Effect size, on the other hand, gives the magnitude of the experimental effect and it is a key component of meta-analysis. In this paper, we define two new estimators, namely, the preliminary test and the shrinkage estimators of the effect size when apriori one suspects that it may be equal to zero. For the shrinkage estimator, we find that the coverage probability of the confidence interval with fixed length, have properties similar to the mean square error comparisons. On the other hand, properties of the coverage probability of the preliminary test estimator is dissimilar. Between the two estimators, the shrinkage estimator, we defined is comparable to the empirical Bayes estimator discussed by Efron (2010), and it may have potential use in the large-scale inference with micro-array data discussed in the book by Efron.

Keywords and Phrases: Test of Hypothesis, Preliminary Test Estimator, Shrinkage Estimator, Coverage Probability, Empirical Bayes Estimators.

AMS Classification: Primary 62F03; Secondary 62F10.

1 Introduction

Let $Y_1^E, \ldots, Y_{n_1}^E$ be the experimental outcomes of size n_1 from $N(\mu^E, \sigma^2)$ and $Y_1^C, \ldots, Y_{n_2}^C$ be the control outcomes of size n_2 from $N(\mu^C, \sigma^2)$. We propose two new estimators of $\delta = \frac{\mu^E - \mu^C}{\sigma}$ in addition to the well-known unbiased estimator, when one may suspect that δ may be equal to zero. Basically, the unbiased estimation of δ is complete when we have

$$\tilde{\delta}_n = \frac{\bar{Y}_{n_1}^E - \bar{Y}_{n_2}^C}{\sigma} \qquad \text{if } \sigma^2 \text{ is known,} \tag{1}$$

$$\tilde{\delta}_n^* = c(\nu)\frac{\bar{Y}_{n_1}^E - \bar{Y}_{n_2}^C}{s_p} \quad \text{if } \sigma^2 \text{ is unknown} \tag{2}$$

$$\text{where } c(\nu) = \frac{\Gamma(\nu/2)}{\sqrt{(\nu/2)}\Gamma(\frac{\nu-1}{2})}, \;\; \nu = n_1 + n_2 - 2.$$

The unbiased estimator in (2) is due to Hedges (1981, 1982). Here, $\Gamma(.)$ is the Gamma function, and the pooled estimator of σ^2 is given by

$$s_p^2 = \nu^{-1}\{\sum_{j=1}^{n_1}(Y_j^E - \bar{Y}_{n_1}^E)^2 + \sum_{j=1}^{n_2}(Y_j^C - \bar{Y}_{n_2}^C)^2\}. \tag{3}$$

Hartung, Knapp and Sinha (2008) survey in detail, the estimation of effect size and Efron (2010) discusses the empirical Bayes estimation of effect size in connection with micro-array data. There is a close connection of the approaches of Efron and the approaches used in this paper. However, we formulate the hypothesis that δ may be equal to zero. That is to say, we suspect that the two means are equal, meaning that there is no effect of the experimental ingredients. To remove this vague hypothesis, we carry on a statistical test of hypothesis:

$$H_0 : \mu^E = \mu^C \;\; \text{Vs} \;\; H_A : \mu^E \neq \mu^C$$

based on the test statistics

$$Z^2 = \frac{(\bar{Y}_{n_1}^E - \bar{Y}_{n_2}^C)^2}{\sigma^2(\frac{1}{n_1} + \frac{1}{n_2})} \text{ if } \sigma^2 \text{ is known.} \tag{4}$$

$$\text{and } F_{1,\nu} = \frac{(\bar{Y}_{n_1}^E - \bar{Y}_{n_2}^C)^2}{s_p^2(\frac{1}{n_1} + \frac{1}{n_2})} \text{ if } \sigma^2 \text{ is unknown.} \tag{5}$$

Now, we propose the following two estimators as in Saleh (2006) (Chapter 3, Sections 3.2, 3.5) depending on whether σ^2 is known or unknown as follows:

$$\left.\begin{aligned} \hat{\delta}_n^{PT} &= \tilde{\delta}_n - \tilde{\delta}_n I(Z^2 \leq c_\alpha). \\ \hat{\delta}_n^s &= \tilde{\delta}_n - d|Z|^{-1}\tilde{\delta}_n, \;\; d > 0. \end{aligned}\right\} \text{ if } \sigma^2 \text{ is known} \tag{6}$$

where d is called the shrinkage constant, free of any parameter of the distribution, $I_x(A)$ stands for the indicator function of the set A, and c_α is the upper α-level critical value of the central chi-square distribution with one degree of freedom.

$$\left.\begin{aligned} \hat{\delta}_n^{*PT} &= \tilde{\delta}_n^* - \tilde{\delta}_n^* I(F_{1,\nu} \le c_\alpha). \\ \hat{\delta}_n^{*s} &= \tilde{\delta}_n^* - d|(F_{1,\nu})^{1/2}|^{-1}\tilde{\delta}_n^*,\ d > 0. \end{aligned}\right\} \text{ if } \sigma^2 \text{ is unknown} \tag{7}$$

where c_α is the upper α-level critical value of the central F-distribution with $(1, \nu)$ degrees of freedom.

The estimators $(\hat{\delta}_n^{PT}, \hat{\delta}_n^{*PT})$ are called "preliminary estimators" (PTE) due to Bancroft (1944), and $(\hat{\delta}_n^s, \hat{\delta}_n^{*s})$ are called "shrinkage type estimators" (SE) as in Saleh (2006).

In this paper, we study the statistical properties of the estimators $(\tilde{\delta}_n, \hat{\delta}_n^{PT}, \hat{\delta}_n^s)$ and $(\tilde{\delta}_n^*, \hat{\delta}_n^{*PT}, \hat{\delta}_n^{*s})$ based on their bias and mean square errors (MSE) which are given in sections 2 and 3. For the corresponding confidence intervals, we consider the coverage probabilities of fixed intervals, as they are given in section 4. In section 5, we study empirical Bayes estimators due to Efron (2010), and compare them with our estimators. Conclusion is given in section 6.

2 Estimators and their Bias and Mean Squared Error Expressions

2.1 Case 1: When σ^2 is Known

In this section, we consider the properties of the estimators when σ^2 is known. Let us call $\tilde{\delta}_n = \frac{\bar{Y}_{n1}^E - \bar{Y}_{n2}^C}{\sigma}$, the unrestricted estimator (UE), then clearly, under the assumption of normal theory given in the introduction, $\tilde{\delta}_n \sim N\{\frac{\mu^E - \mu^C}{\sigma}, (\frac{1}{n_1} + \frac{1}{n_2})\}$ so that $E(\tilde{\delta}_n) = \delta$ giving bias and variance as

$$b_1(\tilde{\delta}_n) = 0 \text{ and the } \mathrm{var}(\tilde{\delta}_n) = (\frac{1}{n_1} + \frac{1}{n_2}) = M_1(\tilde{\delta}_n). \tag{8}$$

$M_1(\tilde{\delta}_n)$ stands for the mean squared error (MSE) of $\tilde{\delta}_n$. For the test of H_0 one uses the test-statistcs Z^2 in (4) where Z^2 has chi-square distribution with one degree of freedom D.F. and noncentrality parameter $\frac{1}{2}\Delta^2$ where Δ^2 defined by $\Delta^2 = (\frac{1}{n_1} + \frac{1}{n_2})^{-1}\delta^2$ using normal theory. Under H_0, it follows a central chi-square distribution. Thus, at the α-level of significance we reject H_0 if $Z^2 \ge c_\alpha$, where c_α is the upper α-level critical value of the central chi-square distribution with one degree of freedom.

Now, consider the PTE defined in (6) and obtain the expected value of $\hat{\delta}_n^{PT}$ using theorem 4 of Saleh (2006, Chapter 2) as

$$E(\hat{\delta}_n^{PT}) = \delta - \delta H_3(c_\alpha, \Delta^2), \tag{9}$$

where $H_3(., \Delta^2)$ is the cdf of a chi-square distribution with 3 D.F. and noncentrality parameter $\frac{1}{2}\Delta^2$. Hence the bias expression is given by

$$b_2(\hat{\delta}_n^{PT}) = -\delta H_3(c_\alpha, \Delta^2). \tag{10}$$

Correspondingly, we obtain the MSE using theorems 4 and 5 of Saleh (2006, Chapter 2) as

$$M_2(\hat{\delta}_n^{PT}) = (\frac{1}{n_1} + \frac{1}{n_2})\{1 - H_3(c_\alpha, \Delta^2) + \Delta^2[2H_3(c_\alpha, \Delta^2) - H_5(c_\alpha, \Delta^2)]\}. \tag{11}$$

Similarly, for the shrinkage type estimator defined in (6), we obtain the bias using theorem 1 of Saleh (2006, Chapter 3, Section 3.5) as

$$\begin{aligned} b_3(\hat{\delta}_n^s) &= -\sqrt{(\frac{1}{n_1} + \frac{1}{n_2})}\ dE[\frac{Z}{|Z|}], \\ &= -\sqrt{(\frac{1}{n_1} + \frac{1}{n_2})}\ d[2\Phi(\Delta) - 1], \\ \text{where } Z &= \frac{\bar{Y}_{n1}^E - \bar{Y}_{n2}^C}{\sigma\sqrt{(\frac{1}{n_1} + \frac{1}{n_2})}}. \end{aligned} \tag{12}$$

Here $\Phi(.)$ is the cdf of N(0,1). The MSE of $\hat{\delta}_n^s$ may be obtained as

$$M_3(\hat{\delta}_n^s) = (\frac{1}{n_1} + \frac{1}{n_2})[1 + d^2 - 2d\sqrt{2/\pi}\ \ e^{-\Delta^2/2}]. \tag{13}$$

The value of d which minimizes $M_3(\hat{\delta}_n^s)$ is $d = \sqrt{2/\pi}\ \ e^{-\Delta^2/2}$, which depends on Δ^2. To make it free of the parameter, Δ^2 we choose $d^* = \sqrt{2/\pi}$. Hence, the resulting MSE becomes

$$M_3(\hat{\delta}_n^s) = (\frac{1}{n_1} + \frac{1}{n_2})\{1 - \frac{2}{\pi}[2e^{-\Delta^2/2} - 1]\}. \tag{14}$$

2.2 Case 2: When σ^2 is Unknown

In this case, $\tilde{\delta}_n^* = c(\nu)\frac{\bar{Y}_{n_1}^E - \bar{Y}_{n_2}^C}{s_p}$, is the unrestricted estimator (UE) of δ, and $\frac{(\frac{1}{n_1}+\frac{1}{n_2})^{-\frac{1}{2}}}{c(\nu)}\tilde{\delta}_n^*$ follows a noncentral t-distribution with ν degrees of freedom D.F., and noncentrality parameter $(\frac{1}{n_1} + \frac{1}{n_2})^{-\frac{1}{2}}\delta$. Thus, using the noncentral t-distribution, we have $E(\tilde{\delta}_n^*) = \delta$ and the bias and variance expressions are given by

$$b_1(\tilde{\delta}_n^*) = 0 \ \text{ and } \operatorname{var}(\tilde{\delta}_n^*) = (\frac{1}{n_1} + \frac{1}{n_2})[\frac{\nu c^2(\nu)}{(\nu - 2)}(1 + \Delta^2) - \Delta^2] = M_1(\tilde{\delta}_n^*) \text{ say}. \tag{15}$$

For the test of $H_0 : \mu^E = \mu^C$ one uses the test-statistcs $F_{1,\nu}$ given by (5) where $F_{1,\nu}$ follows a noncentral F-distribution with $(1, \nu)$ degrees of freedom D.F. and noncentrality parameter $\frac{1}{2}\Delta^2$ where Δ^2 defined by $\Delta^2 = (\frac{1}{n_1} + \frac{1}{n_2})^{-1}\delta^2$ under the normal theory. Under H_0, $F_{1,\nu}$ follows a central F-distribution. Thus, at the α-level of significance we reject H_0 if $F_{1,\nu} \geq c_\alpha$, where c_α is the upper α-level critical value of the central F-distribution with $(1, \nu)$ degrees of freedom.

Now, consider the PTE defined in (7). We may obtained the bias expression of $\hat{\delta}_n^{*PT}$ following Saleh (2006) (Chapter 4, Section 4.2) as

$$b_2(\hat{\delta}_n^{*PT}) = -\delta G_{3,\nu-2}(\frac{\nu-2}{3\nu}c_\alpha, \Delta^2), \tag{16}$$

where $G_{3,\nu-2}(., \Delta^2)$, is the cdf of F-distribution with $(3, \nu - 2)$ D.F. and noncentrality parameter $\frac{1}{2}\Delta^2$ and we may obtain the MSE expression for $\hat{\delta}_n^{*PT}$ as

$$\begin{aligned} M(\hat{\delta}_n^{*PT}) &= (\frac{1}{n_1} + \frac{1}{n_2})\{\frac{\nu c^2(\nu)}{\nu-2}[1 - G_{3,\nu-2}(\frac{\nu-2}{3\nu}c_\alpha, \Delta^2)] \\ &\quad + \Delta^2[(\frac{\nu c^2(\nu)}{\nu-2} - 1) + 2G_{3,\nu-2}(\frac{\nu-2}{3\nu}c_\alpha, \Delta^2) \\ &\quad - \frac{\nu c^2(\nu)}{\nu-2}G_{5,\nu-2}(\frac{\nu-2}{5\nu}c_\alpha, \Delta^2)]\}. \end{aligned} \tag{17}$$

Next, we consider the shrinkage type estimator defined in (7), the bias expression may be shown to be

$$b_3(\hat{\delta}_n^{*s}) = -d\ c(\nu)\ \sqrt{(\frac{1}{n_1} + \frac{1}{n_2})}\ [1 - 2\Phi(-\Delta)].$$

We may calculate the MSE of $\hat{\delta}_n^{*s}$ as follows

$$\begin{aligned} M_3(\hat{\delta}_n^{*s}) &= (\frac{1}{n_1} + \frac{1}{n_2})\{[\frac{\nu c^2(\nu)}{(\nu-2)}(1 + \Delta^2) - \Delta^2] \\ &\quad + d^2c^2(\nu) - 2dc(\nu)[E|Z| - \Delta E(\frac{Z}{|Z|})]\}. \end{aligned}$$

Minimizing w.r.t d, we obtain the optimum value of d as $d = \frac{1}{c(\nu)}\ \sqrt{\frac{2}{\pi}}e^{-\frac{\Delta^2}{2}}$ which depends on Δ^2. To make it free of Δ^2, we choose $d^* = \frac{1}{c(\nu)}\ \sqrt{\frac{2}{\pi}}$. So that the MSE becomes

$$M_3(\hat{\delta}_n^{*s}) = (\frac{1}{n_1} + \frac{1}{n_2})\{[\frac{\nu c^2(\nu)}{(\nu-2)}(1 + \Delta^2) - \Delta^2] - \frac{2}{\pi}[2e^{-\frac{\Delta^2}{2}} - 1]\}. \tag{18}$$

We summarize the results of the previous two subsections in the following two theorems.

Theorem 2.1. *If σ^2 is known, then the bias and MSE expressions of the three estimators are given by*

$$
\begin{aligned}
b_1(\tilde{\delta}_n) &= 0, \\
M_1(\tilde{\delta}_n) &= (\frac{1}{n_1}+\frac{1}{n_2}). \\
b_2(\hat{\delta}_n^{PT}) &= -\delta H_3(c_\alpha,\Delta^2), \\
M_2(\hat{\delta}_n^{PT}) &= (\frac{1}{n_1}+\frac{1}{n_2})\{1-H_3(c_\alpha,\Delta^2)+\Delta^2[2H_3(c_\alpha,\Delta^2)-H_5(c_\alpha,\Delta^2)]\}. \\
b_3(\hat{\delta}_n^{s}) &= -\sqrt{(\frac{1}{n_1}+\frac{1}{n_2})}\ d[2\Phi(\Delta)-1], \\
M_3(\hat{\delta}_n^{s}) &= (\frac{1}{n_1}+\frac{1}{n_2})\{1-\frac{2}{\pi}[2e^{-\Delta^2/2}-1]\}.
\end{aligned}
$$

Theorem 2.2. *If σ^2 is unknown, then the bias and MSE expressions of the three estimators are given by*

$$
\begin{aligned}
b_1(\tilde{\delta}_n^{*}) &= 0, \\
M_1(\tilde{\delta}_n^{*}) &= (\frac{1}{n_1}+\frac{1}{n_2})[\frac{\nu c^2(\nu)}{(\nu-2)}(1+\Delta^2)-\Delta^2]. \\
b_2(\hat{\delta}_n^{*PT}) &= -\delta G_{3,\nu-2}(\frac{\nu-2}{3\nu}c_\alpha,\Delta^2), \\
M(\hat{\delta}_n^{*PT}) &= (\frac{1}{n_1}+\frac{1}{n_2})\{\frac{\nu c^2(\nu)}{\nu-2}[1-G_{3,\nu-2}(\frac{\nu-2}{3\nu}c_\alpha,\Delta^2)] \\
&\quad +\Delta^2[(\frac{\nu c^2(\nu)}{\nu-2}-1)+2G_{3,\nu-2}(\frac{\nu-2}{3\nu}c_\alpha,\Delta^2) \\
&\quad -\frac{\nu c^2(\nu)}{\nu-2}G_{5,\nu-2}(\frac{\nu-2}{5\nu}c_\alpha,\Delta^2)]\}. \\
b_3(\hat{\delta}_n^{*s}) &= -d\ c(\nu)\ \sqrt{(\frac{1}{n_1}+\frac{1}{n_2})}\ [1-2\Phi(-\Delta)], \\
M_3(\hat{\delta}_n^{*s}) &= (\frac{1}{n_1}+\frac{1}{n_2})\{[\frac{\nu c^2(\nu)}{(\nu-2)}(1+\Delta^2)-\Delta^2]-\frac{2}{\pi}[2e^{-\frac{\Delta^2}{2}}-1]\}.
\end{aligned}
$$

3 Properties of the Estimators

We present the properties of the estimators for the unknown variance case only. For the known variance case, the properties of the estimators are very similar. We consider the unrestricted estimator as the basis for comparing the others. Thus, we may define the MSE-based relative efficiency (MRE) of $\hat{\delta}_n^{*PT}$ relative to $\tilde{\delta}_n^*$ as

$$\begin{aligned}\text{MRE}(\hat{\delta}_n^{*PT}, \tilde{\delta}_n^*) &= \frac{MSE(\tilde{\delta}_n^*)}{MSE(\hat{\delta}_n^{*PT})} \\ \text{MRE}(\hat{\delta}_n^{*PT}, \tilde{\delta}_n^*) &= [1 + g_\alpha(\Delta^2)]^{-1}\end{aligned}$$

where $g_\alpha(\Delta^2)$

$$\begin{aligned}= \quad & -(\frac{1}{n_1} + \frac{1}{n_2})[\text{var}(\tilde{\delta}_n^*)]^{-1}\Bigg(\frac{\nu c^2(\nu)}{\nu-2} G_{3,\nu-2}(\frac{\nu-2}{3\nu}c_\alpha, \Delta^2) - \Delta^2 \\ & \times [2G_{3,\nu-2}(\frac{\nu-2}{3\nu}c_\alpha, \Delta^2) - \frac{\nu c^2(\nu)}{\nu-2} G_{5,\nu-2}(\frac{\nu-2}{5\nu}c_\alpha, \Delta^2)]\Bigg).\end{aligned}$$

$\text{MRE}(\hat{\delta}_n^{*PT}, \tilde{\delta}_n^*)$ may be written as a function of (α, Δ^2). Figure 1 below illustrates the graph of $\text{MRE}(\alpha, \Delta^2)$ as a function of Δ^2. For fixed α, it decreases, crossing the 1-line to a minimum at $\Delta^2 = \Delta^2_{min}(\alpha)$, then increases towards the 1-line as $\Delta^2 \to \infty$. The maximum of $\text{MRE}(\alpha, \Delta^2)$ occurs at $\Delta^2 = 0$ with the value

$$\text{MRE}(\alpha, 0) = [1 - (\frac{1}{n_1} + \frac{1}{n_2})[\text{var}(\tilde{\delta}_n^*)]^{-1}\frac{\nu c^2(\nu)}{\nu-2} G_{3,\nu-2}(\frac{\nu-2}{3\nu}c_\alpha, 0)]^{-1} \tag{19}$$

for all $\alpha \in A$, the set of all possible values of α. The value of the max of $\text{MRE}(\alpha, \Delta^2)$ decreases as α increases. Intersection of the graph of $\text{MRE}(\alpha, \Delta^2)$ with the 1-line occurs at Δ_1^2. Thus if $0 \le \Delta^2 \le \Delta_1^2$, then $\hat{\delta}_n^{*PT}$ performs better than $\tilde{\delta}_n^*$, and if $\Delta^2 \ge \Delta_1^2$, then $\tilde{\delta}_n^*$ performs better. The cut-off point Δ_1^2 may be determined by

$$\Delta_1^2 = \frac{\frac{\nu c^2(\nu)}{\nu-2}[G_{3,\nu-2}(\frac{\nu-2}{3\nu}c_\alpha, \Delta^2)]}{2G_{3,\nu-2}(\frac{\nu-2}{3\nu}c_\alpha, \Delta^2) - \frac{\nu c^2(\nu)}{\nu-2}G_{5,\nu-2}(\frac{\nu-2}{5\nu}c_\alpha, \Delta^2)}. \tag{20}$$

In order to obtain a PTE with minimum guaranteed RE, say E_0, we adopt the following procedure : if $0 \le \Delta^2 \le \Delta_1^2$, we always choose $\tilde{\delta}_n^*$ since, $\text{MRE}(\alpha, \Delta^2) \ge 1$ in this interval. However, in general Δ^2 is unknown, and there is no way one can choose an estimator which is uniformly best. Thus, we look for an estimator with minimum guaranteed RE, say, E_0 such that set $A = \{\alpha \backslash MRE(\alpha, \Delta^2) \ge E_0\}$ which is well-defined and satisfies the equation

$$max_\alpha min_{\Delta^2} MRE(\alpha, \Delta^2) = E_0 \text{ for some } \alpha \in A.$$

The solution α^* gives the optimum α-level satisfying the minimum guaranteed RE. Table 1 below may be used to determine the optimum level of significance for the PTE. Table 1 gives the performance of $\hat{\delta}_n^{*s}$ and $\hat{\delta}_n^{*PT}$ (for each selected level of significance)

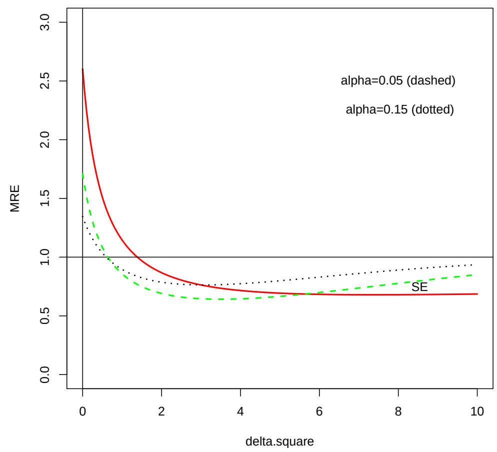

Figure 1: Graph of MSE-based relative efficiency of SE and PTE for $n_1 = 9, n_2 = 10, \alpha = 0.05$ and 0.15

for selected sample sizes. The first two rows of Table 1 contain the maximum and minimum relative efficiency of SE for selected sample sizes. The remaining rows of Table 1 contain the minimum relative efficiency, E_0, of the PTE at $\Delta^2 = \Delta_0^2$ and the maximum relative efficiency, E_{max}, which has been recorded for each α=0.05(0.10)0.45 with the corresponding efficiency ($E_{\Delta_0^2}$) of SE for $\Delta^2 = \Delta_0^2$. To explain how to use this table, for example, let $(n_1, n_2) = (3, 4)$ and assume that we wish to obtain a PTE with at least 86% efficiency. Table 1 gives $\alpha^* = 0.25$ at the intersection of $E_0 = 0.8674$ and $(n_1, n_2) = (3, 4)$. Hence, the optimum level for the PTE in this case is $\alpha = 0.25$ with a maximum possible efficiency 1.2480. Now, we consider the MSE-based relative efficiency of $\hat{\delta}_n^{*s}$ versus $\tilde{\delta}_n^*$ as

$$\mathrm{MRE}(\hat{\delta}_n^{*s}, \tilde{\delta}_n^*) = \{1 - \frac{2}{\pi}[\frac{\nu c^2(\nu)}{(\nu-2)}(1+\Delta^2) - \Delta^2]^{-1}[2e^{-\frac{\Delta^2}{2}} - 1]\}^{-1}.$$

$$\begin{aligned} \text{If } \Delta^2 = 0\ , MRE(\hat{\delta}_n^{*s}, \tilde{\delta}_n^*) &= \frac{\nu\pi c^2(\nu)}{\nu\pi c^2(\nu) - 2(\nu-2)}\cdot \\ \text{and if } \Delta^2 \to \infty\ , \mathrm{MRE}(\hat{\delta}_n^{*s}, \tilde{\delta}_n^*) &= \frac{\nu\pi c^2(\nu)}{\nu\pi c^2(\nu) + 2(\nu-2)}. \end{aligned}$$

Further, as $\nu \to \infty$ and $\Delta^2 = 0, MRE(\hat{\delta}_n^{*s}, \tilde{\delta}_n^*) \to \frac{\pi}{\pi-2}\cdot$ Moreover, as $\nu \to \infty$ and $\Delta^2 \to \infty, RE(\hat{\delta}_n^{*s}, \tilde{\delta}_n^*) \to \frac{\pi}{\pi+2}\cdot$ Figure 1 illustrates the graphs of MSE-based relative efficiency of preliminary test estimators for α=0.05 and 0.15, and that of shrinkage

Table 1: Maximum and Minimum relative efficiency of SE and PTE for selected sample sizes

$\alpha \backslash (n_1, n_2)$		(3,4)	(5,6)	(9,10)	(13,14)
	E^{max}	2.1760	2.4560	2.6030	2.6520
	E^{min}	0.7952	0.7255	0.6798	0.6613
0.05	E_0	0.6286	0.5053	0.4522	0.4360
	E_{max}	2.4266	2.9413	3.2459	3.3546
	$E_{\Delta_0^2}$	0.8168	0.7642	0.7350	0.7257
	Δ_0^2	4.7	4.95	4.7	4.7
0.15	E_0	0.7843	0.6895	0.6417	0.6259
	E_{max}	1.4961	1.6343	1.7113	1.7381
	$E_{\Delta_0^2}$	0.8168	0.7642	0.7350	0.7257
	Δ_0^2	3.4	3.5	3.55	3.55
0.25	E_0	0.8674	0.8004	0.7633	0.7506
	E_{max}	1.2480	1.3110	1.3455	1.3575
	$E_{\Delta_0^2}$	0.8337	0.7860	0.7630	0.7484
	Δ_0^2	2.95	3.05	3.1	3.15
0.35	E_0	0.9201	0.8761	0.8502	0.8411
	E_{max}	1.1334	1.1660	1.1836	1.1897
	$E_{\Delta_0^2}$	0.8469	0.8059	0.7777	0.7665
	Δ_0^2	2.7	2.75	2.85	2.9
0.45	E_0	0.9544	0.9278	0.9115	0.9056
	E_{max}	1.0713	1.0884	1.0976	1.1008
	$E_{\Delta_0^2}$	0.8565	0.8137	0.7903	0.7791
	Δ_0^2	2.55	2.65	2.7	2.75

type estimators. Now, note that the MRE$(\hat{\delta}_n^{*s}, \tilde{\delta}_n^*)$ as a function of Δ^2 crosses the 1-line at $\Delta^2 = \ln 4$. Thus, for $0 \leq \Delta^2 \leq \ln 4$, $\hat{\delta}_n^{*s}$ performs better than $\tilde{\delta}_n^*$, otherwise $\tilde{\delta}_n^*$ performs better outside this interval. The maximum relative efficiency of SE increases as the sample size n $(n = n_1 + n_2)$ increases, and as $n \to \infty$ it tends to $\frac{\pi}{\pi-2}$, while the minimum efficiency decreases as n increases. Finally as $n \to \infty$, it tends to $\frac{\pi}{\pi+2}$. Figure 1 and Table 1, explain that when Δ^2 is near the origin, SE outperforms PTE for most α values. Also, at $\Delta^2 = 0$, the relative efficiency of SE is higher than that of PTE. Further, SE is independent of α. Thus, considering the overall performance of SE relative to PTE, SE is preferable to PTE since it produces interpolated estimators which are free from the level of significance, α.

4 Confidence Intervals

4.1 When σ^2 is Known

We consider the $100(1-\beta)\%$ confidence interval for δ when the variance is known, using $\tilde{\delta}_n$, which may be written as the Neyman-Pearson solution

$$C^0(\tilde{\delta}_n) = \{\delta : \sqrt{\tilde{n}}\ |\delta - \tilde{\delta}_n| \le z_{\beta/2}\} \tag{21}$$

where $z_{\beta/2}$ is the upper $\beta/2$-level critical value from the standard normal distribution, $\tilde{n} = (\frac{1}{n_1} + \frac{1}{n_2})^{-1}$, where

$$P_0\{\sqrt{\tilde{n}}\ |\delta - \tilde{\delta}_n| \le z_{\beta/2}\} = 1 - \beta = P_0\{\tilde{n}(\delta - \tilde{\delta}_n)^2 \le z^2_{\beta/2}\}. \tag{22}$$

Next, we consider the $100(1-\beta)\%$ confidence interval for δ using the PTE, $\hat{\delta}_n^{PT}$ as the solution

$$C^{PT}(\hat{\delta}_n^{PT}) = \{\delta : \sqrt{\tilde{n}}\ |\delta - \hat{\delta}_n^{PT}| \le z_{\beta/2}\}. \tag{23}$$

The computational formula of the coverage probability of the set $C^{PT}(\hat{\delta}_n^{PT})$ is given by

$$\begin{aligned} P_{\Delta^2}\{\sqrt{\tilde{n}}\ |\delta - \hat{\delta}_n^{PT}| \le z_{\beta/2}\} \quad = \quad & H_1(\chi^2_{1,\alpha};\Delta^2)I(\Delta^2 < z^2_{\beta/2}) + (1-\beta) \\ & - \Phi[min\{z_{\beta/2}, z_{\alpha/2} - \Delta\}] \\ & + \Phi[max\{-z_{\beta/2}, -(z_{\alpha/2} + \Delta)\}] \end{aligned} \tag{24}$$

Graphs of coverage probability of 95% and 90% C.I. and the test levels $\alpha = 0.05, 0.1$ and 0.2 of the set $C^{PT}(\hat{\delta}_n^{PT})$ are shown below along with the coverage probability of set based on SE.

From the graphs one may see that from 0 to $z_{\beta/2}$, the coverage probability is greater than or equal to $1-\beta$ and there is a discontinuity at $\Delta = z_{\beta/2}$ when the coverage probability drops to its minimum (depending on α, β) then increases towards $1-\beta$ at $\Delta = z_{\beta/2} + z_{\alpha/2}$ and eventually goes to 1-$\beta/2$ as $\Delta \to \infty$.

Tabular values of the coverage probability are presented in Table 2 for the 95% and 90% confidence sets based on PTE (and the test levels $\alpha = 0.05, 0.1, 0.2$) and SE.

Now, we state the properties of the coverage probability of the set $C^{PT}(\hat{\delta}_n^{PT})$ in the following theorem.

Theorem 4.1.

If (*i*) $0 \le \Delta^2 \le z^2_{\beta/2},\ P\{C^{PT}(\hat{\delta}_n^{PT})\} \ge 1-\beta.$

(*ii*) $z^2_{\beta/2} \le \Delta^2 < (z_{\beta/2} + z_{\alpha/2})^2,\ P\{C^{PT}(\hat{\delta}_n^{PT})\} \le 1-\beta.$

(*iii*) $\Delta^2 = (z_{\beta/2} + z_{\alpha/2})^2,\ P\{C^{PT}(\hat{\delta}_n^{PT})\} = 1-\beta.$

(*iv*) $\Delta^2 = 0$ *and* $min(z_{\beta/2} + z_{\alpha/2}) = z_{\alpha/2},\ P\{C^{PT}(\hat{\delta}_n^{PT})\} = 1-\beta.$

(*v*) $\Delta^2 \to \infty,\ P\{C^{PT}(\hat{\delta}_n^{PT})\} = 1-\beta/2.$

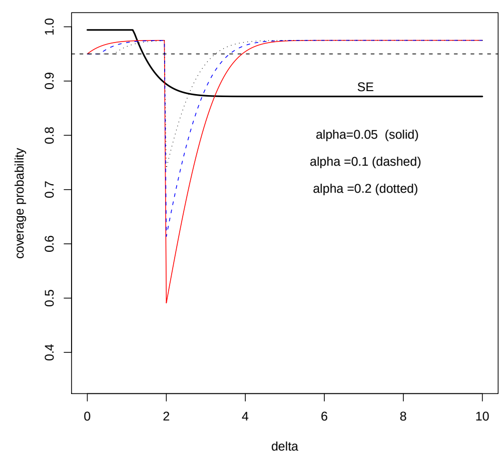

Figure 2: Coverage probabilities of 95% C.I. based on PTE (α= 0.05, 0.1, 0.2) and SE

Proof. Note that

$$\begin{aligned}
(i)\quad P_{\Delta^2}\{\sqrt{\tilde{n}}\ |\delta - \hat{\delta}_n^{PT}| \le z_{\beta/2}\} &= P_{\Delta^2}\{\tilde{n}\Big(\delta - \tilde{\delta}_n I(Z^2 \ge \chi^2_{1,\alpha})\Big)^2 \le z^2_{\beta/2}\} \\
&= P_{\Delta^2}\{\Delta^2 < z^2_{\beta/2}, Z^2 < \chi^2_{1,\alpha}\} \\
&\quad + P\{\tilde{n}(\delta - \tilde{\delta}_n)^2 \le z^2_{\beta/2}, Z^2 \ge \chi^2_{1,\alpha}\} \\
&= H_1(\chi^2_{1,\alpha}; \Delta^2) I(\Delta^2 < z^2_{\beta/2}) \\
&\quad + P\{\tilde{n}(\delta - \tilde{\delta}_n)^2 \le z^2_{\beta/2}, Z^2 \ge \chi^2_{1,\alpha}\} \\
&\ge P_{\Delta^2}\{\tilde{n}(\delta - \tilde{\delta}_n)^2 \le z^2_{\beta/2}, Z^2 \le \chi^2_{1,\alpha}\} \\
&\quad + P\{\tilde{n}(\delta - \tilde{\delta}_n)^2 \le z^2_{\beta/2}, Z^2 \ge \chi^2_{1,\alpha}\} \\
&\ge P_{\Delta^2}\{\tilde{n}(\delta - \tilde{\delta}_n)^2 \le z^2_{\beta/2}\} = 1 - \beta.
\end{aligned}$$

Hence, we have proved that if $\Delta^2 < z^2_{\beta/2}$, then

$$P_{\Delta^2}\{\sqrt{\tilde{n}}\ |\delta - \hat{\delta}_n^{PT}| \le z_{\beta/2}\} \ge 1 - \beta.$$

(ii) If $z^2_{\beta/2} \le \Delta^2 < (z_{\beta/2} + \chi_{1,\alpha})^2$, we have

$$\begin{aligned}
P_{\Delta^2}\{\delta \in C^{PT}(\hat{\delta}_n^{PT})\} &= P_{\Delta^2}\{\tilde{n}(\delta - \tilde{\delta}_n)^2 \le z^2_{\beta/2}; Z^2 \ge \chi^2_{1,\alpha}\} \\
&\le P_{\Delta^2}\{\tilde{n}(\delta - \tilde{\delta}_n)^2 \le z^2_{\beta/2}\} = 1 - \beta.
\end{aligned}$$

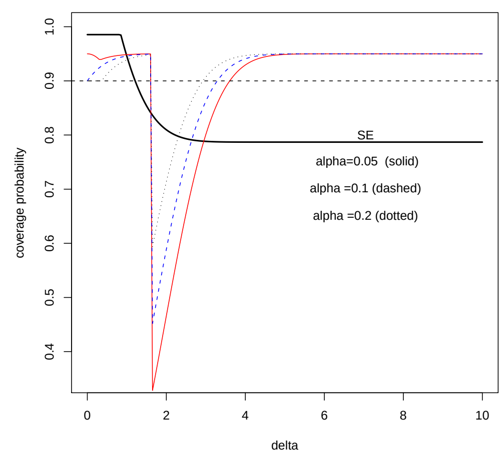

Figure 3: Coverage probabilities of 90% C.I. based on PTE (α=0.05, 0.1, 0.2) and SE

Hence, we have proved that if $z^2_{\beta/2} \leq \Delta^2 < (z_{\beta/2} + \chi_{1,\alpha})^2$, then

$$P_{\Delta^2}\{\sqrt{\tilde{n}}\ |\delta - \hat{\delta}^{PT}_n| \leq z_{\beta/2}\} \leq 1 - \beta.$$

(iii) If $\Delta^2 = (z_{\beta/2} + z_{\alpha/2})^2$, then

$$\begin{aligned} P_{\Delta^2}\{\sqrt{\tilde{n}}\ |\delta - \hat{\delta}^{PT}_n| \leq z_{\beta/2}\} &= P_{\Delta^2}\{\tilde{n}(\delta - \tilde{\delta}_n)^2 \leq z^2_{\beta/2}, Z^2 \geq \chi^2_{1,\alpha}\} \\ &= P_{\Delta^2}\{\sqrt{\tilde{n}}\ |\delta - \tilde{\delta}_n| \leq z_{\beta/2}\} \\ &\quad - P_{\Delta^2}\{\sqrt{\tilde{n}}\ |\delta - \tilde{\delta}_n| \leq z_{\beta/2}, |Z| \leq z_{\alpha/2}\} \\ &= 1 - \beta - \Phi[min\{z_{\beta/2}, z_{\alpha/2} - \Delta\}] \\ &\quad + \Phi[max\{-z_{\beta/2}, -(z_{\alpha/2} + \Delta)\}]. \end{aligned} \tag{25}$$

Substituting the value of $\Delta = z_{\beta/2} + z_{\alpha/2}$, we obtain

$$P_{\Delta^2}\{\sqrt{\tilde{n}}\ |\delta - \hat{\delta}^{PT}_n| \leq z_{\beta/2}\} = 1 - \beta.$$

(iv) If $\Delta^2 = 0$, and $\min(z_{\beta/2} + z_{\alpha/2}) = z_{\alpha/2}$, substituting the value of $\Delta = 0$ in (25) we obtain $P_{\Delta^2}\{\sqrt{\tilde{n}}\ |\delta - \hat{\delta}^{PT}_n| \leq z_{\beta/2}\} = 1 - \beta$.

(v) Further, if $\Delta^2 \to \infty$, from (25) we obtain

$$P_{\Delta^2}\{\sqrt{\tilde{n}}\ |\delta - \hat{\delta}^{PT}_n| \leq z_{\beta/2}\} = 1 - \beta/2.$$

□

Table 2: Coverage probability of 95% and 90% C.I. of the sets $C^{PT}(\hat{\delta}_n^{PT})$ and $C^s(\hat{\delta}_n^s)$

	95%				90 %			
$\Delta\backslash\alpha$	0.05	0.1	0.2	SE	0.05	0.1	0.2	SE
0	0.9500	0.9500	0.9500	0.9942	0.9500	0.9000	0.9000	0.9854
1	0.9734	0.9703	0.9637	0.9942	0.9484	0.9459	0.9387	0.9456
2	0.4909	0.6137	0.7387	0.8943	0.4659	0.5887	0.7137	0.8096
3	0.8258	0.8873	0.9321	0.8729	0.8000	0.8623	0.9071	0.7882
4	0.9543	0.9657	0.9717	0.8716	0.9293	0.9407	0.9467	0.7869
5	0.9736	0.9746	0.9750	0.8716	0.9488	0.9496	0.9498	0.7869
6	0.9750	0.9750	0.9750	0.8716	0.9499	0.9499	0.9499	0.7869
7	0.9750	0.9750	0.9750	0.8716	0.9499	0.9499	0.9500	0.7869
8	0.9750	0.9750	0.9750	0.8716	0.9500	0.9500	0.9500	0.7869
9	0.9750	0.9750	0.9750	0.8716	0.9500	0.9500	0.9500	0.7869
10	0.9750	0.9750	0.9750	0.8716	0.9500	0.9500	0.9500	0.7869

Next, we consider the coverage probability of confidence interval defined by $\hat{\delta}_n^s$ as

$$C^s(\hat{\delta}_n^s) = \{\delta : \sqrt{\tilde{n}}\ |\delta - \hat{\delta}_n^s| \leq z_{\beta/2}\} \tag{26}$$

It may be shown that

$$\begin{aligned} P\{C^s(\hat{\delta}_n^s)\} &= P\{\sqrt{\tilde{n}}\ |\delta - \tilde{\delta}_n + \frac{d}{\sqrt{\tilde{n}}}\frac{\tilde{\delta}_n}{|\tilde{\delta}_n|}| \leq z_{\beta/2}\} \\ &= P\{(d - z_{\beta/2}) \leq (\sqrt{\tilde{n}}\ \ \tilde{\delta}_n - \Delta) \leq (d + z_{\beta/2}); \sqrt{\tilde{n}}\tilde{\delta}_n > 0\} \\ &\quad + P\{-(d + z_{\beta/2}) \leq (\sqrt{\tilde{n}}\ \ \tilde{\delta}_n - \Delta) \leq -(d - z_{\beta/2}); \sqrt{\tilde{n}}\tilde{\delta}_n < 0\} \\ &= 2\Phi(d + z_{\beta/2}) - 1 - \Phi\Big(\max(d - z_{\beta/2}, -\Delta)\Big) \\ &\quad + \Phi\Big(\min(-(d - z_{\beta/2}), -\Delta)\Big). \end{aligned} \tag{27}$$

Here, $d = \sqrt{\frac{2}{\pi}}$. For 95% and 90% C.I. of δ based on $\hat{\delta}_n^s$, we obtain the coverage probability for variation of Δ as given in Table 2.
Thus, for 95% C.I. for δ based on $\hat{\delta}_n^s$, we observe that the coverage probability as a function of Δ is decreasing up to a point say $\Delta = \Delta_{\min}$, then becomes constant 0.8716. Thus, eventual loss of efficiency of this C.I. is 8.25% and gain if Δ is near 0 is 4.65%. Similarly, for 90% C.I. of δ based on $\hat{\delta}_n^s$, the eventual loss of efficiency is 12.56% and gain near the origin 9.5% . Graphs of coverage probabilities of 95% C.I. and 90% C.I. based on $\hat{\delta}_n^s$ are shown in Figure 2 and Figure 3.

Table 3: Efficiency of 95% C.I. of the sets $C^{PT}(\hat{\delta}_n^{PT})$ and $C^s(\hat{\delta}_n^s)$

	α						
Δ	SE	0.1	0.15	0.20	0.25	0.30	0.35
0	1.046	1	1	1	1	1	1
1	1.046	1.022	1.018	1.014	1.009	1.004	1
1.96	0.9477	1.026	1.026	1.025	1.025	1.024	1.024
2.89	0.9197	0.9062	0.9429	0.9648	0.9793	0.9896	1
3	0.9188	0.9340	0.9638	0.9812	0.9924	1	1.006
3.11	0.9184	0.9496	0.9753	0.9900	0.9993	1.006	1.010
3.24	0.9180	0.9692	0.9893	1	1.007	1.012	1.015
3.40	0.9178	0.9846	1	1.008	1.013	1.017	1.019
3.60	0.9176	1.001	1.010	1.015	1.018	1.020	1.022
4	0.9174	1.020	1.020	1.023	1.024	1.024	1.025
5	0.9174	1.026	1.026	1.026	1.026	1.026	1.026
6	0.9174	1.026	1.026	1.026	1.026	1.026	1.026
7	0.9174	1.026	1.026	1.026	1.026	1.026	1.026

Next, we define the efficiency of the $100(1-\beta)\%$ confidence interval as

$$EFF = \frac{\text{coverage probability obtained}}{1-\beta}. \tag{28}$$

Tabular values of the efficiency of 95% and 90% confidence intervals based on PTE and SE for selected Δ values are presented in Table 3 and Table 4. The values of Δ are 0,1, $z_{\beta/2}, z_{\beta/2} + z_{\alpha/2}$ (for each α= 0.35(-0.05)0.1), 4, 5, 6 and 7 .

4.2 When σ^2 is Unknown

For the finite sample 100(1-β)% confidence interval for δ one needs the computational procedures using the noncentral t-distribution or noncentral F-distribution, see for example, Cumming and Finch (2001) and Lecoutre (2007) among others. At the end, they all proposed methods of approximations as it became difficult to specify simple methods. With this backdrop, here we consider the asymptotic distribution of $\tilde{\delta}_n^*$ as $N(\delta, \sigma^2(\delta))$, $\sigma^2(\delta) = \frac{1}{\tilde{n}} + \frac{\delta^2}{2n}, n = n_1 + n_2$ and define the Neyman-Pearson solution as

$$C^0(\tilde{\delta}_n^*) = \{\delta : |\delta - \tilde{\delta}_n^*| \leq z_{\beta/2} \sqrt{\frac{1}{\tilde{n}} + \frac{\tilde{\delta}^{*2}}{2(n-2)}}\}. \tag{29}$$

Table 4: Efficiency of 90% C.I. of the sets $C^{PT}(\hat{\delta}_n^{PT})$ and $C^s(\hat{\delta}_n^s)$

	α						
Δ	SE	0.1	0.15	0.20	0.25	0.30	0.35
0	1.094	1	1	1	1	1	1
1	1.050	1.051	1.047	1.043	1.038	1.032	1.026
1.65	0.9293	1.055	1.054	1.053	1.052	1.051	1.049
2.58	0.8803	0.8597	0.9131	0.9467	0.9698	0.9866	0.9994
2.69	0.8782	0.8937	0.9403	0.9688	0.9882	1	1.012
2.80	0.8772	0.9178	0.9590	0.9839	1	1.012	1.021
2.93	0.8761	0.9489	0.9828	1.003	1.015	1.024	1.031
3.09	0.8755	0.9706	0.9989	1.015	1.025	1.032	1.037
3.30	0.8749	1	1.020	1.031	1.038	1.042	1.045
4	0.8744	1.046	1.049	1.052	1.053	1.054	1.054
5	0.8743	1.055	1.055	1.055	1.055	1.055	1.055
6	0.8743	1.055	1.055	1.055	1.055	1.055	1.055
7	0.8743	1.055	1.055	1.055	1.055	1.055	1.055

However, we notice that as $n \to \infty$

$$\tilde{\delta}^* = c(\nu)\frac{\bar{Y}_{n_1}^E - \bar{Y}_{n_2}^C}{\sigma}\frac{\sigma}{s_p} \xrightarrow{P} \frac{\bar{Y}_{n_1}^E - \bar{Y}_{n_2}^C}{\sigma} \quad \text{as} \quad \frac{s_p}{\sigma} \xrightarrow{a.s} 1. \tag{30}$$

Hence,

$$\lim_{n\to\infty} P\{C^0(\tilde{\delta}_n^*)\} = P\{\sqrt{\tilde{n}}\ |\delta - \tilde{\delta}_n| \le z_{\beta/2}\} = 1 - \beta. \tag{31}$$

Now, we consider 100(1-β)% confidence interval for δ using PTE, $\hat{\delta}_n^{*PT}$ as the Neyman-Pearson solution

$$C^{PT}(\hat{\delta}_n^{*PT}) = \{\delta : |\delta - \hat{\delta}_n^{*PT}| \le z_{\beta/2}\sqrt{\frac{1}{\tilde{n}} + \frac{\tilde{\delta}^{*2}}{2(n-2)}}\ \}. \tag{32}$$

Let us now consider the limiting coverage probability of $C^{PT}(\hat{\delta}_n^{*PT})$. First, note that if $\Delta^2 \le z_{\beta/2}^2(1 + \frac{\tilde{n}\tilde{\delta}^{*2}}{2(n-2)})$, we have

$$\begin{aligned}\lim_{n\to\infty} P\{C^{PT}(\hat{\delta}_n^{*PT})\} &= \lim_{n\to\infty} P_{\Delta^2}\{\tilde{n}(\delta - \hat{\delta}_n^{*PT})^2 \le z_{\beta/2}^2(1 + \frac{\tilde{n}\tilde{\delta}^{*2}}{2(n-2)})\} \\ &= H_1(\chi_{1,\alpha}^2; \Delta^2)I(\Delta^2 < z_{\beta/2}^2) \\ &\quad + P_{\Delta^2}\{-z_{\beta/2} \le Z - \Delta \le z_{\beta/2}, |Z| > z_{\alpha/2}\}.\end{aligned}$$

The R.H.S may be written as

$$\begin{aligned} &= H_1(\chi^2_{1,\alpha};\Delta^2)I(\Delta^2 < z^2_{\beta/2}) + (1-\beta) \\ &\quad - \Phi[min\{z_{\beta/2}, z_{\alpha/2} - \Delta\}] \\ &\quad + \Phi[max\{-z_{\beta/2}, -(z_{\alpha/2} + \Delta)\}]. \end{aligned}$$

For the 100(1-β)% confidence interval for δ based on SE, $\hat{\delta}_n^{*s}$, we define the Neyman-Pearson solution as

$$\begin{aligned} C^s(\hat{\delta}_n^{*s}) &= \{\delta : |\delta - \hat{\delta}_n^{*s}| \le z_{\beta/2} \sqrt{\frac{1}{\tilde{n}} + \frac{\tilde{\delta}^{*2}}{2(n-2)}} \quad \}. \\ &= \{\Delta : \sqrt{\tilde{n}}|\delta - \hat{\delta}_n^{*s}| \le z_{\beta/2} \sqrt{1 + \frac{\tilde{n}\tilde{\delta}^{*2}}{2(n-2)}} \quad \}. \end{aligned}$$

Then, the limiting coverage probability of $C^s(\hat{\delta}_n^{*s})$ is given by

$$\begin{aligned} \lim_{n\to\infty} P\{C^s(\hat{\delta}_n^{*s})\} &= \lim_{n\to\infty} P_{\Delta^2}\{\tilde{n}(\delta - \hat{\delta}_n^{*s})^2 \le z^2_{\beta/2}(1 + \frac{\tilde{n}\tilde{\delta}^{*2}}{2(n-2)})\} \\ &= 2\Phi(d + z_{\beta/2}) - 1 - \Phi\Big(\max(d - z_{\beta/2}, -\Delta)\Big) \\ &\quad + \Phi\Big(\min(-(d - z_{\beta/2}), -\Delta)\Big). \end{aligned}$$

The limiting coverage probabilities are the same as the known variance case and hence the properties specified therein.

Example 4.1. Suppose that a study with $n_1 = n_2 = 15$ yields a sample effect size value of $\tilde{\delta}_n^* = 0.580$, $\hat{\delta}_n^{*PT} = 0$, and $\hat{\delta}_n^{*s} = 0.3284$ then the 90% confidence intervals for δ based on PTE and SE respectively are given by

$$\begin{aligned} \hat{\delta}_n^{*PT} \pm z_{\beta/2} \sqrt{\frac{1}{\tilde{n}} + \frac{\tilde{\delta}_n^{*2}}{2(n-2)}} &= \pm 1.65 \sqrt{\frac{1}{7.5} + \frac{(0.580)^2}{56}} = (-0.616, 0.616). \\ \hat{\delta}_n^{*s} \pm z_{\beta/2}\sqrt{\frac{1}{\tilde{n}} + \frac{\tilde{\delta}_n^{*2}}{2(n-2)}} &= 0.3284 \pm 1.65 \sqrt{\frac{1}{7.5} + \frac{(0.580)^2}{56}} = (-0.2875, 0.9443). \end{aligned}$$

5 Empirical Bayes Estimators and Comparisons

In this section, we discuss the empirical Bayes methods of effect size estimation due to Efron (2010). Let the effect size, δ, be distributed with density $g(\delta)$ and the transformed variable $z|\delta \sim N(\delta, 1)$ where $z = \Phi^{-1}(F_\nu(t))$ and $\Phi(.)$ is the cdf of standard normal distribution, $F_\nu(.)$ is the cdf of central t-distribution with ν D.F. and $t = \frac{\bar{Y}^E_{n_1} - \bar{Y}^C_{n_2}}{s_p}$. Then, we have the following result:

Theorem (Efron, 2010)

Let $f(z)$ be the marginal density of z for the model described above, given by

$$f(z) = \int_{-\infty}^{\infty} \phi(z-\delta)\ g(\delta)\ d\delta \text{ where } \phi(z) = \frac{1}{\sqrt{2\pi}} exp\{\frac{-z^2}{2}\}.$$

Then, the posterior density of δ given z is

$$g(\delta|z) = e^{z\delta-\psi(z)}[e^{-\delta^2/2}g(\delta)], \text{with } \psi(z) = \log(f(z)/\phi(z)).$$

As a consequence

$$E(\delta|z) = \psi^{'}(z) \text{ and } \mathrm{var}(\delta|z) = \psi^{''}(z),$$

where $\psi^{'}(z)$ and $\psi^{''}(z)$ are the first and second derivatives of $\psi(z)$. Let $\ell(z) = \log(f(z))$, then

$$E(\delta|z) = z + \ell^{'}(z) \ \text{ and } \mathrm{var}(\delta|z) = 1 + \ell^{''}(z).$$

Efron uses the R algorithm Ebay to obtain a smooth estimate $\hat{\ell}(z)$ of $\ell(z)$. Based on these results, Efron produced Table 11.2 of the estimators $\hat{\delta}$ of prostate cancer data (given in his book on page 220). Table 5 gives the computational results corresponding to Efron's Table 11.2. Columns 1-3 are the same as Efron's Table 11.2. Column 4 is our unrestricted estimator it is equal to the preliminary test estimator in this case and both are based on converted z-values to adjusted t-values. The fifth column is our shrinkage estimator and is computed based on our formula. The sixth column is a converted Efron's estimator $\hat{\delta}$ to adjusted t-values for comparison. It is clear from the table that our estimators are always higher in magnitude than Efron's. The last two columns are the estimated variance of our shrinkage estimator and the variance of Efron's estimators, respectively. Our results belong to the case of estimating the noncentrality parameter, δ of a noncentral t-distribution with ν D.F.

It is evident that our estimators are competitive to the empirical Bayes estimators and are easy to compute.

6 Conclusion

In this paper, we introduced two new point estimators, namely, the preliminary test estimator (PTE) and shrinkage estimator (SE) of effect size, and the related 100(1-β)% confidence intervals. The PTE as a point estimator has relative efficiency more than one in the interval $0 \leq \Delta^2 \leq 1$, then drops to a minimum depending on the size of the preliminary test and eventually tending to one as $\Delta^2 \to \infty$. The shrinkage estimator, on the other hand, attains relative efficiency more than one in the interval $0 \leq \Delta^2 \leq \ln 4$, then drops smoothly to a stable minimum (for large sample it is $\frac{\pi}{\pi+2}$).

Table 5: Computation of UE, PTE and SE using prostate cancer data

step	z-value	$\hat{\delta}$	$\tilde{\delta}_n^* = \hat{\delta}_n^{*PT}$	$\hat{\delta}_n^{*s}$	$\hat{\delta}(z)$	$\widehat{var}(\hat{\delta}_n^{*s})$	$\widehat{var}(\hat{\delta})$
1	5.29	4.11	5.6557	4.8578	4.119	0.2030	0.87
2	4.83	3.65	5.1013	4.3034	3.625	0.1725	0.89
3	-4.42	-3.57	-4.6225	-3.8246	-3.5399	0.1487	0.92
4	-4.33	-3.52	-4.5191	-3.7212	-3.4872	0.1439	0.92
5	-4.29	-3.47	-4.4734	-3.6755	-3.435	0.1418	0.93
6	-4.14	-3.30	-4.3029	-3.5050	-3.257	0.1341	0.97
7	4.47	3.24	4.6802	3.8823	3.1943	0.1515	0.91
8	4.40	3.16	4.5995	3.8016	3.1114	0.1476	0.92
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
45	-3.38	-2.23	-3.4618	-2.6639	-2.1679	0.1007	1.18
46	3.57	2.22	3.6687	2.8709	2.1579	0.1083	0.97
47	3.56	2.20	3.6578	2.8599	2.1380	0.1079	0.97
48	3.56	2.20	3.6578	2.8599	2.1380	0.1079	0.97
49	-3.33	-2.15	-3.4076	-2.6097	-2.0883	0.0988	1.19
50	3.51	2.15	3.6032	2.8053	2.0883	0.1058	0.97
51	-3.33	-2.14	-3.4076	-2.6097	-2.0783	0.0988	1.19
52	-3.32	-2.12	-3.3968	-2.5989	-2.0585	0.0984	1.19
53	-3.32	-2.12	-3.3968	-2.5989	-2.0585	0.0984	1.19
54	3.47	2.09	3.5595	2.7617	2.0287	0.1042	0.97
55	3.46	2.09	3.5487	2.7508	2.0287	0.1038	0.97

Further, the shrinkage estimator does not depend on the size of the test as does the PTE. This aspect of the shrinkage estimator together with its high efficiency near the origin may be attractive to practitioners. Further, among the two estimators, the shrinkage estimator is comparable to the empirical Bayes estimator proposed by Efron (2010) and it may have potential use in the large scale inference with micro-array data. With respect to the performance of the fixed-width $100(1-\beta)\%$ confidence intervals based on PTE, the relative efficiency of the $100(1-\beta)\%$ is more than one in the interval $[0, z_{\beta/2}]$, and due to discontinuity, drops drastically to a minimum, then increases to one as Δ goes to $(z_{\beta/2} + z_{\alpha/2})$. Finally, the relative efficiency tends to $\frac{(1-\beta/2)}{1-\beta}$ as $\Delta \to \infty$. On the other hand, the relative efficiency of the confidence interval based on the shrinkage estimator is a continuous function of Δ and has value more than one in a shorter interval near the origin than the PTE, then decreases smoothly to a minimum guaranteed efficiency. The relative efficiency based on the SE near the origin is more than the efficiency of PTE based confidence interval. The discontinuity of the efficiency as a function of Δ for the PTE based confidence interval may have a dampening effect on the practitioners and therefore it may be less attractive than the SE based confidence interval.

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