International Journal of Statistical Sciences *ISSN 1683–5603*
Vol. 11(Special Issue), 2011, pp 159-181

Some Aspects of Inference on a Normal Mean with Known Coefficient of Variation

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[Received December 29, 2010; Revised May 30, 2011; Accepted June 25, 2011]

Abstract

We present results pertinent to estimation of a normal mean μ with known coefficient of variation [CV]. Although unbiased estimators of μ based on both the sample mean and the sample sd have been widely discussed, there is an inherent problem with the sample mean when μ is assumed to be positive. We suggest and introduce a sign correction to the sample mean to rectify this problem. Again, if the population mean varies in an unrestricted parameter space in either direction (excluding the value 0), then the sample sd suffers from natural acceptability as an estimator of the population mean. There again, we suggest a similar sign correction. Next, we provide an unbiased estimator for the mean based on the sample sd, adjusted by the sign function of the sample mean and its variance. This is an exact result. We construct a combined unbiased estimator based on the sample mean and the sign-corrected sample sd. We also establish its asymptotic normality and study its behavior for small samples. Lastly we take up the problem of confidence interval estimation and, following Stein, we address the problem of determination of a two-stage procedure with fixed width as well. A result based on the decomposition of sum of squares is used to provide an elegant solution to this problem.

Keywords and Phrases: Coefficient of Variation, Normal Distribution, Stein's Procedure, Decomposition of Total Sum of Squares.

AMS Classification: Primary 62J02; Secondary 62J20.

1 Introduction and Preliminaries

Estimation of mean of a normal population based on a random sample of size n is easily accomplished by taking the sample mean, which is an unbiased estimator for the population mean. It is always desirable to reduce the effects of sampling variation in estimates. When Coefficient of Variation (CV) is known, there is information on normal mean in the variance of the population. It is natural to search for estimators that utilize this information on the coefficient of variation and to produce an estimator that is better than the conventional estimator, i.e. the sample mean.

There is an extensive literature on this topic. Searls (1964) proposed an estimator, which under the assumption of known CV, is biased but more efficient than $\bar{X}$. Khan (1968) considered the problem of estimating the mean μ based on a sample from the population $N(\mu, \sigma^2)$, where $\sigma = k\mu$, k known, $k > 0$, $\mu > 0$ and obtained the best unbiased estimator in the sense of minimum variance among all linear unbiased estimators in terms of the sample mean and the sample standard deviation. Gleser and Healy (1976) went a step further and obtained the uniformly minimum risk estimator under the squared error loss. Sen (1979) proposed a biased but simple and consistent estimator and proved it to be more efficient than the MVUE among a typical class of unbiased estimators derived by Khan (1968). Soofi and Gokhale (1991) considered the same problem, when the coefficient of variation is known, as a constrained optimization of the Kullback-Leibler discrimination information function. Following Kunte (2000), Guo and Pal (2003) derived the expression for the MLE of μ, assumed to be non-zero, and characterized the class of equivariant estimators under the group of scale and direction transformations.

We start with the set-up of a $N(\mu, \sigma^2)$ population with $\sigma^2 = k^2\mu^2$, k is known coefficient of variation, $k > 0$, $\mu > 0$. However, note that even for $\mu > 0$, there is a nontrivial probability that the sample mean is negative, viz., $P(\bar{X} < 0) = 1 - \Phi(\sqrt{n}/k)$. Therefore, in case $\bar{X} < 0$, its acceptability as an estimator of μ is under question. One way to rectify this would be to use conventional truncated estimator, that is, $\tilde{\mu} = 0$ if $\bar{X} < 0$; $\tilde{\mu} = \bar{X}$ if $\bar{X} > 0$. However, as is well known, this turns out to be a biased estimator. Below we suggest a sign correction to $\bar{X}$ for obtaining an unbiased estimator of μ. Let

$$\hat{\mu}_n = \begin{cases} a_n\bar{X} & \text{if} \quad \bar{X} > 0 \\ b_n\bar{X} & \text{if} \quad \bar{X} < 0 \end{cases} \tag{1}$$

where $b_n < 0 < a_n$, and, set

$$\begin{aligned} s_1(n) &= \tfrac{k}{\sqrt{2\pi n}} \exp\{-\tfrac{n}{2k^2}\} + \Phi(\sqrt{n}/k) \\ s_2(n) &= -\tfrac{k}{\sqrt{2\pi n}} \exp\{-\tfrac{n}{2k^2}\} + (1 - \Phi(\sqrt{n}/k)) \\ t_1(n) &= \tfrac{k}{\sqrt{2\pi n}} \exp\{-\tfrac{n}{2k^2}\} + \tfrac{n+k^2}{n}\Phi(\sqrt{n}/k) \\ t_2(n) &= -\tfrac{k}{\sqrt{2\pi n}} \exp\{-\tfrac{n}{2k^2}\} + \tfrac{n+k^2}{n}(1 - \Phi(\sqrt{n}/k)) \\ a_n &= \tfrac{s_1(n)/t_1(n)}{s_1(n)^2/t_1(n) + s_2(n)^2/t_2(n)} \\ b_n &= \tfrac{s_2(n)/t_2(n)}{s_1(n)^2/t_1(n) + s_2(n)^2/t_2(n)} \end{aligned} \tag{2}$$

In Appendix A, the following results are established. For such choice of a_n and b_n and for every $n \geq 1$,

1. $E(\hat{\mu}_n) = \mu$, $0 < \mu < \infty$,

2. $V(\hat{\mu}_n)$ is the least uniformly in $\mu > 0$, for each n among all such unbiased estimators of the form (1).

It follows that $a_n > 0$, $\lim_{n\to\infty} a_n = 1$, and $b_n < 0$, $\lim_{n\to\infty} b_n = -\infty$.

Remark 1. If $\mu > 0$, unless the sample size n is large, $P(\bar{X} < 0) > 0$. It is thus necessary to do the sign correction only for small values of n. This aspect of restricted parameter space is not pursued further in this paper. Henceforth, the parameter space under discussion is $\Omega^* = \{-\infty < \mu < \infty,\ \mu \neq 0\}$. If $\mu = 0$, X is degenerate at $\mu = 0$.

Two unbiased estimators for μ are given in Sections 2 and 3. Asymptotic results and small sample behaviors of the estimator proposed in Section 3 are given in Section 4. Confidence intervals for μ of fixed sample is discussed in Section 5. A proposed two-stage procedure to construct confidence interval for μ and its comparisons with Stein's procedure are presented in Section 6.

We will skip some derivations for brevity. For technical details, we refer to Zhang (2007: Unpublished doctoral dissertation (Chapter 3), UIC, Chicago).

2 Unbiased Estimator for μ

We start with the setup of $X \sim N(\mu, k^2\mu^2)$. Let $X_1, X_2, \ldots, X_n$ be a random sample from this population with $\bar{X}$ as the sample mean, and $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$ as the sample variance. This time, as explained before, $\bar{X}$ is readily acceptable as an estimator of μ. However, since $E(S) \propto |\mu|$, a sign correction is needed to S towards providing an acceptable and yet unbiased estimator for μ. The following result will be used in the sequel.

Lemma 2.1. *An unbiased estimator of μ for $\mu \in \Omega^*$ in terms of S and the sign of $\bar{X}$ is $h(S, \bar{X}) = \mathrm{sgn}(\bar{X})\frac{A_n S\sqrt{n-1}}{kg(n-1)}$, where $g(n-1) = \frac{\sqrt{2}\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})}$, and $A_n = 1/[2\Phi(\sqrt{n}/k) - 1]$.*

Proof. We know from Johnson (1994) that $\frac{\sqrt{n-1}S}{k|\mu|} \sim \chi(n-1)$, $E(\frac{\sqrt{n-1}S}{k|\mu|}) = g(n-1)$ and hence, $E(S) = \frac{k|\mu|g(n-1)}{\sqrt{n-1}}$. Note further that $\bar{X} \sim N(\mu, k^2\mu^2/n)$ and that $\bar{X}$ and S are independent. We now consider the estimator h defined in the statement of the lemma, i.e., explicitly written,

$$h(S, \bar{X}) = \begin{cases} A_n \frac{S\sqrt{n-1}}{kg(n-1)} & \text{if } \bar{X} > 0 \\ -A_n \frac{S\sqrt{n-1}}{kg(n-1)} & \text{if } \bar{X} < 0 \end{cases} \tag{3}$$

Since $\bar{X}$ and S are independent, if $\mu > 0$

$$\begin{aligned} E[h(S,\bar{X})] &= P\{\bar{X}>0\}[A_n\mu\tfrac{kg(n-1)}{\sqrt{n-1}}\tfrac{\sqrt{n-1}}{kg(n-1)}] + P\{\bar{X}<0\}[(-A_n)\mu\tfrac{kg(n-1)}{\sqrt{n-1}}\tfrac{\sqrt{n-1}}{kg(n-1)}] \\ &= A_n\mu[P\{\bar{X}>0\} - P\{\bar{X}<0\}] \\ &= \mu. \end{aligned} \tag{4}$$

Again, if $\mu < 0$,

$$\begin{aligned} E[h(S,\bar{X})] &= P\{\bar{X}>0\}[(-\mu)A_n\tfrac{kg(n-1)}{\sqrt{n-1}}\tfrac{\sqrt{n-1}}{kg(n-1)} \\ &\quad +P\{\bar{X}<0\}[(-A_n)(-\mu)\tfrac{kg(n-1)}{\sqrt{n-1}}\tfrac{\sqrt{n-1}}{kg(n-1)}] \\ &= A_n\mu[P\{\bar{X}<0\} - P\{\bar{X}>0\}] = \mu. \end{aligned} \tag{5}$$

This establishes the claim. □

Next we compute the variance of $h(S,\bar{X})$. We will conveniently skip the suffix n from A_n below. Note that

$$V[h(S,\bar{X})] = V_1(E_2) + E_1(V_2) \tag{6}$$

But,

$$E_2 = E[h(S,\bar{X})|\bar{X}] = \begin{cases} A|\mu| & \text{if } \bar{X}>0 \\ -A|\mu| & \text{if } \bar{X}<0. \end{cases} \tag{7}$$

Then

$$V_1(E_2) = E\{E[h(S,\bar{X})|\bar{X}]\}^2 - \{E[E[h(S,\bar{X})|\bar{X}]]\}^2 = A^2\mu^2 - \mu^2 = \mu^2(A^2-1). \tag{8}$$

Next,

$$V_2 = V[h(S,\bar{X})|\bar{X}] = V[\text{sgn}(\bar{X})\tfrac{AS\sqrt{n-1}}{kg(n-1)}] = \tfrac{A^2(n-1)}{[kg(n-1)]^2}V(S). \tag{9}$$

Since $\frac{\sqrt{n-1}S}{k|\mu|} \sim \chi(n-1)$, $V(S) = \frac{k^2\mu^2}{n-1}\{(n-1) - [g(n-1)]^2\}$, then

$$V_2 = A^2\mu^2(\frac{n-1}{[g(n-1)]^2} - 1) \tag{10}$$

Since V_2 is a constant, $E_1(V_2) = V_2$. Therefore,

$$V[h(S,\bar{X})] = \mu^2(A^2-1) + A^2\mu^2(\tfrac{n-1}{[g(n-1)]^2} - 1) = \mu^2(\tfrac{A^2(n-1)}{[g(n-1)]^2} - 1). \tag{11}$$

3 Unbiased Estimator for μ in Terms of $h(S,\bar{X})$ and $\bar{X}$

$\bar{X}$ is an unbiased estimator of μ with variance $\frac{k^2\mu^2}{n}$. Another unbiased estimator for μ, $h(S,\bar{X})$, is derived from Section 2. Next we construct the best linear unbiased estimator for μ in terms of $\bar{X}$ and $h(S,\bar{X})$.

Lemma 3.1. *If a random variable X follows normal distribution with mean μ and variance σ^2, then $E|X| = \sigma\sqrt{\frac{2}{\pi}}e^{\frac{-\mu^2}{2\sigma^2}} + \mu[2\Phi(\frac{\mu}{\sigma}) - 1]$.*

Proof.

$$\begin{aligned} E[|X|] &= \int_0^\infty \frac{x}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}\mathrm{d}x + \int_{-\infty}^0 \frac{-x}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}\mathrm{d}x, \qquad (\text{let } \tfrac{x-\mu}{\sigma} = t), \\ &= \int_{-\frac{\mu}{\sigma}}^\infty (\sigma t + \mu)\frac{1}{\sqrt{2\pi}}e^{\frac{-t^2}{2}}\mathrm{d}t + \int_{-\infty}^{-\frac{\mu}{\sigma}} (-\sigma t - \mu)\frac{1}{\sqrt{2\pi}}e^{\frac{-t^2}{2}}\mathrm{d}t \\ &= \sigma\sqrt{\frac{2}{\pi}}e^{\frac{-\mu^2}{2\sigma^2}} + \mu[2\Phi(\tfrac{\mu}{\sigma}) - 1].\Box \end{aligned}$$

The value for $\mu = 0$ follows immediately from the above general result, $E(|X|) = \sigma\sqrt{\frac{2}{\pi}}$. The following result is well-known and is stated without proof.

Lemma 3.2. *Random variables X and Y are dependent with the correlation coefficient ρ. Both have the same mean θ and variances σ_1^2 and σ_2^2, respectively. Then the best linear unbiased estimator of θ in terms of X and Y involving ρ, σ_1^2 and σ_2^2 is $Z = aX + bY$, where $a = \frac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$ and $b = \frac{\sigma_1^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$.*

The following result is now immediate.

Lemma 3.3. *Based on a sample of size n from $N(\mu, k^2\mu^2)$, the best linear unbiased estimator of μ in terms of $\bar{X}$ and $h(S, \bar{X})$ is*

$$\mu^* = c_n h(S, \bar{X}) + (1 - c_n)\bar{X} \tag{12}$$

where $c_n = \dfrac{\frac{k^2}{n} - Ak\sqrt{\frac{2}{\pi n}}e^{-n/2k^2}}{\frac{A^2(n-1)}{[g(n-1)]^2} - 1 + \frac{k^2}{n} - 2Ak\sqrt{\frac{2}{\pi n}}e^{-n/2k^2}}$.

Proof.

$$E[\mu^*] = E[c_n h(S, \bar{X}) + (1 - c_n)\bar{X}] = c_n E[h(S, \bar{X})] + (1 - c_n)E(\bar{X}) = \mu. \tag{13}$$

Further, the covariance between $h(S, \bar{X})$ and $\bar{X}$ is computed as

$$\begin{aligned} \mathrm{cov}[h(S, \bar{X}), \bar{X}] &= E[h(S, \bar{X})\bar{X}] - E[h(S, \bar{X})]E(\bar{X}) \\ &= \int_0^\infty Ak|\mu|\bar{x}f(\bar{x})d\bar{x} + \int_{-\infty}^0 -Ak|\mu|\bar{x}f(\bar{x})d\bar{x} - \mu^2 \\ &= Ak|\mu|E|\bar{X}| - \mu^2 = \mu^2 Ak\sqrt{\frac{2}{\pi n}}e^{-n/2} \qquad \text{(By Lemma 3.1)} \end{aligned} \tag{14}$$

The correlation coefficient between $h(S,\bar{X})$ and $\bar{X}$ is

$$\rho = \frac{\text{cov}[h(S,\bar{X}),\bar{X}]}{\sqrt{V[h(S,\bar{X})]V(\bar{X})}} = \frac{A\sqrt{\frac{2}{\pi}}e^{-n/2}}{\sqrt{\frac{A^2(n-1)}{[g(n-1)]^2}-1}}. \quad (15)$$

The relationship between ρ and sample size n is shown in Figure 1. For a fixed k, as sample size gets larger, the correlation coefficient between $h(S,\bar{X})$ and $\bar{X}$ gets smaller. Furthermore,

$$\lim_{n\to\infty} \text{cov}[h(S,\bar{X}),\bar{X}] = \lim_{n\to\infty} \mu^2 Ak\sqrt{\frac{2}{\pi n}}e^{-n/2} = 0. \quad (16)$$

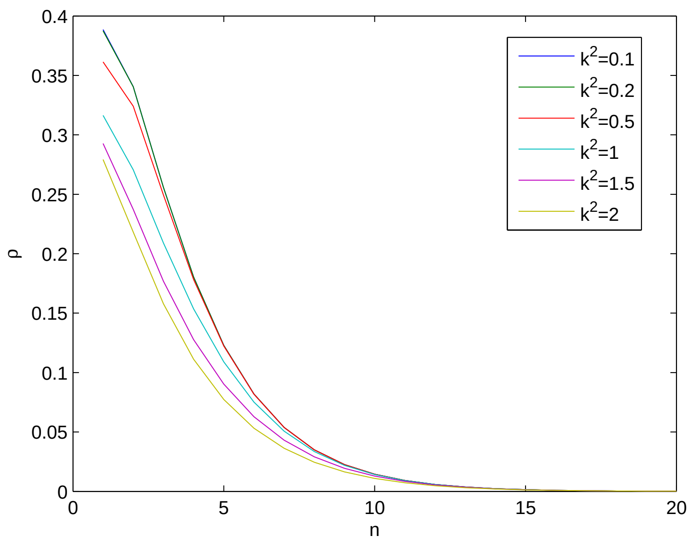

Figure 1: The relationship between ρ and n for different values of k^2

By Lemma 3.2, the coefficient of the best linear unbiased estimator of μ in terms of $\bar{X}$ and $h(S,\bar{X})$ is $c_n = \frac{\frac{k^2}{n} - Ak\sqrt{\frac{2}{\pi n}}e^{-n/2k^2}}{\frac{A^2(n-1)}{[g(n-1)]^2} - 1 + \frac{k^2}{n} - 2Ak\sqrt{\frac{2}{\pi n}}e^{-n/2k^2}}$, upon simplification. $\square$

We may compute the variance of μ^* as

$$V(\mu^*) \quad = c_n^2 V[h(S,\bar{X})] + (1-c_n)^2 V(\bar{X}) + 2c_n(1-c_n)\text{cov}[h(S,\bar{X}),\bar{X}] = v(n)\mu^2 \quad (17)$$

where $v(n) = c_n^2[\frac{A^2(n-1)}{[g(n-1)]^2} - 1] + (1-c_n)^2\frac{k^2}{n} + 2c_n(1-c_n)Ak\sqrt{\frac{2}{\pi n}}e^{-n/2}$.

Lemma 3.4. *(Anis, 2008)*

1. $\lim_{n\to\infty} \alpha_n = 1,\ \lim_{n\to\infty} n(1-\alpha_n) = \frac{1}{4},\ \lim_{n\to\infty} n(1-\alpha_n^2) = \frac{1}{2}.$

where $\alpha_n = \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})}\sqrt{\frac{2}{n-1}} = \frac{g(n-1)}{\sqrt{n-1}}$.

The ratio of the coefficients of the best linear unbiased estimator of μ in terms of $h(S,\bar{X})$ and $\bar{X}$ is

$$\frac{c_n}{1-c_n} = \frac{\frac{k^2}{n} - Ak\sqrt{\frac{2}{\pi n}}e^{-n/2k^2}}{\frac{A^2(n-1)}{[g(n-1)]^2} - 1 - Ak\sqrt{\frac{2}{\pi n}}e^{-n/2k^2}} \tag{18}$$

As $n \to \infty, \frac{c_n}{1-c_n} \to 2k^2$. The proof is given in the following. By Lemma 3.4 (1) and (3),

$$\lim_{n\to\infty}\frac{c_n}{1-c_n} = \lim_{n\to\infty}\frac{\frac{k^2}{n} - Ak\sqrt{\frac{2}{\pi n}}e^{-n/2}}{\frac{A^2(n-1)}{[g(n-1)]^2} - 1 - Ak\sqrt{\frac{2}{\pi n}}e^{-n/2}} = \lim_{n\to\infty}\frac{k^2\alpha_n^2}{n(1-\alpha_n^2)} = 2k^2. \quad (19)$$

4 Asymptotic Normality and Small Sample Behavior of The BLUE

As we discuss in Section 3, $\mu^* = c_n h(S,\bar{X}) + (1-c_n)\bar{X}$ is the best linear unbiased estimator(BLUE) for μ. It is natural to consider the properties of this estimator. Then we could base statistical inference on it. First note that μ^* is asymptotically normally distributed as proved in Appendix B.

Since $\lim_{n\to\infty} \mathrm{cov}[h(S,\bar{X}),\bar{X}] = 0$ and $V[h(S,\bar{X})] = \frac{\mu^2}{2n} + O(\frac{1}{n^2})$, it is asymptotically normal with asymptotic variance $\frac{\mu^2}{2n}$. Then the asymptotic variance of μ^* is $\frac{k^2\mu^2}{n(1+2k^2)}$. The variance ratio of $h(S,\bar{X})]$ and $\bar{X}$ is $\frac{1}{1+2k^2}$ asymptotically. As for its behavior of small samples, it is examined in Table 1. Further, it is shown in Appendix C that $P\{|\frac{\mu^*-\mu}{\sqrt{V(\mu^*)}}| \le z\}$ is independent of μ. Then in Table 1, $\mu = 1$ is taken for illustration. 10,000 samples from $N(\mu, k^2\mu^2)$ are drawn for small and moderate sample size for $k^2 = 0.1, 1,$ and 2. Vide, Zhang (2007) for other values of k. When $k \le 1$, the simulated probabilities are close to the probabilities according to the standard normal distribution. As for higher values of z which are often used in constructing confidence intervals and hypothesis testings, $P\{|\frac{\mu^*-\mu}{\sqrt{V(\mu^*)}}| \le z\}$ are very close to the probability according to the standard normal distribution even for small sample size. When $k = 2$, the $P\{|\frac{\mu^*-\mu}{\sqrt{V(\mu^*)}}| \le z\}$ is very close to the probabilities according to the standard normal distribution for large value of $n(\ge 50)$.

Table 1: $P\{|\frac{\mu^*-\mu}{\sqrt{V(\mu^*)}}| \leq z\}$, $k^2 = 0.1$

z	$n = 5$	10	15	20	25	30	$2\Phi(z) - 1$
0.5	0.378	0.384	0.386	0.381	0.375	0.379	0.383
1	0.679	0.685	0.681	0.687	0.675	0.676	0.683
1.5	0.863	0.864	0.867	0.871	0.860	0.865	0.866
1.96	0.948	0.949	0.950	0.950	0.947	0.946	0.950
2	0.952	0.953	0.954	0.955	0.951	0.951	0.954
2.5	0.988	0.986	0.987	0.988	0.987	0.986	0.988
2.58	0.990	0.989	0.989	0.990	0.990	0.989	0.990
3	0.997	0.997	0.998	0.998	0.997	0.997	0.997

$P\{|\frac{\mu^*-\mu}{\sqrt{V(\mu^*)}}| \leq z\}$, $k^2 = 1$

z	$n = 5$	10	15	20	25	30	$2\Phi(z) - 1$
0.5	0.431	0.391	0.387	0.381	0.383	0.384	0.383
1	0.746	0.696	0.687	0.678	0.685	0.689	0.683
1.5	0.910	0.877	0.872	0.860	0.867	0.869	0.866
1.96	0.968	0.958	0.952	0.952	0.947	0.950	0.950
2	0.970	0.960	0.957	0.951	0.954	0.954	0.955
2.5	0.984	0.988	0.987	0.987	0.990	0.989	0.988
2.58	0.985	0.992	0.990	0.992	0.991	0.991	0.990
3	0.988	0.997	0.997	0.997	0.998	0.997	0.997

$P\{|\frac{\mu^*-\mu}{\sqrt{V(\mu^*)}}| \leq z\}$, $k^2 = 2$

z	$n = 20$	30	40	50	60	70	$2\Phi(z) - 1$
0.5	0.503	0.461	0.426	0.395	0.382	0.383	0.383
1	0.823	0.781	0.729	0.704	0.685	0.686	0.683
1.5	0.950	0.932	0.901	0.883	0.868	0.868	0.866
1.96	0.982	0.981	0.967	0.959	0.953	0.947	0.950
2	0.982	0.982	0.971	0.963	0.957	0.951	0.954
2.5	0.987	0.995	0.993	0.991	0.989	0.987	0.988
2.58	0.987	0.995	0.995	0.993	0.992	0.990	0.990
3	0.987	0.997	0.998	0.998	0.997	0.996	0.997

5 Confidence Interval for μ for A Fixed Sample

5.1 The Estimators under Study

Traditionally, the confidence interval for the mean μ of a normal distribution is constructed based on t distribution with degrees of freedom $n-1$, that is,

$$\bar{X} \pm \frac{S}{\sqrt{n}} t(1-\alpha/2, n-1). \tag{20}$$

where $t(1-\alpha/2, n-1)$ is the lower $1-\alpha/2$ percentage point of the t distribution of $n-1$ degrees of freedom. The length of this confidence interval is

$$L_1 = 2t(1-\alpha/2, n-1)\frac{1}{\sqrt{n}}S. \tag{21}$$

The expected length of this confidence interval is

$$E(L_1) = 2t(1-\alpha/2, n-1)\frac{1}{\sqrt{n}}\frac{g(n-1)}{\sqrt{n-1}}k|\mu|. \tag{22}$$

When the sample is from $N(\mu, k^2\mu^2)$, L_1 does not utilize the information of μ in the population variance. Another statistic $\frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{k^2\mu^2}$ utilizes this information and it follows χ^2 distribution with degrees of freedom $n-1$. Based on this statistic, the confidence interval for $|\mu|$ is

$$\left(\frac{\sqrt{n-1}S}{k\sqrt{\chi^2(1-\alpha/2, n-1)}}, \frac{\sqrt{n-1}S}{k\sqrt{\chi^2(\alpha/2, n-1)}}\right) \tag{23}$$

where $\chi^2(\alpha, m)$ is the lower α percentage point of the χ^2 distribution of m degrees of freedom. An adjustment for the confidence interval of μ is, if $\bar{X} > 0$,

$$\frac{1}{k\Phi(\sqrt{n}/k)}\left(\frac{\sqrt{n-1}S}{\sqrt{\chi^2(1-\alpha/2, n-1)}}, \frac{\sqrt{n-1}S}{\sqrt{\chi^2(\alpha/2, n-1)}}\right); \tag{24}$$

and if $\bar{X} < 0$,

$$\frac{1}{k\Phi(\sqrt{n}/k)}\left(-\frac{\sqrt{n-1}S}{\sqrt{\chi^2(\alpha/2, n-1)}}, -\frac{\sqrt{n-1}S}{\sqrt{\chi^2(1-\alpha/2, n-1)}}\right). \tag{25}$$

Next we justify that this proposed confidence interval provides $100(1-\alpha)\%$ confidence.

If $\mu > 0$, confidence coefficient is computed as

$$\begin{aligned} & \frac{1}{k\Phi(\sqrt{n}/k)} P(\bar{X} > 0) P(\frac{\sqrt{n-1}S}{\sqrt{\chi^2(1-\alpha/2, n-1)}} < \mu < \frac{\sqrt{n-1}S}{\sqrt{\chi^2(\alpha/2, n-1)}}) \\ & + \frac{1}{k\Phi(\sqrt{n}/k)} P(\bar{X} < 0) P(-\frac{\sqrt{n-1}S}{\sqrt{\chi^2(\alpha/2, n-1)}} < \mu < -\frac{\sqrt{n-1}S}{\sqrt{\chi^2(1-\alpha/2, n-1)}}) \\ = & \frac{1}{\Phi(\sqrt{n}/k)} P(\bar{X} > 0) P(\sqrt{\chi^2(\alpha/2, n-1)} < \frac{\sqrt{n-1}S}{k\mu} < \sqrt{\chi^2(1-\alpha/2, n-1)}) \\ = & \frac{1}{\Phi(\sqrt{n}/k)} P(\bar{X} > 0)(1-\alpha) = 1-\alpha. \end{aligned} \tag{26}$$

Again, if $\mu < 0$, the same is computed as

$$\begin{aligned} & \frac{1}{k\Phi(\sqrt{n}/k)} P(\bar{X} > 0) P(\frac{\sqrt{n-1}S}{\sqrt{\chi^2(1-\alpha/2, n-1)}} < \mu < \frac{\sqrt{n-1}S}{\sqrt{\chi^2(\alpha/2, n-1)}}) \\ & + \frac{1}{k\Phi(\sqrt{n}/k)} P(\bar{X} < 0)(-\frac{\sqrt{n-1}S}{\sqrt{\chi^2(\alpha/2, n-1)}} < \mu < -\frac{\sqrt{n-1}S}{\sqrt{\chi^2(1-\alpha/2, n-1)}}) \end{aligned} \tag{27}$$

Interestingly enough, the length of this confidence interval is the same for $\mu > 0$ or for $\mu < 0$ and is given by

$$L_2 = \frac{1}{k\Phi(\sqrt{n}/k)} (\frac{\sqrt{n-1}S}{\sqrt{\chi^2(\alpha/2, n-1)}} - \frac{\sqrt{n-1}S}{\sqrt{\chi^2(1-\alpha/2, n-1)}}). \tag{28}$$

The expected length of this confidence interval is

$$E(L_2) = \frac{g(n-1)}{\Phi(\sqrt{n}/k)} (\frac{1}{\sqrt{\chi^2(\alpha/2, n-1)}} - \frac{1}{\sqrt{\chi^2(1-\alpha/2, n-1)}})|\mu|. \tag{29}$$

Now we compare the expected lengths of the confidence intervals based on t distribution and the one based on χ^2 distribution respectively. Define,

$$R_{21}(n) = \frac{E(L_1)}{E(L_2)} = \frac{2kt(1-\alpha/2, n-1)}{\frac{\sqrt{n(n-1)}}{\Phi(\sqrt{n}/k)} (\frac{1}{\sqrt{\chi^2(\alpha/2, n-1)}} - \frac{1}{\sqrt{\chi^2(1-\alpha/2, n-1)}})} \tag{30}$$

From Johnson (1994), by Fisher (1922)'s approximation,

$$\chi^2(\alpha, \nu) \simeq \frac{1}{2}(U_\alpha + \sqrt{2\nu - 1})^2 \tag{31}$$

Table 2: R_{21} for small and moderate sample size, $\alpha = 0.05$

n	$k^2 = 0.1$	0.2	0.5	1	1.5	2
5	0.345	0.488	0.771	1.078	1.292	1.456
10	0.398	0.562	0.889	1.256	1.532	1.756
15	0.414	0.586	0.927	1.311	1.604	1.848
20	0.423	0.598	0.945	1.337	1.637	1.889
25	0.428	0.605	0.956	1.353	1.657	1.913
30	0.431	0.610	0.964	1.363	1.669	1.927

where U_α is the lower α percentage point of the standard normal distribution. ν is the degrees of freedom of the χ^2 distribution. By Peiser(1943),

$$t(\alpha, \nu) \simeq U_\alpha + \frac{U_\alpha^3 + U_\alpha}{4\nu} \tag{32}$$

$$\begin{aligned}
\lim_{n\to\infty} R_{21}(n) &= \lim_{n\to\infty} \frac{2kt(1-\alpha/2, n-1)}{\frac{\sqrt{n(n-1)}}{\Phi(\sqrt{n}/k)}\left(\frac{1}{\sqrt{\chi^2(\alpha/2,n-1)}} - \frac{1}{\sqrt{\chi^2(1-\alpha/2,n-1)}}\right)} \\
&= \lim_{n\to\infty} \frac{k\left(1 + \frac{U_{1-\alpha/2}^2+1}{4(n-1)}\right)}{\frac{1}{\sqrt{\frac{2(n-1)}{n}} - \frac{1}{\sqrt{2n(n-1)}} - \frac{U_{1-\alpha/2}^2}{\sqrt{2n(n-1)}})}} \\
&= \sqrt{2}k
\end{aligned} \tag{33}$$

So the expected length of L_1 is $\sqrt{2}k$ of the expected length of L_2 asymptotically. L_2 has shorter length than L_1 even for small and moderate sample size n and $k > 1$ as shown in Table 2.

5.2 An Unbiased Estimator for μ: a New Perspective

Now we consider another unbiased estimator for μ based on the sum of squares decomposition. The total sample of size n is divided into two parts with n_0 and n_1 observations, respectively. Let

$$\begin{aligned}
SS_n &= \textstyle\sum_{i=1}^n (X_i - \bar{X}_n)^2, & SS_0 &= \textstyle\sum_{i=1}^{n_0} (X_i - \bar{X}_0)^2, \\
W &= \textstyle\sum_{i=n_0+1}^n (X_i - \bar{X}_1)^2 + \frac{n_0 n_1}{n}(\bar{X}_0 - \bar{X}_1)^2, & \bar{X}_n &= \textstyle\sum_{i=1}^n X_i/n, \\
\bar{X}_0 &= \textstyle\sum_{i=1}^{n_0} X_i/n_0, & \bar{X}_1 &= \textstyle\sum_{i=n_0+1}^n X_i/n_1, \\
n &= n_0 + n_1.
\end{aligned} \tag{34}$$

The sum of squares of the whole sample, SS_n is decomposed as $SS_n = SS_0 + W$. Then

$$\frac{SS_n}{k^2\mu^2} \sim \chi^2(n-1), \ \frac{SS_0}{k^2\mu^2} \sim \chi^2(n_0-1), \ \frac{W}{k^2\mu^2} \sim \chi^2(n_1). \tag{35}$$

It is easy to see that SS_0 and W are independent; $\bar{X}_n$ and W are independent. We now propose an unbiased estimator for μ in terms of $\bar{X}_n$ and W, viz.,

$$h(W, \bar{X}_n) = \frac{\text{sgn}(\bar{X}_n) A\sqrt{W}}{kg(n_1)} \tag{36}$$

where $g(n_1) = \frac{\sqrt{2}\Gamma(\frac{n_1+1}{2})}{\Gamma(\frac{n_1}{2})}$, and $A = 1/[2\Phi(\sqrt{n}/k) - 1]$. By Lemma 3.4, it is an unbiased estimator for μ . By Lemma 3.3, the best linear unbiased estimator of μ in terms of $\bar{X}_n$ and $h(W, \bar{X}_n)$ is

$$\hat{\mu}^* = d_n h(W, \bar{X}_n) + (1 - d_n)\bar{X}_n \tag{37}$$

where $d_n = \frac{\frac{k^2}{n} - Ak\sqrt{\frac{2}{\pi n}}e^{-n/2}}{\frac{A^2 n_1}{[g(n_1)]^2} - 1 + \frac{k^2}{n} - 2Ak\sqrt{\frac{2}{\pi n}}e^{-n/2}}$. Its variance is

$$\begin{aligned} V(\hat{\mu}^*) &= d_n^2 V[h(W, \bar{X}_n)] + (1 - d_n)^2 V(\bar{X}_n) + 2d_n(1 - d_n)Cov[\bar{X}_n, h(W, \bar{X}_n)] \\ &= v(n, n_1)\mu^2 \end{aligned} \tag{38}$$

where $v(n, n_1) = d_n^2[\frac{A^2 n_1}{[g(n_1)]^2} - 1] + (1 - d_n)^2 \frac{k^2}{n} + 2d_n(1 - d_n)Ak\sqrt{\frac{2}{\pi n}}e^{-n/2}$.

5.3 Confidence Interval for μ Based on $\hat{\mu}^*$

For given n_0, n_1 goes to the infinity as n goes to the infinity. Therefore, $\hat{\mu}^*$ is asymptotically normally distributed as μ^* in Section 3. Since W and SS_0 along with $\bar{X}_n$ and SS_0 are independent,

$$t_{\hat{\mu}^*} = \frac{\frac{\hat{\mu}^* - \mu}{\sqrt{v(n, n_1)\mu^2}}}{\sqrt{\frac{SS_0}{k^2\mu^2(n_0 - 1)}}} = \frac{(\hat{\mu}^* - \mu)k}{\sqrt{v(n, n_1)}S_0} \tag{39}$$

follows asymptotically t distribution with degrees of freedom $n_0 - 1$, where $S_0 = \sqrt{\frac{SS_0}{n_0 - 1}}$. The confidence interval based on $t_{\hat{\mu}^*}$ is thus given by

$$\hat{\mu}^* \pm t(1 - \alpha/2, n_0 - 1)\sqrt{v(n, n_1)}\frac{S_0}{k}. \tag{40}$$

The expected length of this confidence interval is

$$E(L_3) = 2t(1 - \alpha/2, n_0 - 1)\sqrt{v(n, n_1)}g(n_0 - 1)/\sqrt{n_0 - 1}|\mu|. \tag{41}$$

To compare expected length of confidence intervals (22), (29) and (41), define

$$R_{13}(n) = \frac{E(L_3)}{E(L_1)} = \frac{t(1-\alpha/2, n_0-1)\sqrt{v(n, n_1)}/kg(n_0-1)/\sqrt{n_0-1}}{t(1-\alpha/2, n-1)\frac{1}{\sqrt{n}}g(n-1)/\sqrt{n-1}} \tag{42}$$

Table 3: R_{13}

	$k^2=0.1$		$k^2=0.2$		$k^2=0.5$		$k^2=1$		$k^2=1.5$		$k^2=2$	
n	n_0	R_{13}	n_0	R_{13}	n_0	R_{13}	n_0	R_{13}	n_0	R_{13}	n_0	R_{13}
10	9	1.008	9	1.000	7	0.948	6	0.849	6	0.797	6	0.782
15	14	1.000	11	0.983	8	0.900	7	0.782	7	0.708	7	0.676
20	16	0.996	12	0.969	9	0.872	8	0.749	8	0.667	8	0.619
25	18	0.991	14	0.958	10	0.854	9	0.728	9	0.644	8	0.588
30	19	0.987	15	0.950	11	0.841	10	0.713	9	0.629	9	0.570
40	22	0.978	17	0.937	13	0.822	11	0.693	11	0.609	10	0.549
50	25	0.974	19	0.928	15	0.809	13	0.679	12	0.596	12	0.537
60	27	0.970	21	0.927	16	0.800	14	0.669	13	0.586	13	0.528
70	29	0.967	22	0.916	17	0.793	15	0.662	14	0.579	14	0.521

Table 4: R_{23}

	$k^2=0.1$		$k^2=0.2$		$k^2=0.5$		$k^2=1$		$k^2=1.5$		$k^2=2$	
n	n_0	R_{23}	n_0	R_{23}	n_0	R_{23}	n_0	R_{23}	n_0	R_{23}	n_0	R_{23}
10	9	0.401	9	0.562	7	0.843	6	1.067	6	1.222	6	1.373
15	14	0.415	11	0.576	8	0.834	7	1.025	7	1.136	7	1.250
20	16	0.421	12	0.579	9	0.825	8	1.001	8	1.092	8	1.170
25	18	0.424	14	0.580	10	0.817	9	0.984	9	1.067	8	1.125
30	19	0.425	15	0.579	11	0.810	10	0.972	9	1.049	9	1.098
40	22	0.426	17	0.577	13	0.800	11	0.953	11	1.026	10	1.069
50	25	0.426	19	0.575	15	0.792	13	0.940	12	1.009	12	1.050
60	27	0.426	21	0.572	16	0.786	14	0.929	13	0.997	13	1.037
70	29	0.426	22	0.571	17	0.781	15	0.922	14	0.988	14	1.026

and

$$R_{23}(n) = \frac{E(L_3)}{E(L_2)} = \frac{2t(1-\alpha/2, n_0-1)g(n_0-1)/\sqrt{n_0-1}\sqrt{v(n,n_1)}}{\frac{g(n-1)}{\Phi(\sqrt{n}/k)}\left(\frac{1}{\sqrt{\chi^2(\alpha/2,n-1)}}-\frac{1}{\sqrt{\chi^2(1-\alpha/2,n-1)}}\right)} \tag{43}$$

Tables 3 and 4 show R_{13} and R_{23} for different values of n. The value of n_0 for each fixed n give the smallest value of R_{13} and R_{23}. The expected length of L_3 is less than or very close to the expected length of L_1 for different values of coefficient of variation. When $k^2 < 1$, the expected length of L_3 is shorter than that of L_2. When $k^2 > 1$, the expected length of L_3 is longer than that of L_2. When $k^2 = 1$, $n \geq 25$, the expected length of L_3 is shorter than that of L_2; $n < 25$, the expected length of L_3 is close to that of L_2. Therefore, according to the expected length, L_2 is recommended to construct the $100(1-\alpha)\%$ confidence interval for μ when k^2 is greater than 1; L_3 is recommended to construct the $100(1-\alpha)\%$ confidence interval for μ when k^2 is less than or equal to 1 and sample size is large; when k^2 is extremely small, there is trivial difference between L_1 and L_3. From the perspective of simplicity, L_1 is recommended.

Another statistic $\frac{\sum_{i=1}^{n}(X_i-\mu)^2}{k^2\mu^2}$ follows χ^2 distribution with degrees of freedom n. The confidence interval based on this statistics cannot be expressed in a closed form to give rise to a closed interval and the quadratic equations derived for the confidence interval may not have real roots. It cannot be used to construct the confidence interval for μ. Another reason that $\frac{\sum_{i=1}^{n}(X_i-\mu)^2}{k^2\mu^2}$ is not utilized is that there is trivial difference between degrees of freedom for n and $n-1$ as $n \to \infty$.

6 Fixed Width Confidence Interval for μ

6.1 Stein's Procedure

Stein (1945) proposed a two-stage procedure to construct a fixed width ($2d$) confidence interval for μ. It is also summarized in Ghosh, Mukhopadhyay, and Sen (1997). Let $X_i (i = 1, 2, \cdots)$ be independent normal variables with mean μ and variance σ^2 (unknown). Take a sample of size n_0 observations $X_1, \cdots, X_{n_0}$ and compute the sample variance give by

$$S_0^2 = \frac{1}{n_0}\sum_{i=1}^{n_0}(X_i - \bar{X}_0)^2. \tag{44}$$

Take additional observations $X_{n_0+1}, \cdots, X_n$, and let $\bar{X}_n = \frac{\sum_{i=1}^{n} Xi}{n}$. Then

$$T' = \frac{(\bar{X}_n - \mu)\sqrt{n}}{\sqrt{S_0^2}} \tag{45}$$

follows t distribution with degrees of freedom $n_0 - 1$. The total sample size n is chosen as

$$n = \max\{[\frac{t^2(1-\alpha/2, n_0-1)S_0^2}{d^2}] + 1, n_0\} \tag{46}$$

A $100(1-\alpha)\%$ confidence interval for μ of specified length $2d$ is then given by

$$(\bar{X} - d, \bar{X} + d) \tag{47}$$

6.2 Proposed Procedure

When a $100(1-\alpha)\%$ confidence interval is constructed for μ based on a sample from $N(\mu, k^2\mu^2)$, the information on μ in the variance can be utilized. As discussed in Section 5.3, $t_{\hat{\mu}^*}$ in Equation (39) follows asymptotic t distribution with degrees of freedom $n_0 - 1$. The confidence interval based on $\hat{\mu}^*$ is

$$\hat{\mu}^* \pm t(1-\alpha/2, n_0-1)\sqrt{v(n, n_1)}\frac{S_0}{k} \tag{48}$$

Similarly to the Stein's procedure, a two-stage procedure to construct a fixed width ($2d$) confidence interval is based on Equation (48). The total sample size is chosen as follows: if $n_0 \geq [\frac{t(1-\alpha/2,n_0-1)S_0}{d}]^2$, then no more observations will be taken, the total sample size is $n = n_0$; otherwise, the total sample size n is chosen such that

$$d = t(1-\alpha/2, n_0 - 1)\sqrt{v(n, n_1)}\frac{S_0}{k} \tag{49}$$

holds. The confidence interval for μ based on $\hat{\mu}^*$ is

$$(\hat{\mu}^* - d, \hat{\mu}^* + d) \tag{50}$$

From Seelbinder(1953), the expected sample size $E_s(n)$ of Stein's procedure is

$$E_s(n) = [n_0 - \frac{t^2(1-\alpha/2, n_0-1)}{c^2}]F(\chi_0^2) + \frac{t^2(1-\alpha/2, n_0-1)}{c^2}(1+K) \tag{51}$$

where,

$$\begin{aligned}
c &= \tfrac{d}{k\mu}, & \chi_0^2 &= \tfrac{c^2 n_0(n_0-1)}{t^2(1-\alpha/2,n_0-1)},\\
F(\chi_0^2) &= I(\tfrac{\chi_0^2}{\sqrt{2(n_0-1)}}, (\tfrac{n_0-1}{2}-1)), & I(u,p) &= \int_0^{u\sqrt{p+1}} \tfrac{e^{-\nu}\nu^p}{\Gamma(p+1)}\mathrm{d}\nu,\\
\nu &= \tfrac{x^2}{2}, & x &\sim \chi^2(n_0-1),\\
p &= \tfrac{n_0-1}{2} - 1, & u &= \tfrac{\chi_0^2}{\sqrt{2(n_0-1)}},\\
K &= \frac{(\frac{\chi_0^2}{2})^{(n_0-1)/2}}{\frac{n_0-1}{2}\Gamma(\frac{n_0-1}{2})\exp(\frac{\chi_0^2}{2})}.
\end{aligned}$$

Procedures (47) and (50) construct the $100(1-\alpha)\%$ confidence intervals of same width $2d$. Then the relationship between the expected sample sizes of Stein's procedure $E_s(n)$ and proposed procedure $E_p(n)$ is

$$v(E_p(n), n_1) = \frac{k^2}{E_s(n)} \tag{52}$$

The comparison of the expected sample sizes of the two procedures for $\alpha = 0.05$ is given in Table 5. The blank space left thereafter in the table of expected sample sizes will remain equal to n_0. As the asymptotic normality and small sample behavior are discussed in Section 4, n_0 is assumed to take values larger or equal to 50 for $k^2 = 2$. n_0 starts from 10 for other values of k^2. For given n_0, the expected sample size from the proposed procedure is less than that from Stein's procedure. For the moderate $c(\geq 0.2)$, as n_0 gets bigger, the expected sample sizes of the two procedures are the same eventually. When c takes small values, say 0.01, or 0.05, there is huge expected sample size gain by the proposed procedure.

Table 5: Expected Sample Size Comparison of the Two Procedures

k^2=0.1

	$c = 0.01$		$c = 0.05$		$c = 0.1$		$c = 0.2$		$c = 0.3$		$c = 0.4$		$c = 0.5$		$c = 0.6$		$c = 0.7$		$c = 0.8$	
n_0	$^*E_s$	$^*E_p$	E_s	E_p	E_s	E_p	E_s	E_p	E_s	E_p	E_s	E_p	E_s	E_p	E_s	E_p	E_s	E_p	E_s	E_p
10	51174	354	2047	354	512	354	128	109	57	50	32	29	21	20						
20	43807	364	1752	364	438	364	110	95	49	44	28	27								
30	41830	374	1673	374	418	354	105	93	47	45										
40	40913	384	1637	384	409	348	102	92												
50	40384	394	1615	394	404	345	101	93												
60	40040	404	1602	404	400	344	100	94												
70	39798	414	1592	414	398	344	100	95												
80	39619	424	1585	424	396	344	100	97												
90	39481	434	1579	434	395	345	101	100												
100	39371	444	1575	444	394	345														
120	39208	464	1568	464	392	347														
240	38807	584	1552	584	388	364														

$^*E_s = E_s(n), ^*E_p = E_p(n)$

k^2=1

	$c = 0.01$		$c = 0.05$		$c = 0.1$		$c = 0.2$		$c = 0.3$		$c = 0.4$		$c = 0.5$		$c = 0.6$		$c = 0.7$		$c = 0.8$	
n_0	E_s	E_p	E_s	E_p	E_s	E_p	E_s	E_p	E_s	E_p	E_s	E_p	E_s	E_p	E_s	E_p	E_s	E_p	E_s	E_p
10	51174	354	2047	354	512	178	128	50	57	26	32	18	21	14	15	12	12	11		
20	43807	364	1752	364	438	160	110	51	49	30	28	23								
30	41830	374	1673	374	418	160	105	56	47	36										
40	40913	384	1637	384	409	164	102	61	47	43										
50	40384	394	1615	394	404	169	101	68	52	51										
60	40040	404	1602	404	400	174	100	74												
70	39798	414	1592	414	398	180	100	81												
80	39619	424	1585	424	396	186	100	87												
90	39481	434	1579	434	395	192	101	94												
100	39371	444	1575	444	394	199	105	102												
120	39208	464	1568	464	392	211														
240	38807	584	1552	584	388	290														

k^2=2

	$c = 0.01$		$c = 0.05$		$c = 0.1$		$c = 0.2$		$c = 0.3$		$c = 0.4$		$c = 0.5$		$c = 0.6$		$c = 0.7$		$c = 0.8$	
n_0	E_s	E_p	E_s	E_p	E_s	E_p	E_s	E_p	E_s	E_p	E_s	E_p	E_s	E_p	E_s	E_p	E_s	E_p	E_s	E_p
50	40384	394	1615	364	404	121	101	61	52	51										
60	40040	404	1602	369	400	129	100	69												
70	39798	414	1592	375	398	136	100	77												
80	39619	424	1585	382	396	144	100	85												
90	39481	434	1579	389	395	152	101	93												
100	39371	444	1575	396	394	159	105	102												
120	39208	464	1568	410	392	175														
240	38807	584	1552	503	388	270														

Acknowledgement

The authors sincerely thank Professor Nitis Mukhophadhyay for useful comments on an earlier version of the manuscript. The comments by two anonymous referees are also gratefully acknowledged.

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Appendix A

Sign correction of unbiased estimator $\bar{X}$ when $X \sim N(\mu, k^2\mu^2)$, $\mu > 0$.

We will conveniently drop the suffix n from a_n, b_n, $s_1(n)$, $s_2(n)$, $t_1(n)$, and $t_2(n)$ in (2). Let

$$\hat{\mu} = \begin{cases} a\bar{X} & \text{if } \bar{X} > 0 \\ b\bar{X} & \text{if } \bar{X} < 0 \end{cases} \tag{A.1}$$

be an unbiased estimator of μ, where $b < 0 < a$. First note that,

$$\begin{aligned} E(\hat{\mu}) &= a\int_0^\infty \bar{x}\frac{1}{\sqrt{2\pi k^2\mu^2/n}}\exp\{-\frac{(\bar{x}-\mu)^2}{2k^2\mu^2/n}\}\mathrm{d}\bar{x} \\ &+b\int_{-\infty}^0 \bar{x}\frac{1}{\sqrt{2\pi k^2\mu^2/n}}\exp\{-\frac{(\bar{x}-\mu)^2}{2k^2\mu^2/n}\}\mathrm{d}\bar{x} \end{aligned} \tag{A.2}$$

$$\begin{aligned} E(\hat{\mu}^2) &= a\int_0^\infty \bar{x}^2\frac{1}{\sqrt{2\pi k^2\mu^2/n}}\exp\{-\frac{(\bar{x}-\mu)^2}{2k^2\mu^2/n}\}\mathrm{d}\bar{x} \\ &+b\int_{-\infty}^0 \bar{x}^2\frac{1}{\sqrt{2\pi k^2\mu^2/n}}\exp\{-\frac{(\bar{x}-\mu)^2}{2k^2\mu^2/n}\}\mathrm{d}\bar{x} \end{aligned} \tag{A.3}$$

Upon substituting $u = \frac{(\bar{x}-\mu)\sqrt{n}}{k\mu}$, we obtain

$$\begin{aligned} E(\hat{\mu}) &= a[\int_{-\sqrt{n}/k}^{\infty} u\frac{k\mu}{\sqrt{n}}\frac{1}{\sqrt{2\pi}}\exp\{-\frac{u^2}{2}\}\mathrm{d}u + \mu\Phi(\sqrt{n}/k)] \\ &\quad +b[\int_{-\infty}^{-\sqrt{n}/k} u\frac{k\mu}{\sqrt{n}}\frac{1}{\sqrt{2\pi}}\exp\{-\frac{u^2}{2}\}\mathrm{d}u + \mu(1-\Phi(\sqrt{n}/k))] \\ &= \mu a[\frac{k}{\sqrt{2\pi n}}\exp\{-\frac{n}{2k^2}\} + \Phi(\sqrt{n}/k)] + b\mu[-\frac{k}{\sqrt{2\pi n}}\exp\{-\frac{n}{2k^2}\} + (1-\Phi(\sqrt{n}/k))] \\ &= \mu(as_1 + bs_2) \end{aligned} \tag{A.4}$$

Therefore, the choices of a and b are restricted to $as_1 + bs_2 = 1$ such that $b < 0 < a$, where s_1 and s_2 are as defined in (2). Next, we compute

$$\begin{aligned}
V(\hat{\mu}) &= E(\hat{\mu}^2) - \mu^2 \\
&= a^2[\int_0^{\infty} [(\bar{x}-\mu)^2 + 2(\bar{x}-\mu)\mu + \mu^2] \frac{1}{\sqrt{2\pi k^2\mu^2/n}} \exp\{-\frac{(\bar{x}-\mu)^2}{2k^2\mu^2/n}\} d\bar{x}] \\
&\quad + b^2[\int_{-\infty}^{0} [(\bar{x}-\mu)^2 + 2(\bar{x}-\mu)\mu + \mu^2] \frac{1}{\sqrt{2\pi k^2\mu^2/n}} \exp\{-\frac{(\bar{x}-\mu)^2}{2k^2\mu^2/n}\} d\bar{x}] - \mu^2 \\
&= \mu^2\{a^2[\frac{k}{\sqrt{2\pi n}} \exp\{-\frac{n}{2k^2}\} + \frac{n+k^2}{n}\Phi(\sqrt{n}/k)] \\
&\quad + b^2[-\frac{k}{\sqrt{2\pi n}} \exp\{-\frac{n}{2k^2}\} + \frac{n+k^2}{n}(1-\Phi(\sqrt{n}/k))] - 1\} \\
&= \mu^2(a^2 t_1 + b^2 t_2 - 1) \qquad \text{(A.5)}
\end{aligned}$$

By Cauchy-Schwarz Inequality,

$$\begin{aligned}
(t_1 a^2 + t_2 b^2)(\tfrac{s_1^2}{t_1} + \tfrac{s_2^2}{t_2}) &\geq (as_1 + bs_2)^2 \\
\text{i.e.} \quad (t_1 a^2 + t_2 b^2) &\geq \frac{1}{(\frac{s_1^2}{t_1} + \frac{s_2^2}{t_2})} \qquad \text{(A.6)}
\end{aligned}$$

"=" holds iff $\sqrt{t_1}a = c\frac{s_1}{\sqrt{t_1}}$ and $\sqrt{t_2}b = c\frac{s_2}{\sqrt{t_2}}$ with c a constant. This leads to the choices of a and b for minimum $V(\hat{\mu})$ as

$$\begin{aligned}
a &= \frac{s_1/t_1}{s_1^2/t_1 + s_2^2/t_2} \\
b &= \frac{s_2/t_2}{s_1^2/t_1 + s_2^2/t_2} \qquad \text{(A.7)}
\end{aligned}$$

where $a > 0$. Next we show below that $b < 0$ which establishes our claim. For given k, let $l(x) = \frac{k}{\sqrt{2\pi x}} \exp\{-\frac{x}{2k^2}\} + \Phi(\sqrt{x}/k)$, $x \in \mathbb{R}^+$. Since

$$\begin{aligned}
l'(x) &= \frac{k}{\sqrt{2\pi}}[(-\tfrac{1}{2})x^{-3/2}\exp\{\tfrac{-x}{2k^2}\} + \tfrac{1}{\sqrt{x}}(-\tfrac{1}{2k^2})\exp\{\tfrac{-x}{2k^2}\}] + \frac{k}{\sqrt{2\pi}}\exp\{\tfrac{-x}{2}\}(\tfrac{1}{2k})x^{-1/2} \\
&= -\frac{k}{\sqrt{2\pi}}(\tfrac{1}{2})x^{-3/2}\exp\{-x/2\} < 0 \qquad \text{(A.8)}
\end{aligned}$$

$l(x)$ is a decreasing function. In addition, $\lim_{x\to\infty} l(x) = 1$. Therefore, then $l(x) \geq 1$. Hence, for any positive integer n, $s_2 = 1 - l(n) < 0$. Additionally, $t_2 > 0$; therefore, $b < 0$. □

$$\lim_{n\to\infty} s_1(n) = \lim_{n\to\infty} \frac{k}{\sqrt{2\pi n}} \exp\{-\frac{n}{2k^2}\} + \Phi(\sqrt{n}/k) = 1 \qquad \text{(A.9)}$$

Note that $\lim_{n\to\infty} a_n = 1$, and $\lim_{n\to\infty} b_n s_2(n) = 0$.

Appendix B

Proof of the claim that the proposed estimator μ^* follows asymptotic normal distribution.

We know that $\frac{\sqrt{n-1}S}{k|\mu|} \sim \chi(n-1)$, then

$$E(S) = \frac{k|\mu|g(n-1)}{\sqrt{n-1}}. \tag{B.1}$$

$$V(S) = \frac{k^2\mu^2}{n-1}\{(n-1) - [g(n-1)]^2\} \tag{B.2}$$

where $g(n-1) = \frac{\sqrt{2}\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})}$. Then

$$Z_n = \frac{S - \frac{k|\mu|g(n-1)}{\sqrt{n-1}}}{k|\mu|\sqrt{1 - \frac{[g(n-1)]^2}{n-1}}} \; as\tilde{y}m \; N(0,1) \tag{B.3}$$

S can be written as

$$S = k|\mu|[Z_n\sqrt{1 - \frac{[g(n-1)]^2}{n-1}} + \frac{g(n-1)}{\sqrt{n-1}}] \tag{B.4}$$

where $Z_n \sim N(0,1)$. Then $h(S,\bar{X})$ can be written as

$$\begin{aligned} h(S,\bar{X}) &= \mathrm{sgn}(\bar{X})\tfrac{AS\sqrt{n-1}}{kg(n-1)} \\ &= \tfrac{\mathrm{sgn}(\bar{X})A|\mu|[Z_n(\sqrt{n-1-[g(n-1)]^2}) + g(n-1)]}{g(n-1)} \end{aligned} \tag{B.5}$$

Let

$$\begin{aligned} Z_n^* &= \tfrac{\mu^* - \mu}{\sqrt{v(n)}|\mu|} \\ &= \tfrac{c_n h(S,\bar{X}) + (1-c_n)\bar{X} - \mu}{\sqrt{v(n)}|\mu|} \\ &= \beta_n \mathrm{sgn}(\bar{X})Z_n + \gamma_n \mathrm{sgn}(\bar{X}) + \delta_n\bar{X} + \theta_n \end{aligned} \tag{B.6}$$

where,

$$\begin{aligned} \beta_n &= \tfrac{c_n A\sqrt{n-1-[g(n-1)]^2}}{g(n-1)\sqrt{v(n)}} & \gamma_n &= \tfrac{c_n A}{\sqrt{v(n)}} \\ \delta_n &= \tfrac{1-c_n}{\sqrt{v(n)}|\mu|} = \tfrac{\delta_n^*}{|\mu|} & \theta_n &= -\tfrac{\mu}{\sqrt{v(n)}|\mu|} = \tfrac{\theta_n^*\mu}{|\mu|} \end{aligned} \tag{B.7}$$

where $\delta_n^* = \frac{1-c_n}{\sqrt{v(n)}}$, and $\theta_n^* = -\frac{1}{\sqrt{v(n)}}$. By Lemma 2.1,

$$\lim_{n\to\infty}\{n - [g(n)]^2\} = \frac{1}{2}, \lim_{n\to\infty}\{\frac{g(n)}{\sqrt{n}}\} = 1, \lim_{n\to\infty} c_n = \frac{2k^2}{2k^2+1} \tag{B.8}$$

$$\begin{aligned}\lim_{n\to\infty}\{\beta_n^2+\delta_n^2k^2\mu^2/n\} &= \lim_{n\to\infty}\{[\frac{c_nA\sqrt{n-1-[g(n-1)]^2}}{g(n-1)\sqrt{v(n)}}]^2+[\frac{1-c_n}{\sqrt{v(n)}|\mu|}]^2\frac{k^2\mu^2}{n}\} \\ &= \lim_{n\to\infty}\{\frac{c_n^2(\frac{1}{2})}{c_n^2(\frac{1}{2})+(1-c_n)^2k^2}+\frac{(1-c_n)^2k^2}{c_n^2(\frac{1}{2})+(1-c_n)^2k^2}\}=1\end{aligned} \tag{B.9}$$

If $\mu > 0$,

$$\begin{aligned}\lim_{n\to\infty}\{-\gamma_n-\theta_n-\delta_n\mu\} &= \lim_{n\to\infty}\{-\gamma_n-\theta_n^*-\delta_n^*\} \\ &= \lim_{n\to\infty}\{-\frac{c_nA}{\sqrt{v(n)}}+\frac{1}{\sqrt{v(n)}}-\frac{1-c_n}{\sqrt{v(n)}}\} \\ &= \lim_{n\to\infty}\{\frac{\frac{1}{\sqrt{2\pi}}e^{-n/2k^2}\frac{1}{2k}n^{-1/2}}{-\frac{1}{2}n^{-3/2}}\}=0\end{aligned} \tag{B.10}$$

$$\text{If } \mu<0, \quad \lim_{n\to\infty}\{\gamma_n-\theta_n-\delta_n\mu\}=\lim_{n\to\infty}\{\gamma_n+\theta_n^*+\delta_n^*\}=0. \tag{B.11}$$

For any real number z,

$$\begin{aligned}P\{Z_n^*\le z\} &= P\{\beta_n\text{sgn}(\bar{X})Z_n+\gamma_n\text{sgn}(\bar{X})+\delta_n\bar{X}+\theta_n\le z\} \\ &= P\{\bar{X}>0\}P\{\beta_nZ_n+\gamma_n+\delta_n\bar{X}+\theta_n\le z|\bar{X}>0\} \\ &\quad +P\{\bar{X}<0\}P\{-\beta_nZ_n-\gamma_n+\delta_n\bar{X}+\theta_n\le z|\bar{X}<0\}\end{aligned} \tag{B.12}$$

If $\mu > 0$,

$$\begin{aligned}P\{Z_n^*\le z\} &= P\{\bar{X}>0\}P\{\beta_nZ_n+\gamma_n+\delta_n\bar{X}+\theta_n\le z|\bar{X}>0\} \\ &\quad +P\{\bar{X}<0\}P\{\beta_nZ_n+\gamma_n+\delta_n\bar{X}+\theta_n\le z|\bar{X}<0\} \\ &\quad -P\{\bar{X}<0\}P\{\beta_nZ_n+\gamma_n+\delta_n\bar{X}+\theta_n\le z|\bar{X}<0\} \\ &\quad +P\{\bar{X}<0\}P\{-\beta_nZ_n-\gamma_n+\delta_n\bar{X}+\theta_n\le z|\bar{X}<0\} \\ &= \Phi[\frac{z-\gamma_n-\theta_n-\delta_n\mu}{\sqrt{\beta_n^2+\delta_n^2k^2\mu^2/n}}]+P\{\bar{X}<0\}\{-P[\beta_nZ_n+\gamma_n+\delta_n\bar{X}+\theta_n\le z|\bar{X}<0] \\ &\quad +P[-\beta_nZ_n-\gamma_n+\delta_n\bar{X}+\theta_n\le z|\bar{X}<0]\}\end{aligned} \tag{B.13}$$

Since $\lim_{n\to\infty}P\{\bar{X}<0\}=\lim_{n\to\infty}1-\Phi(\sqrt{n}/k)=0$, and $-P[\beta_nZ_n+\gamma_n+\delta_n\bar{X}+\theta_n\le z|\bar{X}<0]$

$+ P[-\beta_n Z_n - \gamma_n + \delta_n \bar{X} + \theta_n \leq z | \bar{X} < 0]$ is bounded,

$$\begin{aligned}
\lim_{n\to\infty} P\{Z_n^* \leq z\} &= \lim_{n\to\infty} \Phi[\frac{z - \gamma_n - \theta_n - \delta_n \mu}{\sqrt{\beta_n^2 + \delta_n^2 \mu^2 / n}}] \\
&\quad + \lim_{n\to\infty} P\{\bar{X} < 0\}\{-P[\beta_n Z_n + \gamma_n + \delta_n \bar{X} + \theta_n \leq z | \bar{X} < 0] \\
&\quad + P[-\beta_n Z_n - \gamma_n + \delta_n \bar{X} + \theta_n \leq z | \bar{X} < 0]\} \\
&= \Phi(z) \qquad \text{(B.14)}
\end{aligned}$$

If $\mu < 0$,

$$\begin{aligned}
P\{Z_n^* \leq z\} &= P\{\bar{X} > 0\} P\{\beta_n Z_n + \gamma_n + \delta_n \bar{X} + \theta_n \leq z | \bar{X} > 0\} \\
&\quad + P\{\bar{X} < 0\} P\{-\beta_n Z_n - \gamma_n + \delta_n \bar{X} + \theta_n \leq z | \bar{X} < 0\} \\
&\quad + P\{\bar{X} > 0\} P\{-\beta_n Z_n - \gamma_n + \delta_n \bar{X} + \theta_n \leq z | \bar{X} > 0\} \\
&\quad - P\{\bar{X} > 0\} P\{-\beta_n Z_n - \gamma_n + \delta_n \bar{X} + \theta_n \leq z | \bar{X} > 0\} \\
&= \Phi[\frac{z + \gamma_n - \theta_n - \delta_n \mu}{\sqrt{\beta_n^2 + \delta_n^2 k^2 \mu^2 / n}}] + P\{\bar{X} > 0\}\{P[\beta_n Z_n + \gamma_n + \delta_n \bar{X} + \theta_n \leq z | \bar{X} > 0] \\
&\quad - P[-\beta_n Z_n - \gamma_n + \delta_n \bar{X} + \theta_n \leq z | \bar{X} > 0]\} \qquad \text{(B.15)}
\end{aligned}$$

Since $\lim_{n\to\infty} P\{\bar{X} > 0\} = \lim_{n\to\infty} 1 - \Phi(\sqrt{n}/k) = 0$, and $P[\beta_n Z_n + \gamma_n + \delta_n \bar{X} + \theta_n \leq z | \bar{X} > 0] - P[-\beta_n Z_n - \gamma_n + \delta_n \bar{X} + \theta_n \leq z | \bar{X} > 0]$ is bounded,

$$\begin{aligned}
\lim_{n\to\infty} P\{Z_n^* \leq z\} &= \lim_{n\to\infty} \Phi[\frac{z + \gamma_n - \theta_n - \delta_n \mu}{\sqrt{\beta_n^2 + \delta_n^2 k^2 \mu^2 / n}}] \\
&\quad + \lim_{n\to\infty} P\{\bar{X} > 0\}\{P[\beta_n Z_n + \gamma_n + \delta_n \bar{X} + \theta_n \leq z | \bar{X} > 0] \\
&\quad - P[-\beta_n Z_n - \gamma_n + \delta_n \bar{X} + \theta_n \leq z | \bar{X} > 0]\} \\
&= \Phi(z) \qquad \text{(B.16)}
\end{aligned}$$

Therefore, $\lim_{n\to\infty} P\{Z_n^* \leq z\} = \Phi(z)$.□

Appendix C

Establishing the claim that the probability $P(|\frac{\mu^*-\mu}{\sqrt{v(n)|\mu|}}| \leq z)$ is independent of μ. Assume $\mu > 0$. Let, for given k, $Y = \frac{X-\mu}{k\mu}$. Then $X = \mu Y + k\mu$ and $Y \sim N(0,1)$.

$$\begin{aligned}
Z^*_{n+} &= \frac{\mu^* - \mu}{\sqrt{v(n)}\mu} \\
&= \frac{c_n h(S,\bar{X}) + (1-c_n)\bar{X} - \mu}{\sqrt{v(n)}\mu} \\
&= \frac{c_n \cdot \text{sgn}(\bar{Y}+k)\frac{AS_Y\sqrt{n-1}}{kg(n-1)} + (1-c_n)(\bar{Y}+k) - 1}{\sqrt{v(n)}} \\
&= \frac{c_n[\text{sgn}(\bar{Y}+k)\frac{AS_Y\sqrt{n-1}}{kg(n-1)} - 1] + (1-c_n)(\bar{Y}+k-1)}{\sqrt{v(n)}} \qquad \text{(C.1)}
\end{aligned}$$

where $\bar{Y} = \frac{\sum_{i=1}^n Y_i}{n}$ and $S_Y = \frac{1}{n-1}\sum_{i=1}^n (Y_i - \bar{Y})^2$. The distribution of Z^*_{n+} is independent of μ when μ is positive. If $\mu < 0$,

$$\begin{aligned}
Z^*_{n-} &= \frac{\mu^* - \mu}{\sqrt{v(n)}(-\mu)} \\
&= \frac{c_n h(S,\bar{X}) + (1-c_n)\bar{X} - \mu}{\sqrt{v(n)}(-\mu)} \\
&= \frac{c_n \cdot \text{sgn}[\mu(\bar{Y}+k)]\frac{A(-\mu)S_Y\sqrt{n-1}}{kg(n-1)} + (1-c_n)(\bar{Y}+k-1)\mu - \mu}{\sqrt{v(n)}(-\mu)} \\
&= \frac{-c_n[\text{sgn}(\bar{Y}+k)\frac{AS_Y\sqrt{n-1}}{kg(n-1)} - 1] - (1-c_n)(\bar{Y}+k-1)}{\sqrt{v(n)}} \qquad \text{(C.2)}
\end{aligned}$$

The distribution of Z^*_{n-} is independent of μ when μ is positive. Therefore,

$$\begin{aligned}
P\{|Z^*_{n+}| \leq z\} &= P\{|\frac{c_n[\text{sgn}(\bar{Y}+k)\frac{AS_Y\sqrt{n-1}}{kg(n-1)} - 1] + (1-c_n)(\bar{Y}+k-1)}{\sqrt{v(n)}}| \leq z\} \\
&= P\{|Z^*_{n-}| \leq z\}.\square \qquad \text{(C.3)}
\end{aligned}$$