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Concomitants of Record Values from Marshall and Olkin's Bivariate Exponential Distributions

Jagdish Saran and Aparna Pandey
Department of Statistics
University of Delhi
Delhi - 110 007, India

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Abstract

In this paper the probability density function (pdf) of the concomitants of n-th record values for Marshall and Olkin's bivariate exponential distributions have been derived. The moments of these concomitants have also been derived. Further, their means and variances are tabulated.

Keywords and Phrases: Record values, Concomitants of Record Values, Moments, Moment Generating Function, Bivariate Exponential Distributions.

AMS Classification: 62G30.

1 Introduction

The study of concomitants of record values was initiated by Houchens (1984) and is precisely parallel to that associated with concomitants of order statistics (see David and Nagaraja (1998)). Various developments on concomitants of record values and their properties have been reviewed by a number of authors including Kamps (1995a), Ahsanullah and Nevzorov (2001) and Ahsanullah (2005).

Let us consider i.i.d. bivariate observations $(X_1, Y_1), (X_2, Y_2), \ldots$ from some bivariate population with common joint cdf $F_{X,Y}(x, y)$. For convenience, we will assume that F is absolutely continuous with joint pdf $f_{X,Y}(x, y)$ and the family of conditional densities of Y given $X = x$ is denoted by $F_{Y|X}(y|x)$. To define the n-th lower record R'_n of the sequence $X_1, X_2, \ldots$, let the sequence of record times be defined as follows:

$$T_0 = 1\,, \quad \text{with probability} \;\; 1,$$

and for $n > 1$,

$$T_n = \min\{j : j > T_{n-1}, X_j < X_{T_{n-1}}\}.$$

Then $R'_n = X_{T_n}$, $n = 0, 1, 2, \ldots$ denotes the corresponding lower record value sequence. The corresponding random variable Y, observed with the X value which qualified as the n-th lower record is called the n-th record concomitant and is denoted by $R'_{[n]}$. The probability density function (pdf) of the n-th record concomitant is given by

$$f_{R'_{[n]}}(y) = \frac{1}{n!} \int_{-\infty}^{\infty} f_{Y|X}(y|x) f_X(x)[-\log F_X(x)]^n dx, \tag{1}$$

(cf. Arnold, Balakrishnan and Nagaraja (1998, p.272)), where $f_X(x)$ and $F_X(x)$ denote the pdf and the cdf of X, respectively.

Further, let $E(R'^{(p)}_{[n]})$ denote the p-th moment of $R'_{[n]}$ which can be computed using the following result:

$$E(R'^{(p)}_{[n]}) = \frac{1}{n!} \int_{-\infty}^{\infty} E[Y^p | X = x] f_X(x)[-\log F_X(x)]^n dx. \tag{2}$$

In the present paper we consider Marshall and Olkin's bivariate exponential distributions (Johnson and Kotz (1972), p.266) with cdf given by

$$F_{X,Y}(x, y) = \exp\{\lambda_1 x + \lambda_2 y + \lambda_{12} \min(x, y)\}, \quad x, y < 0;\ \lambda_1, \lambda_2, \lambda_{12} > 0, \tag{3}$$

and derive expressions for the pdf and moments of the n-th record concomitant from this distribution. Further, the means and variances for $n = 0, 1, 2, \ldots, 10$ have been computed for the case $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_{12} = 1$. We shall also derive an expression for the moment generating function of $R'_{[n]}$ for the considered distribution. Finally, some results for the concomitants of the k-lower record values have been obtained.

The cdf given in (3) can be rewritten as (cf. Beg and Balasubramanian (1996, 1997))

$$F_{X,Y}(x, y) = \begin{cases} \exp\{\lambda_1 x + \lambda_2 y + \lambda_{12} x\}, & x \le y < 0, \\ \exp\{\lambda_1 x + \lambda_2 y + \lambda_{12} y\}, & 0 > x > y, \end{cases} \tag{4}$$

and the pdf is given by

$$f_{X,Y}(x, y) = \begin{cases} \lambda_2(\lambda_1 + \lambda_{12}) \exp\{\lambda_1 x + \lambda_2 y + \lambda_{12} x\}, & x < y < 0, \\ \lambda_1(\lambda_2 + \lambda_{12}) \exp\{\lambda_1 x + \lambda_2 y + \lambda_{12} y\}, & 0 > x > y, \\ \lambda_{12} \exp\{\lambda_1 + \lambda_2 + \lambda_{12}\} x, & x = y. \end{cases} \tag{5}$$

The conditional pdf of Y given X is as follows

$$f_{Y|X}(y|x) = \begin{cases} \lambda_2 \exp\{\lambda_2 y\}, & x < y < 0, \\ \dfrac{\lambda_1(\lambda_2+\lambda_{12})}{(\lambda_1+\lambda_{12})} \exp\{(\lambda_2+\lambda_{12})y - \lambda_{12}x\}, & 0 > x > y, \\ \dfrac{\lambda_{12}}{(\lambda_1+\lambda_{12})} \exp\{\lambda_2 x\}, & x = y. \end{cases} \tag{6}$$

The marginal cdf of X is given by

$$F_X(x) = \exp\{(\lambda_1+\lambda_{12})x\} \quad , -\infty < x \le 0. \tag{7}$$

2 Probability Density Function of $R'_{[n]}$

The pdf of n-th record concomitant for Marshall and Olkin's bivariate exponential distributions (4) can be obtained from (1) on using (6) and (7), and is given by

$$\begin{aligned} f_{R'_{[n]}}(y) &= \frac{(-1)^n}{n!}\int_{-\infty}^{y} \lambda_2 \exp\{\lambda_2 y\}(\lambda_1+\lambda_{12})^{n+1}\exp\{(\lambda_1+\lambda_{12})x\}x^n dx \\ &+ \frac{(-1)^n}{n!}\int_{y}^{0} \lambda_1(\lambda_2+\lambda_{12}) \exp\{(\lambda_2+\lambda_{12})y - \lambda_{12}x\}\,(\lambda_1+\lambda_{12})^n \\ &\times \exp\{(\lambda_1+\lambda_{12})x\}x^n dx \\ &+ \frac{(-1)^n}{n!}\lambda_{12} \exp\{\lambda_2 y\}\,(\lambda_1+\lambda_{12})^n \exp\{(\lambda_1+\lambda_{12})y\}\,y^n. \end{aligned} \tag{8}$$

Further, on solving the integrals in (8) and simplifying the resulting expression, we obtain

$$\begin{aligned} f_{R'_{[n]}}(y) &= \frac{(-1)^n}{n!}\lambda_2(\lambda_1+\lambda_{12})^{n+1}\exp\{\lambda_2 y\} \\ &\times \left[\frac{y^n e^{(\lambda_1+\lambda_{12})y}}{(\lambda_1+\lambda_{12})} - \frac{n y^{n-1} e^{(\lambda_1+\lambda_{12})y}}{(\lambda_1+\lambda_{12})^2} + \ldots + (-1)^n \frac{n! e^{(\lambda_1+\lambda_{12})y}}{(\lambda_1+\lambda_{12})^{n+1}}\right] \\ &+ \frac{(-1)^n}{n!}\lambda_1(\lambda_2+\lambda_{12})(\lambda_1+\lambda_{12})^n \exp\{(\lambda_2+\lambda_{12})y\} \\ &\times \left[-\frac{y^n e^{\lambda_1 y}}{\lambda_1} + \frac{n y^{n-1} e^{\lambda_1 y}}{\lambda_1^2} - \ldots + (-1)^{n+1}\frac{n! e^{\lambda_1 y}}{\lambda_1^{n+1}}\right] \\ &+ \frac{1}{\lambda_1^n}(\lambda_2+\lambda_{12})(\lambda_1+\lambda_{12})^n \exp\{(\lambda_2+\lambda_{12})y\} \\ &+ \frac{(-1)^n}{n!}\lambda_{12}(\lambda_1+\lambda_{12})^n y^n \exp\{(\lambda_1+\lambda_2+\lambda_{12})y\}. \end{aligned} \tag{9}$$

It can easily be verified that $\int f_{R'_{[n]}}(y)dy = 1$.

3 Moments of $R'_{[n]}$

On using (6) and (7) in (2), we obtain

$$
\begin{aligned}
E(R'^{(p)}_{[n]}) = {} & \frac{(-1)^{(n)}}{n!}\lambda_2(\lambda_1+\lambda_{12})^{n+1}\Bigg[\int_{-\infty}^{0}\frac{y^{n+p}e^{(\lambda_1+\lambda_2+\lambda_{12})y}}{(\lambda_1+\lambda_{12})}dy \\
& - \int_{-\infty}^{0}\frac{ny^{n+p-1}e^{(\lambda_1+\lambda_2+\lambda_{12})y}}{(\lambda_1+\lambda_{12})^2}dy + \ldots + (-1)^n\int_{-\infty}^{0}\frac{n!y^p e^{(\lambda_1+\lambda_2+\lambda_{12})y}}{(\lambda_1+\lambda_{12})^{n+1}}dy\Bigg] \\
& + \frac{(-1)^{(n)}}{n!}\lambda_1(\lambda_2+\lambda_{12})(\lambda_1+\lambda_{12})^n\Bigg[-\int_{-\infty}^{0}\frac{y^{n+p}e^{(\lambda_1+\lambda_2+\lambda_{12})y}}{\lambda_1}dy \\
& + \int_{-\infty}^{0}\frac{ny^{n+p-1}e^{(\lambda_1+\lambda_2+\lambda_{12})y}}{\lambda_1^2}dy - \ldots + (-1)^{n+1}\int_{-\infty}^{0}\frac{n!y^p e^{(\lambda_1+\lambda_2+\lambda_{12})y}}{\lambda_1^{n+1}}dy\Bigg] \\
& + \frac{1}{\lambda_1^n}(\lambda_2+\lambda_{12})(\lambda_1+\lambda_{12})^n\int_{-\infty}^{0}y^p e^{(\lambda_2+\lambda_{12})y}dy \\
& + \frac{(-1)^n}{n!}\lambda_{12}(\lambda_1+\lambda_{12})^n\int_{-\infty}^{0}y^{n+p}e^{(\lambda_1+\lambda_2+\lambda_{12})y}dy, \qquad (10)
\end{aligned}
$$

for $p = 1, 2, \ldots$.

On simplifying the expression in (10), we get

$$
\begin{aligned}
E(R'^{(p)}_{[n]}) = {} & \frac{(-1)^p}{n!}\lambda_2(\lambda_1+\lambda_{12})^{n+1}\Bigg[\frac{\Gamma(n+p+1)}{(\lambda_1+\lambda_{12})(\lambda_1+\lambda_2+\lambda_{12})^{n+p+1}} \\
& + \frac{n\Gamma(n+p)}{(\lambda_1+\lambda_{12})^2(\lambda_1+\lambda_2+\lambda_{12})^{n+p}} + \ldots + \frac{n!}{(\lambda_1+\lambda_{12})^{n+1}}\frac{\Gamma(p+1)}{(\lambda_1+\lambda_2+\lambda_{12})^{p+1}}\Bigg] \\
& + \frac{(-1)^{p+1}}{n!}\lambda_1(\lambda_1+\lambda_{12})^n(\lambda_2+\lambda_{12})\Bigg[\frac{\Gamma(n+p+1)}{\lambda_1(\lambda_1+\lambda_2+\lambda_{12})^{n+p+1}} \\
& + \frac{n}{\lambda_1^2}\frac{\Gamma(n+p)}{(\lambda_1+\lambda_2+\lambda_{12})^{n+p}} + \ldots + \frac{n!}{\lambda_1^{n+1}}\frac{\Gamma(p+1)}{(\lambda_1+\lambda_2+\lambda_{12})^{p+1}}\Bigg] \\
& + \frac{(-1)^p}{\lambda_1^n}(\lambda_1+\lambda_{12})^n(\lambda_2+\lambda_{12})\frac{\Gamma(p+1)}{(\lambda_2+\lambda_{12})^{p+1}} \\
& + \frac{(-1)^p\lambda_{12}(\lambda_1+\lambda_{12})^2}{n!}\frac{\Gamma(n+p+1)}{(\lambda_1+\lambda_2+\lambda_{12})^{n+p+1}}, \qquad (11)
\end{aligned}
$$

for $p = 1, 2, \ldots$.

The means and variances of $R'_{[n]}$ have been tabulated in Table 1 and Table 2, respectively.

Table 1: Mean of the concomitant of record values

n	0	1	2	3	4	5	6	7	8	9	10
mean	−0.5000	−0.6667	−0.7778	−0.8519	−0.9012	−0.9342	−0.9561	−0.9707	−0.9805	−0.9871	−0.9961

Table 2: Variance of the concomitant of record values

n	0	1	2	3	4	5	6	7	8	9	10
variance	0.2500	0.3333	0.4320	0.5336	0.6281	0.7103	0.7786	0.8333	0.8760	0.9085	0.9329

4 Moment Generating Function of $R'_{[n]}$

The moment generating function $M_{R'_{[n]}}(t)$ of $R'_{[n]}$ is given by

$$\begin{aligned}
M_{R'_{[n]}}(t) &= E(\exp(tR'_{[n]})) \\
&= \frac{(-1)^n}{n!}\lambda_2(\lambda_1+\lambda_{12})^{n+1}\left[\int_{-\infty}^{0}\frac{y^n e^{\{(\lambda_1+\lambda_2+\lambda_{12})+t\}y}}{(\lambda_1+\lambda_{12})}dy\right. \\
&\left. -n\int_{-\infty}^{0}\frac{y^{n-1}e^{\{(\lambda_1+\lambda_2+\lambda_{12})+t\}y}}{(\lambda_1+\lambda_{12})^2}dy+\ldots+(-1)^n n!\int_{-\infty}^{0}\frac{e^{\{(\lambda_1+\lambda_2+\lambda_{12})+t\}y}}{(\lambda_1+\lambda_{12})^{n+1}}dy\right] \\
&+\frac{(-1)^n}{n!}\lambda_1(\lambda_2+\lambda_{12})(\lambda_1+\lambda_{12})^n\left[-\int_{-\infty}^{0}\frac{y^n e^{\{(\lambda_1+\lambda_2+\lambda_{12})+t\}y}}{\lambda_1}dy\right. \\
&\left. +n\int_{-\infty}^{0}\frac{y^{n-1}e^{\{(\lambda_1+\lambda_2+\lambda_{12})+t\}y}}{\lambda_1^2}dy-\ldots+(-1)^{n+1}n!\int_{-\infty}^{0}\frac{e^{\{(\lambda_1+\lambda_2+\lambda_{12})+t\}y}}{\lambda_1^{n+1}}dy\right] \\
&+\frac{1}{\lambda_1^n}(\lambda_2+\lambda_{12})(\lambda_1+\lambda_{12})^n\int_{-\infty}^{0}e^{(\lambda_2+\lambda_{12}+t)y}\,dy+\frac{(-1)^n}{n!}\lambda_{12}(\lambda_1+\lambda_{12})^n \\
&\times\int_{-\infty}^{0}y^n e^{\{(\lambda_1+\lambda_2+\lambda_{12})+t\}y}\,dy,
\end{aligned}$$

which on simplification reduces to

$$\begin{aligned}
M_{R'_{[n]}}(t) &= \frac{1}{n!}\lambda_2(\lambda_1+\lambda_{12})^{n+1}\left[\frac{\Gamma(n+1)}{(\lambda_1+\lambda_{12})(\lambda_1+\lambda_2+\lambda_{12}+t)^{n+1}}\right. \\
&\left.+\frac{n\Gamma(n)}{(\lambda_1+\lambda_{12})^2(\lambda_1+\lambda_2+\lambda_{12}+t)^n}+\ldots+\frac{n!}{(\lambda_1+\lambda_{12})^{n+1}}\frac{1}{(\lambda_1+\lambda_2+\lambda_{12}+t)}\right] \\
&+\frac{1}{n!}\lambda_1(\lambda_1+\lambda_{12})^n(\lambda_2+\lambda_{12})\left[-\frac{1}{\lambda_1}\frac{\Gamma(n+1)}{(\lambda_1+\lambda_2+\lambda_{12}+t)^{n+1}}\right. \\
&\left.-\frac{n}{\lambda_1^2}\frac{\Gamma(n)}{(\lambda_1+\lambda_2+\lambda_{12}+t)^n}-\ldots-\frac{n!}{\lambda_1^{n+1}}\frac{1}{(\lambda_1+\lambda_2+\lambda_{12}+t)}\right]
\end{aligned}$$

$$+\frac{1}{\lambda_1^n}(\lambda_1+\lambda_{12})^n(\lambda_2+\lambda_{12})\frac{1}{(\lambda_2+\lambda_{12}+t)}+\frac{\lambda_{12}(\lambda_1+\lambda_{12})^n}{n!}$$
$$\times\frac{\Gamma(n+1)}{(\lambda_1+\lambda_2+\lambda_{12}+t)^{n+1}}. \qquad (12)$$

Differentiating (12) p times with respect to t, we get

$$\frac{d^p}{dt^p}M_{R'_{[n]}}(t)=\frac{(-1)^p}{n!}\lambda_2(\lambda_1+\lambda_{12})^{n+1}\left[\frac{\Gamma(n+p+1)}{(\lambda_1+\lambda_{12})(\lambda_1+\lambda_2+\lambda_{12}+t)^{n+p+1}}\right.$$
$$+\frac{n\Gamma(n+p)}{(\lambda_1+\lambda_{12})^2(\lambda_1+\lambda_2+\lambda_{12}+t)^{n+p}}+\ldots+\frac{n!}{(\lambda_1+\lambda_{12})^{n+1}}$$
$$\left.\times\frac{\Gamma(p+1)}{(\lambda_1+\lambda_2+\lambda_{12}+t)^{p+1}}\right]+\frac{(-1)^{p+1}}{n!}\lambda_1(\lambda_1+\lambda_{12})^n(\lambda_2+\lambda_{12})$$
$$\times\left[\frac{\Gamma(n+p+1)}{\lambda_1(\lambda_1+\lambda_2+\lambda_{12}+t)^{n+p+1}}+\frac{n}{\lambda_1^2}\frac{\Gamma(n+p)}{(\lambda_1+\lambda_2+\lambda_{12}+t)^{n+p}}+\ldots\right.$$
$$\left.+\frac{n!}{\lambda_1^{n+1}}\frac{\Gamma(p+1)}{(\lambda_1+\lambda_2+\lambda_{12}+t)^{p+1}}\right]+\frac{(-1)^p}{\lambda_1^n}(\lambda_1+\lambda_{12})^n(\lambda_2+\lambda_{12})$$
$$\times\frac{\Gamma(p+1)}{(\lambda_2+\lambda_{12}+t)^{p+1}}+\frac{(-1)^p\lambda_{12}(\lambda_1+\lambda_{12})^n}{n!}\frac{\Gamma(n+p+1)}{(\lambda_1+\lambda_2+\lambda_{12}+t)^{n+p+1}},$$

which on setting $t=0$ gives an expression for $E(R'^{(p)}_{[n]})$, thus verifying (11).

5 Concomitants of K-Record Values

To define the k-record values, let $T_{0(k)}=k$ and $T_{n(k)}=\min\{j:j>T_{n-1(k)}, X_{j-k+1:j}<X_{T_{n-1(k)}-k+1:T_{n-1(k)}}\}$ where $X_{j:n}$ is the j-th order statistic of the random sample $X_1,X_2,\ldots,X_n$. Then $R'_{n(k)}=X_{T_{n(k)}-k+1:T_{n(k)}}$ is defined as the n-th k-lower record (see Dziubdziela and Kopocinski (1976)). These statistics are contained in the model of generalized order statistics (Kamps (1995b)).

Let $R'_{[n(k)]}$ denote the concomitant of the n-th k-lower record value. On using the density function $f_{R'_{[n(k)]}}(x)$ of $R'_{[n(k)]}$ given by

$$f_{R'_{[n(k)]}}(x)=\int_{-\infty}^{\infty}f_{Y|X}(y|x)f_{R'_{n(k)}}(x)dx,$$

where $f_{R'_{n(k)}}(x)$, the density function of $R'_{n(k)}$ is

$$f_{R'_{n(k)}}(x)=\frac{k^n}{n!}[-\log F_X(x)]^n[F_X(x)]^{k-1}f_X(x),$$

(cf. Pawlas and Szynal (1998)) and applying the same procedure as used in Sections 2 and 3, we obtain expressions for the pdf and the moments of $R'_{[n(k)]}$ as given in the following two theorems.

Theorem 1.

$$f_{R'_{[n(k)]}}(y) = \frac{(-k)^n}{n!}\lambda_2(\lambda_1+\lambda_{12})^{n+1}\exp\{\lambda_2 y\}\left[\frac{y^n e^{k(\lambda_1+\lambda_{12})y}}{k(\lambda_1+\lambda_{12})}\right.$$
$$\left. -\frac{ny^{n-1}e^{k(\lambda_1+\lambda_{12})y}}{k^2(\lambda_1+\lambda_{12})^2} + \ldots + (-1)^n\frac{n!e^{k(\lambda_1+\lambda_{12})y}}{k^{n+1}(\lambda_1+\lambda_{12})^{n+1}}\right]$$
$$+\frac{(-k)^n}{n!}\lambda_1(\lambda_2+\lambda_{12})(\lambda_1+\lambda_{12})^n\exp\{(\lambda_2+\lambda_{12})y\}$$
$$\times\left[-\frac{y^n e^{(k\lambda_1+(k-1)\lambda_{12})y}}{(k\lambda_1+(k-1)\lambda_{12})} + \frac{ny^{n-1}e^{(k\lambda_1+(k-1)\lambda_{12})y}}{(k\lambda_1+(k-1)\lambda_{12})^2} - \ldots\right.$$
$$\left. +(-1)^{n+1}\frac{n!e^{(k\lambda_1+(k-1)\lambda_{12})y}}{(k\lambda_1+(k-1)\lambda_{12})^{n+1}}\right]$$
$$+\frac{k^n\lambda_1}{(k\lambda_1+(k-1)\lambda_{12})^{n+1}}(\lambda_2+\lambda_{12})(\lambda_1+\lambda_{12})^n\exp\{(\lambda_2+\lambda_{12})y\}$$
$$+\frac{(-k)^n}{n!}\lambda_{12}(\lambda_1+\lambda_{12})^n y^n\exp\{k(\lambda_1+\lambda_{12})+\lambda_2\}y.$$

Theorem 2.

$$E(R'^{(p)}_{[n(k)]}) = \frac{(-1)^p k^n}{n!}\lambda_2(\lambda_1+\lambda_{12})^{n+1}\left[\frac{\Gamma(n+p+1)}{k(\lambda_1+\lambda_{12})(\lambda_2+k(\lambda_1+\lambda_{12}))^{n+p+1}}\right.$$
$$+\frac{n\Gamma(n+p)}{k^2(\lambda_1+\lambda_{12})^2(\lambda_2+k(\lambda_1+\lambda_{12}))^{n+p}} + \ldots$$
$$\left.+\frac{n!}{k^{n+1}(\lambda_1+\lambda_{12})^{n+1}}\frac{\Gamma(p+1)}{(\lambda_2+k(\lambda_1+\lambda_{12}))^{p+1}}\right]$$
$$+\frac{(-1)^{p+1}\lambda_1 k^n(\lambda_1+\lambda_{12})^n(\lambda_2+\lambda_{12})}{n!}$$
$$\times\left[\frac{\Gamma(n+p+1)}{(k\lambda_1+(k-1)\lambda_{12})(\lambda_2+k(\lambda_1+\lambda_{12}))^{n+p+1}}\right.$$
$$+\frac{n\Gamma(n+p)}{(k\lambda_1+(k-1)\lambda_{12})^2(\lambda_2+k(\lambda_1+\lambda_{12}))^{n+p}} + \ldots +$$
$$\left.+\frac{n!}{(k\lambda_1+(k-1)\lambda_{12})^{n+1}}\frac{\Gamma(p+1)}{(\lambda_2+k(\lambda_1+\lambda_{12}))^{p+1}}\right]$$
$$+\frac{(-1)^p k^n\lambda_1}{(k\lambda_1+(k-1)\lambda_{12})^{n+1}}(\lambda_1+\lambda_{12})^n(\lambda_2+\lambda_{12})\frac{\Gamma(p+1)}{(\lambda_2+\lambda_{12})^{p+1}}$$
$$+\frac{k^n(-1)^p\lambda_{12}(\lambda_1+\lambda_{12})^n}{n!}\frac{\Gamma(n+p+1)}{(k(\lambda_1+\lambda_{12})+\lambda_2)^{n+p+1}}, \quad \textit{for} \;\; p=1,2,\ldots\;.$$

Remark 1. Putting $k = 1$ in Theorems 1 and 2, we deduce the expressions for the pdf and moments of concomitants of lower record values already established in (9) and (11), respectively.

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