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Non-parametric Testing in the Survival/Sacrifice Model

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Abstract

In this paper we develop tests when the available data are sampled under censorship and sacrificing. For the underlying test process we derive an i.i.d. representation which is useful to justify a distributional approximation through a wild bootstrap.

Keywords and Phrases: Sacrifice Data, Kaplan-Meier Estimator, I. I. D. Representation, Wild Bootstrap.

AMS Classification: 62G10; 62N01; 62N03.

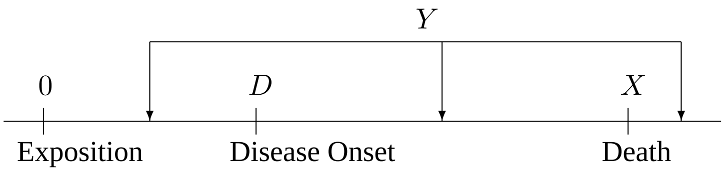
1 Introduction

When analyzing lifetime data a major difficulty is caused by the fact that due to time limitations placed on a study or follow-up losses, information may only be available in incomplete form. Undoubtedly, the best-studied example is the right-censorship case handled in the landmark paper from Kaplan and Meier (1958).

Here, one is interested in the distribution of a random variable X denoting the "lifetime" of the study subjects. Due to the reasons mentioned above, it can occur that instead of X, one observes a "censored" quantity $Z = \min(X,\ C)$ together with an indicator $\delta = 1_{\{X \leq C\}}$ of "failure". Here C is a censoring variable, the length of time the subject takes part in the study, which, in the case $\delta = 0$, is observed instead of the target X. Given a sample $(Z_i,\ \delta_i)\ ,\ 1 \leq i \leq n$, of such data, a portion is reported as X_i, while the rest equal C_i.

A similar but more complicated situation arises when, for example in an animal experiment, the goal is not to analyze the lifetime X, but the time D elapsed from the exposition to some risk (e.g., exposure to a carcinogenic substance) until the onset of disease (start of tumor development). However, the onset of the disease remains unobservable unless the animal is sacrificed, so that one faces the following situation:

> Letting Y denote the time of sacrifice, we may expect three different cases as indicated in the illustration below.



$Y < D$	The sacrificed subject is examined, but no evidence of the disease is detectable.
$Y \geq D$	The sacrificed subject is examined, and the disease is found to be present.
$Y \geq X$ ($\geq D$)	The subject dies of the disease *before* the predetermined time of sacrifice.

From this we see that D is never actually observable. Instead we observe the variables:

$Z = \min(X,\ Y)$	The subject's lifespan – ended by disease or sacrifice.
$\delta = 1_{\{X \leq Y\}}$	An indicator of whether the subject died of the disease or was sacrificed.
$\mu = 1_{\{D \leq Y\}}$	An indicator of whether the disease was present at the time of sacrifice.

One possible goal is to estimate the distribution function $A(x) = \mathbb{P}(D \leq x)$ of D. However, here we will be concerned with testing possible specifications of A, i.e., given a simple or composite model $\mathcal{M}$ of distribution functions, to test for

$$\mathsf{H}_0 : A \in \mathcal{M}\ .$$

Before beginning, we introduce some notation for several distribution functions (d.f.) and sub-d.f. Define

$$H(t) = \mathbb{P}(Z \le t)$$
$$H^0(t) = \mathbb{P}(Z \le t,\ \delta = 0) \qquad H^1(t) = \mathbb{P}(Z \le t, \delta = 1)$$
$$H^{01}(t) = \mathbb{P}(Z \le t,\ \delta = 0,\ \mu = 1)\ ,$$

all unknown, and for the d.f. of Y and X, let

$$G(t) = \mathbb{P}(Y \le t) \qquad \text{and} \qquad F(t) = \mathbb{P}(X \le t)\ .$$

We will also work under the assumption that $(D,\ X)$ and Y are independent. This independence yields a first very important equation, namely

$$1 - H = (1 - F)(1 - G)\ . \tag{1.1}$$

Furthermore,

$$H^0(t) = \int_0^t (1 - F(y))\ G(dy)\ , \tag{1.2}$$

and similarly for H^1. Finally,

$$\begin{aligned} H^{01}(t) &= \int \mathbb{P}(\,Z \le t,\ X > y,\ D \le y \mid Y = y)\, G(dy) \\ &= \int_0^t \mathbb{P}(X > y,\ D \le y)\, G(dy) \\ &= \int_0^t [\mathbb{P}(D \le y) - \mathbb{P}(D \le y,\ X \le y)]\ G(dy) \\ &= \int_0^t A(y)\, G(dy) - \int_0^t F(y)\, G(dy), \end{aligned} \tag{1.3}$$

where the final equality follows from $D \le X$.

Since we are concerned with testing, we need to introduce estimators for some of the above quantities. Given a sample $(Z_i,\ \delta_i,\ \mu_i),\ 1 \le i \le n$, of independent replicates of $(Z,\ \delta,\ \mu)$, we can use the classic non-parametric estimator of $H^{01}(t)$

$$H_n^{01}(t) = \frac{1}{n} \sum_{i=1}^{n} 1_{\{Z_i \le t,\ \delta_i = 0,\ \mu_i = 1\}}$$

for H^{01}. Furthermore, recalling $Y \sim G$ and $X \sim F$, we can consistently estimate F and G with their Kaplan-Meier estimators

$$\hat{F}_n = \sum_{i=1}^{n} W_{ni}^F \cdot \delta_{Z_{i:n}} \qquad \text{and} \qquad \hat{G}_n = \sum_{i=1}^{n} W_{ni}^G \cdot \delta_{Z_{i:n}}\ .$$

Here $\delta_{Z_{i:n}}$ is the Dirac measure concentrated at the i-th order statistic $Z_{i:n}$, while

$$W_{ni}^F = \frac{\delta_{[i:n]}}{n-i+1} \prod_{j=1}^{i-1} \left(\frac{n-j}{n-j+1} \right)^{\delta_{[j:n]}}$$

and

$$W_{ni}^G = \frac{1-\delta_{[i:n]}}{n-i+1} \prod_{j=1}^{i-1} \left(\frac{n-j}{n-j+1} \right)^{1-\delta_{[j:n]}}$$

are the Kaplan-Meier weights attached to the ith order statistic $Z_{i:n}$ under $\hat{F}_n$ and $\hat{G}_n$, respectively. $\delta_{[i:n]}$ is taken to mean the δ-concomitant of $Z_{i:n}$. If we plug this into (1.3), we obtain the empirical analogue

$$\int_0^t A(y)\, \hat{G}_n(dy) \sim H_n^{01}(t) + \int_0^t \hat{F}_n(y)\, \hat{G}_n(dy) \; . \tag{1.4}$$

The right hand side is completely known (i.e., computable from the sample data), but the left hand side contains the unknown d.f. A. We can, however, use the relationship in (1.4) to construct tests for various hypotheses about A. In this work we will consider the simple hypothesis

$$\mathsf{H}_0 : A = A_0 \; ,$$

where A_0 is completely specified.

2 Testing The Simple Hypothesis $\mathsf{H}_0 : A = A_0$

Recalling (1.4)

$$\int_0^t A \, d\hat{G}_n \sim H_n^{01}(t) + \int_0^t \hat{F}_n \, d\hat{G}_n \; ,$$

we see that under H_0 the left hand side becomes computable as well.
Now, $\int_0^t A_0(y)\, \hat{G}_n(dy)$ constitutes a Kaplan-Meier integral (process) for which a linear expansion (uniformly in t) has already been obtained in Stute (1995).

H_n^{01} is a simple empirical sub-d.f., but the integral $\int_0^t \hat{F}_n(y)\, \hat{G}_n(dy)$ will require quite a bit more work to handle. Motivated by (1.4) we are led then to consider the process

$$C_n(t) = \sqrt{n} \left(\int_0^t A_0 \, d\hat{G}_n - H_n^{01}(t) - \int_0^t \hat{F}_n \, d\hat{G}_n \right) \; . \tag{2.1}$$

Tests of H_0 will then be based on C_n, with H_0 being rejected when $C_n(t)$ at a given point t or an appropriate discrepancy of the function C_n exceeds a critical value. To obtain such values we need the limit distribution of C_n. For this, in the following, we will concentrate on expanding C_n uniformly in t into a sum of centered, independent processes plus remainder under H_0. That is

$$C_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \zeta(t, Z_i, \delta_i, \mu_i) + o_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) , \tag{2.2}$$

where

$$\xi(t, Z_i, \delta_i, \mu_i) = -K_i(t) + L_i(t) - M_i(t) - N_i(t)$$

are centered i.i.d. processes to be specified later. This will give us

Theorem 2.1. *Under* H_0*,*

$$C_n \longrightarrow C \quad \textit{in distribution,}$$

where C is a centered Gaussian process (with complicated covariance).

Though the limit process is complicated, the linear expansion (2.2) is useful because it allows us to approximate relevant distributions through a wild bootstrap. This will be illustrated in Section 3.

In order to show (2.2) and specify K_i, L_i, M_i and N_i, we first note that by (1.3)

$$C_n(t) = \sqrt{n}\left(\int_0^t A_0 \, d\hat{G}_n - \int_0^t A_0 \, dG - \left(H_n^{01}(t) - H^{01}(t)\right) - \left(\int_0^t \hat{F}_n \, d\hat{G}_n - \int_0^t F \, dG\right)\right) . \tag{2.3}$$

Obviously,

$$\begin{aligned} H_n^{01}(t) - H^{01}(t) &= \frac{1}{n} \sum_{i=1}^{n} \left(1_{\{Z_i \le t,\ \delta_i = 0,\ \mu_i = 1\}} - H^{01}(t)\right) \\ &\equiv \frac{1}{n} \sum_{i=1}^{n} K_i(t) \end{aligned} \tag{2.4}$$

is already a sum of centered independent processes.

As mentioned before, a representation of the Kaplan-Meier integral $\int_0^t A_0 \, d\hat{G}_n$ as a linear expansion (uniformly in t) plus remainder is found in Stute (1995). We will rely heavily on this representation in the following.

Let $\tau_H = \inf\{x : \ H(x) = 1\} \le \infty$. In the following we will consider the process $C_n(t)$ only for $t \le T$ for some $T < \tau_H$. We will also work under the assumption that F and G (and thus H) are all continuous. In this case, Stute (1995), Theorem 1.1 and the remark on p. 438

therein give us

$$
\begin{aligned}
\int_0^t A_0\, d\hat{G}_n - \int_0^t A_0\, dG &= \frac{1}{n}\sum_{i=1}^n 1_{[0,t]}(Z_i)A_0(Z_i)\gamma_0(Z_i)(1-\delta_i) - \int_0^t A_0 dG \\
&\quad + \frac{1}{n}\sum_{i=1}^n \gamma_1(Z_i,t)\delta_i - \frac{1}{n}\sum_{i=1}^n \gamma_2(Z_i,t) + R_n(t) \\
&\equiv \frac{1}{n}\sum_{i=1}^n L_i(t) + R_n(t), \qquad (2.5)
\end{aligned}
$$

where $\sup\limits_{0\le t\le T} |R_n(t)| = o_{\mathbb{P}}(\frac{1}{\sqrt{n}})$ and

$$\gamma_0(x) = \exp\left\{\int_0^x \frac{H^1(dz)}{1-H(z)}\right\} \quad = \quad \frac{1}{1-F(x)} \quad \text{for } x < \tau_H\ , \qquad (2.6)$$

$$\gamma_1(x,t) = \frac{1}{1-H(x)}\int 1_{\{x<w\}}1_{[0,t]}(w)A_0(w)\gamma_0(w)\ H^0(dw)\ , \qquad (2.7)$$

and

$$\gamma_2(x,t) = \iint \frac{1_{\{v<x,\ v<w\}}1_{[0,t]}(w)A_0(w)\gamma_0(w)}{(1-H(v))^2}\ H^1(dv)H^0(dw)\ . \qquad (2.8)$$

It remains then to find a representation of $\int_0^t \hat{F}_n\, d\hat{G}_n - \int_0^t F\, dG$ as a sum of independent processes plus remainder. Firstly,

$$\int_0^t \hat{F}_n\, d\hat{G}_n - \int_0^t F\, dG = \int_0^t \left(\hat{F}_n - F\right) d\hat{G}_n + \int_0^t F\, d\hat{G}_n - \int_0^t F\, dG\ . \qquad (2.9)$$

Since $1_{[0,t]}\cdot F$ is deterministic, $\int_0^t F\, d\hat{G}_n$ can be handled exactly as $\int_0^t A_0\, d\hat{G}_n$ in (2.5), yielding

$$
\begin{aligned}
\int_0^t F d\hat{G}_n - \int_0^t F dG &= \frac{1}{n}\sum_{i=1}^n 1_{[0,t]}(Z_i)F(Z_i)\gamma_0(Z_i)(1-\delta_i) - \int_0^t F dG \\
&\quad + \frac{1}{n}\sum_{i=1}^n \tilde{\gamma}_1(Z_i,t)\delta_i - \frac{1}{n}\sum_{i=1}^n \tilde{\gamma}_2(Z_i,t) + \tilde{R}_n(t) \\
&\equiv \frac{1}{n}\sum_{i=1}^n M_i(t) + \tilde{R}_n(t), \qquad (2.10)
\end{aligned}
$$

where $\sup_{0\le t\le T} |\tilde{R}_n(t)| = o_{\mathbb{P}}(\frac{1}{\sqrt{n}})$ and

$$\tilde{\gamma}_1(x,t) = \frac{1}{1-H(x)} \int 1_{\{x<w\}} 1_{[0,t]}(w) F(w) \gamma_0(w) H^0(dw)$$
$$\tilde{\gamma}_2(x,t) = \iint \frac{1_{\{v<x, v<w\}} 1_{[0,t]}(w) F(w) \gamma_0(w)}{(1-H(v))^2} H^1(dv) H^0(dw).$$

Thus, we need to focus on finding a linearization with remainder of the first term $\int_0^t \left(\hat{F}_n - F\right) d\hat{G}_n$.

Lemma 2.1. *We have, uniformly in* $0 \le t \le T$*,*

$$n^{1/2} \int_0^t (\hat{F}_n - F) d\hat{G}_n = \frac{1}{\sqrt{n}} \sum_{i=1}^n N_i(t) + o_{\mathbb{P}}(n^{-1/2}), \tag{2.11}$$

where

$$\begin{aligned} N_i(t) = {} & \frac{\delta_i}{1-G(Z_i)} [G(t) - G(Z_i)] 1_{\{Z_i \le t\}} - \int_0^t F dG \\ & + (1-\delta_i) \int_0^t [F(x) - F(Z_i)] 1_{\{Z_i \le x\}} G(dx) \\ & - \int_0^t \int \frac{1-F(v)}{[1-H(v)]^2} 1_{\{v<Z_i\}} [F(x) - F(v)] 1_{\{v \le x\}} G(dv) G(dx). \end{aligned}$$

It is indeed easy to check that $N_i(t)$ has expectation zero.

PROOF. Write

$$n^{1/2} \int_0^t (\hat{F}_n - F) d\hat{G}_n = n^{1/2} \int_0^t (\hat{F}_n(x) - F(x)) G(dx) + n^{1/2} \int_0^t (\hat{F}_n - F)(d\hat{G}_n - dG).$$

If we apply Theorem 1.1. in Stute (1995) for indicators, i.e., for $\hat{F}_n(x)$, and integrate out, we obtain the right-hand side of (2.11). It thus suffices to show that

$$n^{1/2} \int_0^t (\hat{F}_n - F)(d\hat{G}_n - dG) = o_{\mathbb{P}}(1). \tag{2.12}$$

For this, introduce the Kaplan-Meier process

$$\hat{\alpha}_n(x) = n^{1/2} [\hat{F}_n(x) - F(x)], \qquad 0 \le x \le T.$$

It follows from Breslow and Crowley (1974) or Stute (1995), Theorem 1.1., that $\hat{\alpha}_n$ is asymptotically C-tight, i.e., for given positive ε and ρ there exists $\delta > 0$ such that for all large n

$$\mathbb{P}\left(\sup_{|x-\tilde{x}|\leq\delta} |\hat{\alpha}_n(x) - \hat{\alpha}_n(\tilde{x})| \geq \varepsilon\right) \leq \rho.$$

Moreover, the sample paths of $\hat{\alpha}_n$ are uniformly bounded with large probability. For functions satisfying such oscillation bounds Rao (1962) has shown that a uniform Glivenko-Cantelli result holds. Since in our case, we have these bounds only with large probability, a modification of Rao's (1962) arguments together with the SLLN for Kaplan-Meier due to Stute and Wang (1993) yields

$$\int_0^t \hat{\alpha}_n(d\hat{G}_n - dG) = o_{\mathbb{P}}(1).$$

This completes the proof of Lemma 2.1. □

3 A Simulation

In order to finish constructing an asymptotic level α test for H_0, it still remains to determine critical values for a test statistic $\Phi\left(C_n\right)$, where Φ is a functional operating in the Skorokhod space $\mathcal{D}\left([0,T]\right)$ such that larger values of $\Phi\left(C_n\right)$ support a rejection of H_0, for example

$$\Phi\left(C_n\right) = \sup\left\{|C_n(t)| : 0 \leq t \leq T\right\}\ .$$

As mentioned before, however, due to the complexity of the covariance structure of the limit process C of C_n, it is not feasible to use the covariance function of C – which is *not* distribution-free – to study its distribution in order to determine critical values. Neither is it feasible to collect further statistics $\Phi\left(C_{n,\,j}\right),\ 1 \leq j \leq m$, since we only have one sample $(Z_i,\ \delta_i,\ \mu_i),\ 1 \leq i \leq n$, all elements of which are needed in the calculation of C_n. Thus, one approach is to use a wild bootstrap procedure to approximate critical values of the limit distribution.

3.1 The Simulation Procedure

After a sample $(Z_i,\ \delta_i,\ \mu_i),\ 1 \leq i \leq n$, has been collected (or in the case of a simulation, computer generated) we are in a position to calculate the process C_n at any point $t \in [0,T]$ and we can write

$$C_n(t) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\zeta\left(t,\ Z_i,\ \delta_i,\ \mu_i\right) + o_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right)$$

with

$$\zeta\left(t,\ Z_i,\ \delta_i,\ \mu_i\right) \equiv -K_i(t) + L_i(t) - M_i(t) - N_i(t), \ \ \text{for } 1 \leq i \leq n\ .$$

At this point, we generate m sets of n i.i.d. normally distributed random variables $\xi_1^{(j)}, \cdots,$ $\xi_n^{(j)},\ 1 \leq j \leq m$, with mean 0 and variance 1, independent of Z, δ, and μ as well, and use them to weight the leading part of the C_n sum, which gives us m new processes

$$C_n^{(j)}(t) \equiv \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \xi_i^{(j)} \cdot \zeta\left(t,\ Z_i,\ \delta_i,\ \mu_i\right), \quad 1 \leq j \leq m\,,$$

and thus m new statistics

$$\Phi\left(C_n^{(j)}\right), \quad 1 \leq j \leq m\,.$$

Conditioned on the sample $(Z_i,\ \delta_i,\ \mu_i),\ 1 \leq i \leq n$, these m processes are each centered Gauss processes with covariance asymptotically identical to C.

Therefore, rejecting H_0 when

$$\Phi\left(C_n\right) > \Phi\left(C_n^{(j)}\right)$$

for $\lfloor (1-\alpha)\cdot m \rfloor$ of the $j \in \{1, \ldots, m\}$ can be used as an asymptotic level α test for H_0. We will simulate the n observations $(Z_1,\ \delta_1,\ \mu_1), \ldots, (Z_n,\ \delta_n,\ \mu_n)$ with the following model:

Let G and A_0 be exponential distributions with parameters λ_1 and $\lambda_2 > 0$ respectively. Since it is important for our model as described in Section 1 that $X \geq D$ wp1, we set $X = D + X_0$, with $X_0 \sim \text{Exp}(\lambda_3)$, independent of D, being the subject's lifetime *after* disease onset. We need only generate n realizations of $Y \sim G$, $D \sim A_0$, and $X_0 \sim \text{Exp}(\lambda_3)$ respectively to obtain our sample $(Z_i,\ \delta_i,\ \mu_i),\ 1 \leq i \leq n$.

Based on this sample and the known distributions $G = \text{Exp}(\lambda_1)$, $A_0 = \text{Exp}(\lambda_2)$, and $F = \text{Exp}(\lambda_2) * \text{Exp}(\lambda_3)$, we are in a position to calculate $\Phi\left(C_n\right)$ as well as the bootstrap values and to make a decision for or against rejection of H_0 based on the criteria above.

3.2 Results

The approach outlined above was implemented with the program *R 2.5.1.* After setting $\lambda_1 = \frac{1}{30}$, $\lambda_2 = \frac{1}{5}$, and $\lambda_3 = \frac{1}{15}$, the following sample of size $n = 35$ was generated and shown in Table 1.

First, this sample was used to generate bootstrap values together with the "true" distributions F, G, H, and H^{01}, as they appear in the K_i, L_i, M_i and N_i. For our purposes, we choose the test statistic

$$\Phi\left(C_n\right) = |C_n(t_1)|$$

for a fixed $t_1 \in [0, T]$. Since the process C_n is (neglecting remainder terms) centered under H_0, we have $\mathsf{E}\left[C_n(t_1)\right] = 0$, meaning that larger values of $|\Phi\left(C_n\right)|$ support rejection of H_0. As an example, Figure 1 shows an empirical d.f. of the first set of $m = 50$ bootstrap values generated with the sample above and the original statistic $C_n(t_1)$ for $t_1 = 10$.

Table 1: A sample of size $n = 35$ drawn under censorship and sacrificing

i	Z_i	δ_i	μ_i	i	Z_i	δ_i	μ_i
1	45.6309040	1	1	19	14.7202004	1	1
2	12.1424422	0	1	20	18.6115287	0	1
3	4.3995801	0	1	21	10.6941886	0	0
4	6.5584534	1	1	22	20.6657402	0	1
5	2.6857854	0	0	23	23.3815238	1	1
6	15.1280329	1	1	24	8.3022079	1	1
7	24.8203449	1	1	25	13.4059135	1	1
8	33.0821729	1	1	26	1.3350373	1	1
9	4.4641163	1	1	27	16.5287285	0	1
10	4.6959084	0	1	28	30.4298429	1	1
11	10.4468399	1	1	29	0.1181929	0	0
12	2.7262884	1	1	30	18.6238358	0	1
13	20.2133506	0	0	31	5.2754026	1	1
14	34.5137933	1	1	32	5.7268069	0	1
15	29.2987938	1	1	33	3.2786180	1	1
16	23.8979167	0	1	34	12.7667918	0	1
17	14.8551747	1	1	35	5.4127871	0	1
18	9.1511955	1	1				

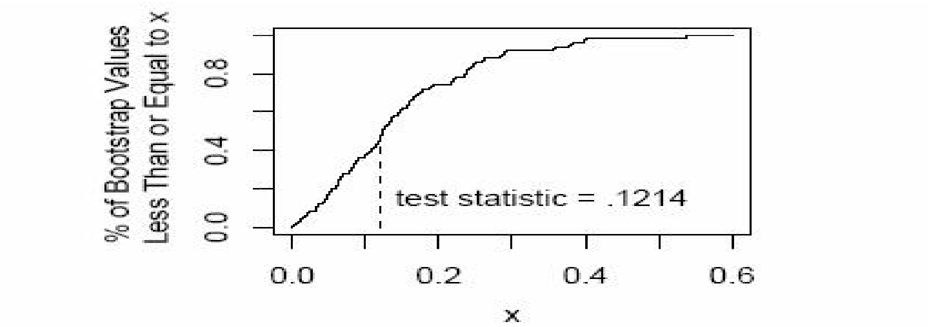


Figure 1: Empirical distribution of 50 bootstrap values with $t_1 = 10$

With $m = 50$, we can test, as described in the previous section, for $\mathsf{H}_0 : A = A_0$ at an α-level of 0.1 by rejecting H_0 when our sample statistic is larger than $\lfloor (1-\alpha) \cdot m \rfloor = 45$ or more of the bootstrap values. Obviously, this first sample did not lead to a rejection of H_0 at α = 0.1, since the statistic for this sample 0.1214 is situated close to the median of the bootstrap values. While operating under the null hypothesis with $A = A_0$, we expect rejection of H_0 to occur in about $100 \times \alpha-$ percent of cases when the test functions correctly. Generating 49 further samples like the one above, then calculating the statistic $|C_n(t_1)|$ for each sample along with $m = 50$ new bootstrap statistics as above yielded 50 repetitions of the test procedure as outlined in the previous section. Of these 50 repetitions, 3 led to rejections of H_0 at $\alpha = 0.1$. Table 2 shows some selected α and corresponding rejection rates.

Table 2: Nominal α levels and actual rejection rates for $t_1 = 10$

α	Rejection Rate
0.05	0.02
0.1	0.06
0.2	0.12
0.25	0.16
0.3	0.26
0.4	0.36
0.5	0.52

Repeating the entire procedure with $t_2 = 25$ yielded similar rates shown in Table 3.
The values above indicate that the asymptotic test functioned correctly under $\mathsf{H}_0 : A = A_0$ with arbitrarily chosen α. Presumably, the nominal and actual rejection rates would match more closely, were a larger sample size numerically feasible.

Clearly it is not possible to implement the test in this fashion in practice, since the true F, G, H, and H^{01} are unknown. For this reason, in a second simulation we switch to the use of the estimators $\hat{F}_n$ for F, $\hat{G}_n$ for G, H_n for H, and H_n^{01} for H^{01}, which due to their adequate rates of convergence, do not affect the convergence of the leading terms of C_n to C when substituted for the true distributions.

Table 3: Nominal α levels and actual rejection rates for $t_2 = 25$

α	Rejection Rate
0.05	0.08
0.1	0.08
0.2	0.14
0.25	0.2
0.3	0.24
0.4	0.3
0.5	0.32

This substitution gives us a viable way to implement the test in practice since the leading terms of C_n are then able to be calculated based solely on the sample together with A_0. Using the same procedure as above with $n = 40$, $m = 50$, and $t_1 = 10$ led to the rejection rates shown in Table 4, indicating that the test functioned properly.

Table 4: Nominal α levels and actual rejection rates for $t_1 = 10$ using empirical distributions $\hat{F}_n$, $\hat{G}_n$, H_n, and H_n^{01} in place of F, G, H, and H^{01}

α	Rejection Rate
0.05	0.02
0.1	0.08
0.2	0.16
0.25	0.2
0.3	0.26
0.4	0.32
0.5	0.38

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