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Semi-Nonparametric Approximations to the Distribution of Certain Portmanteau Statistics

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Abstract

This paper proposes accurate semi-nonparametric approximations to the distribution of certain portmanteau statistics that are expressible as sums of ratios of quadratic forms. Two methodologies, namely the symbolic computational approach and a recursive formula expressing joint moments in terms of joint cumulants, are being utilized to determine the exact moments of the portmanteau statistics. The density functions of those statistics are then approximated on the basis of those moments in terms of gamma density functions and Laguerre polynomials. As verified by a simulation study, the proposed approximations prove more accurate than those that are based on the asymptotic chi-square distribution, especially in the case of time series of short or moderate length.

Keywords and Phrases: Portmanteau Statistics, Ratios of Quadratic Forms, Moments, Gamma Distribution, Laguerre Polynomials, Semi-nonparametric Density Approximations.

AMS Classification: Primary 62E17; Secondary 62M10.

1 Introduction

Box and Pierce (1970), Ljung and Box (1978), Pukkila (1982, 1984), and Dufour and Roy (1985) proposed various portmanteau statistics to detect serial correlation up to a certain lag in a given sequence. Those statistics are in fact linear combinations of squared serial correlations. It has been observed that their distributions deviate markedly from the asymptotic chi-square distribution for series of short or moderate length. Typically, if a portmanteau statistic involves the first m serial correlations from a time series of length n, asymptotic quantities are inaccurate for finite series unless n is much larger than m. The aim of this paper is to provide accurate density approximations which hold even for short time series.

First, we introduce some notation in connection with the concept of serial covariance. Given a series of residuals $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ having a joint normal distribution with mean vector $\boldsymbol{\mu} = \mathbf{0}$ and positive definite covariance matrix $\Sigma = \sigma^2 I$, the *centered serial covariance* at lag k is defined as

$$\bar{c}_k = \frac{1}{n} \sum_{i=1}^{n-k} (\varepsilon_i - \bar{\varepsilon})(\varepsilon_{i+k} - \bar{\varepsilon}) \tag{1.1}$$

for $k = 0, 1, \ldots, n-1$, where $\bar{\varepsilon} = \sum_{i=1}^{n} \varepsilon_i / n$. Throughout this paper, n will denote the length of the series.

In matrix notation, one has

$$\bar{c}_k = \frac{1}{n} \boldsymbol{\varepsilon}' B_k \boldsymbol{\varepsilon} \tag{1.2}$$

where

$$\begin{aligned}
\boldsymbol{\varepsilon}' &= (\varepsilon_1, \ldots, \varepsilon_n), \\
B_k &= V A_k V, \\
V &= (I - \frac{1}{n}\ \boldsymbol{\delta}\, \boldsymbol{\delta}'), \\
\boldsymbol{\delta} &= (1, 1, \ldots, 1)', \\
A_k &= \frac{1}{2} M_k\,, \\
M_k &= L_k + L_k'\,,
\end{aligned}$$

and L_k is a null matrix with the zeros in its kth subdiagonal replaced by ones. The *centered lag-k serial correlation* coefficient is then given by

$$\bar{r}_k = \frac{\bar{c}_k}{\bar{c}_o} = \frac{\boldsymbol{\varepsilon}' B_k \boldsymbol{\varepsilon}}{\boldsymbol{\varepsilon}' V \boldsymbol{\varepsilon}}\ . \tag{1.3}$$

Note that $B_o = V A_o V = V I V = V$, V being idempotent. This definition of the serial correlation coefficient was used by Anderson (1971) and Anderson (1990), among others. The case $k = 1$ is treated in Hannan (1970), and Moran (1948) obtained the first four moments for this case. The *noncentered lag-k serial covariance* is given by

$$c_k = \frac{\sum_{i=1}^{n-k} \varepsilon_i \varepsilon_{i+k}}{n} = \frac{\boldsymbol{\varepsilon}' A_k \boldsymbol{\varepsilon}}{n}, \tag{1.4}$$

the corresponding *noncentered lag–k serial correlation* being

$$r_k = \frac{c_k}{c_o}. \tag{1.5}$$

Merikoski and Pukkila (1983) made use of this simpler representation in connection with a moment problem.

Definition (1.1) can be viewed as definition (1.4) applied to the centered (or mean-corrected) series $y_1 - \bar{y}, \ldots, y_n - \bar{y}$. In particular, for a series of residuals resulting from fitting a model to data, the mean is zero. So, then, the properties of the noncentered serial covariances (1.4), as opposed to (1.1), legitimately become of interest.

As pointed out by Box and Jenkins (1976), it is usual to verify the adequacy of a candidate fit to the data by testing for the 'whiteness' of the resulting residual series. Consider a discrete time series, $\{Z_t\}$, generated by a stationary autoregressive moving average process of order (p, q),

$$(1 - \phi_1 B - \cdots - \phi_p B^p)\, Z_t = (1 - \theta_1 B - \cdots - \theta_q B^q)\, \varepsilon_t, \tag{1.6}$$

where B is the backward shift operator such that, for any function $f(\cdot)$, $B^s f(t) = f(t - s)$, and ε_t denotes the random error at time t.

We assume that p and q in model (1.6) have been correctly identified and that the coefficients ϕ_i and θ_j have been efficiently estimated, and denote the residual series by $\{\hat{\varepsilon}_1, \ldots, \hat{\varepsilon}_n\}$. In order to test for residual serial correlations up to lag m, Box and Pierce (1970) proposed the following overall or portmanteau statistic

$$T(m, n) = n \sum_{k=1}^{m} r_k^2, \qquad m > p + q, \tag{1.7}$$

which is asymptotically distributed as a chi-square random variable with m degrees of freedom. Unfortunately, $T(m, n)$ converges only rather slowly with increasing n to its asymptotic distribution, so that the test performs poorly for all but large data sets.

Ljung and Box (1978) and Davies, Triggs and Newbold (1977) gave some indication of the shortcomings of considering (1.7) for finite series and discussed the improvement of using

$$T'(m, n) = \sum_{k=1}^{m} \frac{n(n+2)}{n-k}\, r_k^2. \tag{1.8}$$

This modification takes account of the fact that $\text{Var}[r_k] = (n-k)/(n(n+2))$, see for example Moran (1948). The first two moments of $T'(m,n)$ are closer to those of the targeted chi-square, but otherwise the problem of slow convergence is not being addressed. Clearly when n is large, $T(m,n) \approx T'(m,n)$.

As shown in Section 2, on writing the portmanteau statistics as linear combinations of squares of ratios of quadratic forms, one can express their moments in terms of certain linear combinations of joint moments of the quadratic forms. A symbolic computational methodology for determining such joint moments as well as a recursive formula expressing joint moments in terms of joint cumulants are introduced in Section 3. A semi-nonparametric density approximant which is based on the first s moments of the portmanteau statistics and expressed in terms of Laguerre polynomials, is then presented in Section 4; a gamma approximation is obtained as a particular case. Bounds for the supports of the distributions of the statistics are determined in Section 5. Certain percentiles of $T'(m,n)$ are evaluated under various approximations and compared with those obtained by simulation for selected values of m and n in Section 6.

2 A Representation of the Moments

A useful representation of the moments of the portmanteau statistics as specified by (1.7) or (1.8) is derived in this section. Let

$$Q_i = \mathbf{X}' A_i \mathbf{X}$$

where $\mathbf{X} \sim \mathcal{N}_n(\mathbf{0}, I)$, that is, $\mathbf{X}$ has an n-variate normal distribution with $\mathbf{0} = (0,0,\ldots,0)'$ as its mean vector and I, the identity matrix of order n, as its covariance matrix, and $A_i = L_i + L_i'$, L_i being as defined in (1.2). Consider a statistic having the following structure

$$T \;=\; \sum_{i=1}^{m} c_i \frac{Q_i^2}{Q_0^2} \tag{2.1}$$

where the c_i's are known constants and $Q_0 = \mathbf{X}' A_0 \mathbf{X}$ with $A_0 = I$. The multinomial expansion of T^r yields the following representations of the rth moment of T:

$$E(T^r) = E\left(\sum_{i=1}^{m} c_i \frac{Q_i^2}{Q_0^2}\right)^r$$

$$
\begin{aligned}
&= \left\{ \sum_{r_1=0}^{r} \sum_{r_2=0}^{r-r_1} \cdots \sum_{r_{m-1}=0}^{r-r_1-\ldots-r_{m-2}} \binom{r}{r_1, \ldots, r_m} \right\} E\left[\left(c_1 \frac{Q_1^2}{Q_0^2} \right)^{r_1} \cdots \left(c_m \frac{Q_m^2}{Q_0^2} \right)^{r_m} \right] \\
&= \left\{ \sum_{r_1=0}^{r} \sum_{r_2=0}^{r-r_1} \cdots \sum_{r_{m-1}=0}^{r-r_1-\ldots-r_{m-2}} \binom{r}{r_1, \ldots, r_m} \right\} \frac{\Gamma(\frac{n}{2})}{2^{2r}\Gamma(\frac{n}{2}+2r)} \\
&\qquad \times E(c_1^{r_1} Q_1^{2r_1} \cdots c_m^{r_m} Q_m^{2r_m}) \qquad (2.2)
\end{aligned}
$$

where

$$
\binom{r}{r_1, \ldots, r_m} = \frac{r!}{r_1! \cdots r_m!}
$$

and

$$
r_m = r - \sum_{j=1}^{m-1} r_j .
$$

The last equality in (2.2) is obtained by expressing $1/Q_0^{2r} = 1/(\mathbf{x}'\mathbf{x})^{2r}$ as

$$
\int_0^\infty \frac{z^{2r-1} e^{-z(\mathbf{x}'\mathbf{x})}}{\Gamma(2r)} \, \mathrm{d}z \, ,
$$

noting that the integrand is proportional to a gamma density function with parameters $2r$ and $(\mathbf{x}'\mathbf{x})^{-1}$. Thus,

$$
\begin{aligned}
E\left[\left(c_1 \frac{Q_1^2}{Q_0^2} \right)^{r_1} \cdots \left(c_m \frac{Q_m^2}{Q_0^2} \right)^{r_m} \right] &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{c_1^{r_1} Q_1^{2r_1} \cdots c_m^{r_m} Q_m^{2r_m}}{Q_0^{2r}} \, \frac{e^{-\frac{1}{2}\mathbf{x}'\mathbf{x}}}{(2\pi)^{n/2}} \, \mathrm{d}\mathbf{x} \\
&= \int_0^\infty \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} c_1^{r_1} Q_1^{2r_1} \cdots c_m^{r_m} Q_m^{2r_m} \frac{(1+2z)^{n/2}}{(1+2z)^{n/2}} \\
&\qquad \times \frac{e^{-\frac{1}{2}(\mathbf{x}'\mathbf{x})(1+2z)} z^{2r-1}}{(2\pi)^{n/2}\Gamma(2r)} \, \mathrm{d}\mathbf{x} \, \mathrm{d}z \, ,
\end{aligned}
$$

which, on letting $\mathbf{y} = (1+2z)^{1/2}\mathbf{x}$ is seen to be equal to

$$\begin{aligned}
& \int_0^\infty \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty \frac{z^{2r-1}(\mathbf{y}'A_1\mathbf{y})^{2r_1}\cdots(\mathbf{y}'A_m\mathbf{y})^{2r_m}}{\Gamma(2r)(1+2z)^{2r+\frac{n}{2}}} \;\; \frac{e^{-\frac{1}{2}\mathbf{y}'\mathbf{y}}}{(2\pi)^{n/2}}\,\mathrm{d}\mathbf{y}\,\mathrm{d}z \\
= \;& \frac{1}{\Gamma(2r)}\left(\int_0^\infty z^{2r-1}(1+2z)^{-(\frac{n}{2}+2r)}\mathrm{d}z\right)\,(\prod_{i=1}^m c_i^{r_i})\,E(Q_1^{2r_1}\cdots Q_m^{2r_m}) \\
= \;& \frac{1}{\Gamma(2r)}\left(\int_0^\infty \frac{1}{2^{2r-1}}u^{2r-1}(1+u)^{-(\frac{n}{2}+2r)}\frac{1}{2}\,\mathrm{d}u\right)\,(\prod_{i=1}^m c_i^{r_i})\,E(Q_1^{2r_1}\cdots Q_m^{2r_m}) \\
= \;& \frac{\Gamma(n/2)}{2^{2r}\Gamma(2r+n/2)}\,(\prod_{i=1}^m c_i^{r_i})\,E(Q_1^{2r_1}\cdots Q_m^{2r_m}), \qquad (2.3)
\end{aligned}$$

noting that

$$\int_0^\infty y^{\alpha-1}(1+y)^{\alpha+\beta}\mathrm{d}y = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\,.$$

For $T(m,n)$, the original portmanteau statistic, $c_i = n,\ i = 1,2,\ldots,m$, whereas $c_i = n(n+2)/(n-i),\ i = 1,2,\ldots,m$ in $T'(m,n)$, the modified portmanteau statistic.

3 Methodologies for Evaluating the Moments

Two techniques are proposed for determining the exact moments of the test statistics, namely the symbolic computational approach and the application of a general recursive formula which expresses joint moments in terms of joint cumulants.

3.1 The Symbolic Computational Approach

By making use of symbolic computational packages such as Maple or *Mathematica*, one can define an expected value operator $\mathcal{E}$ having the following properties:

$$\mathcal{E}[\sum_{i=1}^p \alpha_i Y_i] \;=\; \sum_{i=1}^p \alpha_i\,\mathcal{E}(Y_i)$$

and

$$\mathcal{E}(\prod_{i=1}^p Y_i^{s_i}) \;=\; \prod_{i=1}^p \mathcal{E}(Y_i^{s_i})\,,$$

where the α_i's and s_i's are constants and the Y_i's are independently distributed random variables, $i = 1,\ldots,p$. After expressing the quadratic forms $Q_k^{2r_k}$ in (2.2) as

$$Q_k^{2r_k} = (\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^{(k)} X_i X_j)^{2r_k} \tag{3.1}$$

where the $a_{ij}^{(k)}$'s are the elements of the matrix A_k defined in (1.2), and expanding, one obtains a linear combination of products of powers of independent standard Gaussian random variables, which on application of the expected value operator yields the rth moment of $T(m,n)$ or $T'(m,n)$.

For example, the second moment of $T(2,3)$ can be evaluated as follows. On applying (2.2), one has

$$\frac{\Gamma(\frac{3}{2})}{2^4\Gamma(\frac{3}{2}+4)}\, 3^2 \left\{ \binom{2}{0,2} E(Q_1^0\, Q_2^4) + \binom{2}{1,1} E(Q_1^2\, Q_2^2) + \binom{2}{2,0} E(Q_1^4\, Q_2^0) \right\} \tag{3.2}$$

where $Q_k = \sum_{i=1}^{3-k} X_i X_{i+k}$, so that $Q_1^2 = (X_1X_2 + X_2X_3)^2$ and $Q_2^2 = (X_1X_3)^2$, which on expanding and simplifying gives

$$\frac{1}{105}\, 2!\, E(\frac{1}{2}X_1^4X_2^4 + 2X_1^3X_2^4X_3 + X_1^4X_2^2X_3^2 + 3X_1^2X_2^4X_3^2 + 2X_1^3X_2^2X_3^3 + 2X_1X_2^4X_3^3$$
$$+\frac{1}{2}X_1^4X_3^4 + X_1^2X_2^2X_3^4 + \frac{1}{2}X_2^4X_3^4),$$

where $X_1,\ X_2$ and X_3 are independently distributed $\mathcal{N}(0,1)$ random variables whose kth moment is 0 when k is odd and $2^{k/2}\Gamma\left(k+1/2\right)/\sqrt{\pi}$ when k is even. The second moment so obtained is $19/35$.

Similarly, it can be verified that the first, third, fourth and fifth moments of $T(2,3)$ are respectively $3/5,\ 3051/5005,\ 65853/85085,\ 341469/323323$.

3.2 General Recursive Formula for Obtaining Joint Moments from Joint Cumulants

Letting $Q_i = \mathbf{X}' A_i \mathbf{X},\ \ i = 1,\ldots,\eta,$ where A_i is a symmetric matrix and $\mathbf{X} \sim \mathcal{N}_n(\mathbf{0}, V)$, the joint cumulant generating function of $Q_1,\ldots,Q_\eta$ is

$$\mathcal{K}_{Q_1,\ldots,Q_\eta}(t_1,\ldots,t_\eta) = \ln|I-W|^{-\frac{1}{2}} \ = \ \frac{1}{2}\sum_{j=1}^{\infty}\mathrm{tr}(W^j)/j \tag{3.3}$$

where $W = 2\sum_{i=1}^{\eta}(t_i V A_i)$. This is explained for instance in Mathai and Provost (1992, Section 3.3). The joint moments, $E[(Q_1 - E(Q_1))^{s_1}\cdots(Q_\eta - E(Q_\eta))^{s_\eta}] \equiv \mu_{s_1,\ldots,s_\eta}$, (we note that $E(Q_i)$ is equal to zero for $i = 1,\ldots,\eta$ in this case) can then be determined from the joint cumulants by making use of the following recursive relationship derived by Smith (1995):

$$\mu_{s_1,s_2,\ldots,s_{m+1}} = \sum_{i_1=0}^{s_1}\cdots\sum_{i_m=0}^{s_m}\sum_{i_{m+1}=0}^{s_{m+1}-1}\binom{s_1}{i_1}\cdots\binom{s_m}{i_m}\binom{s_{m+1}-1}{i_{m+1}} \times K_{s_1-i_1,s_2-i_2,\ldots,s_{m+1}-i_{m+1}}\,\mu_{i_1,i_2,\ldots,i_{m+1}},\quad m=1,2,\ldots,\eta-1, \tag{3.4}$$

where $\mu_{0,0,\ldots,0} = 1$ and $K_{a_1,\ldots,a_{m+1}}$ denotes the joint cumulant of orders $a_1,\ldots,a_{m+1}$ of $Q_1,\ldots,Q_{m+1}$, which is equal to

$$\frac{\partial^{a_1+\cdots+a_{m+1}}}{\partial t_1^{a_1}\cdots\partial t_{m+1}^{a_{m+1}}}\,\mathcal{K}_{Q_1,\ldots,Q_{m+1}}(t_1,\ldots,t_{m+1}) \text{ evaluated at } t_i=0,\ i=1,\ldots,m+1.$$

Thus, the rth moment of $T(m,n)$ or $T'(m,n)$ can be determined from the joint moments $E(Q_1^{2r_1}\cdots Q_{n-k}^{2r_{n-k}})$. For example, for $n=3$ and $k=2$, (3.2) becomes $(1/105)\,(E(Q_1^0\,Q_2^4)+ 2E(Q_1^2\,Q_2^2)+E(Q_1^4\,Q_2^0))$ where

$$\begin{aligned} E(Q_1^0\,Q_2^4) &= \mu_{0,4} \\ &= \sum_{j=0}^{3}\binom{3}{j}K_{0,n-j}\,\mu_{0,j} = 9 \end{aligned}$$

with $\mu_{0,0}=1$ and

$$\begin{aligned} K_{0,\ell} &= \frac{\partial^\ell}{\partial t_1^\ell}\left(\frac{1}{2}\sum_{j=1}^{\infty}\frac{\mathrm{tr}(2t_1A_2)^j}{j}\right) \text{ evaluated at } t_1=0 \\ &= 2^{\ell-1}(\ell-1)!\,\mathrm{tr}A_2^\ell\,; \\ 2E(Q_1^2\,Q_2^2) &= 2\mu_{2,2} \\ &= 2\sum_{i=0}^{2}\sum_{j=0}^{1}\binom{2}{i}\binom{1}{j}K_{2-i,2-j}\,\mu_{i,j} = 12 \end{aligned}$$

where

$$\begin{aligned} \mu_{r,t} &= \sum_{i=0}^{r}\sum_{j=0}^{t-1}\binom{r}{i}\binom{t-1}{j}K_{r-i,t-j}\,\mu_{i,j}\,, \\ K_{h,\ell} &= \frac{\partial^{h+\ell}}{\partial t_1^h\partial t_2^\ell}\,\mathcal{K}_{Q_1,Q_2}(t_1,t_2) \text{ evaluated at } t_1=0,\ t_2=0 \\ &= \frac{h!\,\ell!\,2^{h+\ell}}{2(h+\ell)}\,\mathrm{tr}\sum_{(h,\ell)}(A_1A_2), \end{aligned}$$

and the notation $\sum_{(h,\ell)}(A_1 A_2)$ stands for the sum of all the possible distinct permutations of a product of h matrices A_1 and ℓ matrices A_2; and

$$\begin{aligned} E(Q_1^4\, Q_2^0) &= \mu_{4,0} \\ &= \sum_{i=0}^{3} \binom{3}{i} K_{n-i,0}\, \mu_{i,0} = 36. \end{aligned}$$

This yields $E(T(2,3)^2) = 19/35$. It should be noted that the moments so obtained agree exactly with those determined by means of the symbolic computational approach which, incidentally, was found to be computationally more intensive. The *Mathematica* code for evaluating the moments is available on request.

4 A Semi-Nonparametric Density Approximation Technique

The density functions of numerous statistics whose asymptotic distribution is chi-square can be accurately approximated from their exact moments by means of sums involving Laguerre polynomials. For instance, Laguerre series expansions for the density functions of quadratic forms in normal variables and non-central χ^2 and F random variables were respectively obtained by Gurland (1955) and Tiku (1965). A general methodology for obtaining such expansions is described below. It is assumed that all the moments are finite and that the moment sequence uniquely determines the distribution.

Consider a random variable Y defined on the interval $(0, \infty)$, whose tail behavior is congruent to that of a gamma distribution. Let its jth moment, $E(Y^j)$, be denoted by $\mu_Y[j],\ j = 0, 1, 2, \ldots,$

$$c = \frac{\mu_Y[2] - \mu_Y[1]^2}{\mu_Y[1]}\,, \tag{4.1}$$

$$\alpha = \frac{\mu_Y[1]}{c} - 1 \tag{4.2}$$

and

$$X = \frac{Y}{c}\,. \tag{4.3}$$

The parameters c and α are chosen so that the leading term of the resulting approximating sum given in (4.8) will in fact be a Gamma density function whose first and second moments agree with those of Y.

Denoting the jth moment of the random variable $X,\ E[X^j] = \mu_Y[j]/c^j$, by μ_j, its density function can be expressed as

$$f(x) = \sum_{k=0}^{\infty} a_k \varphi_k(x) \tag{4.4}$$

where

$$\varphi_k(x) \;=\; x^{\alpha} e^{-x} L_k(\alpha) \tag{4.5}$$

and

$$a_k \;=\; (-1)^k \sum_{j=0}^{k} (-1)^i \frac{k!}{j!\,(k-j)!\,\Gamma(\alpha+k-j+1)}\,\mu_{k-j}\,, \tag{4.6}$$

$L_k(\alpha)$ denoting a Laguerre polynomials of degree k with parameter α, which can be obtained as follows:

$$\begin{aligned} L_k(\alpha) \;&=\; \frac{x^{-\alpha}e^{x}}{k!}\,\frac{\partial^k}{\partial x^k}\left(x^{k+\alpha}e^{-x}\right) \\ &=\; (-1)^k \sum_{i=0}^{k} \frac{(-1)^i\Gamma\left(\alpha+k+1\right)}{i!(k-i)!\Gamma\left(\alpha+k-i+1\right)}\,x^{k-i}, \;\; k=0,1,\ldots, \end{aligned} \tag{4.7}$$

where $\alpha > -1$, see for instance Gradshteyn and Ryzhik (1980). Density approximants for $T(m,n)$ or $T'(m,n)$ are determined from their first s moments by truncating the series given in Equation (4.4). Then, on making the change of variables $Y = cX$, one obtains the following density approximation for Y:

$$f_{Y_n}(y) \;=\; \frac{y^{\alpha}e^{-y/c}}{c^{\alpha+1}\Gamma(\alpha+1)} \sum_{k=0}^{s} \Gamma(\alpha+1)\,a_k\,\varphi_k(y/c)\,, \quad y>0. \tag{4.8}$$

The distributions of the statistics $T(m,n)$ or $T'(m,n)$ can also be approximated by a gamma distribution whose parameters, as estimated by the method of moments, are $\alpha+1$ and c where c and α are as defined in (4.1) and (4.2), respectively. This approximation can also be obtained by letting $s=2$ in (4.8).

5 An Upper Bound for the Portmanteau Statistics

The following result, which is stated for instance in Mathai and Provost (1992, Section 2.4), provides an upper bound for the ratios of certain quadratic forms.

Let B be any $n\times n$ positive definite matrix, A be an $n\times n$ symmetric matrix and the eigenvalues of $B^{-1}A$ be $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$; then assuming that $\mathbf{y}$ is not equal to the null vector,

$$\overset{sup}{\mathbf{y}}\left\{\frac{\mathbf{y}'A\mathbf{y}}{\mathbf{y}'B\mathbf{y}}\right\} = \lambda_1\,.$$

Thus, an upper bound for the support of portmanteau statistics of the form

$$\sum_{i=1}^{m} c_i \left(\frac{\boldsymbol{\varepsilon}' A_i \boldsymbol{\varepsilon}}{\boldsymbol{\varepsilon}'\boldsymbol{\varepsilon}} \right)^2 , \quad c_i > 0, \;\; i = 1, \ldots, m,$$

where $\boldsymbol{\varepsilon} \sim \mathcal{N}_n(\mathbf{0}, I)$, is given by $\sum_{i=1}^{m} c_i \lambda_{(i)}^2$ where $\lambda_{(i)}$ is the largest eigenvalue of $A_i,\ i = 1, 2, \ldots, m$. For example, an upper bound for $T'(3,5)$ is given by $(35/4)(\sqrt{3}/2)^2 + (35/3)(1/\sqrt{2})^2 + (35/2)(1/2)^2 = 16.7708$. Clearly, such portmanteau statistics are always nonnegative.

6 Numerical Results

The 50^{th}, 90^{th}, 95^{th} and 99^{th} percentiles of the modified portmanteau statistic $T'(m,n)$, as determined by simulation on the basis of 100,000 replications and under the gamma approximation, the Laguerre polynomial approximation based on the eight moments and the asymptotic chi-square distribution with m degrees of freedom, are reported in Tables 1 to 4 for selected values of n and m. As expected, the Laguerre polynomial approximants generally yield more accurate percentiles than the approximate chi-square distributions or the two-moment gamma approximations.

TABLE 1 : 50^{th} Percentiles of $T'(m,n)$

n	m	Simulated	Laguerre	Gamma	Asymp. χ^2
5	3	2.69165	2.70916	2.58221	2.36597
6	2	1.53712	1.52138	1.49215	1.38629
6	3	2.57850	2.59577	2.49961	2.36597
7	2	1.50337	1.49804	1.47183	1.38629
7	3	2.46886	2.48237	2.43169	2.36597
8	1	0.51086	0.48987	0.55621	0.45494
8	2	1.48937	1.48413	1.45901	1.38629
10	2	1.47624	1.46878	1.44314	1.38629
10	3	2.37574	2.38270	2.35094	2.36597
20	3	2.36643	2.37911	2.32377	2.36597

TABLE 2 : 90^{th} Percentiles of $T'(m,n)$

n	m	Simulated	Laguerre	Gamma	Asymp. χ^2
5	3	5.44501	5.43122	5.63530	6.25139
6	2	4.29576	4.30120	4.37435	4.60517
6	3	5.69489	5.67796	5.88723	6.25139
7	2	4.32332	4.32674	4.42068	4.60517
7	3	5.93427	5.91915	6.07815	6.25139
8	1	2.70660	2.74693	2.55310	2.70554
8	2	4.35381	4.35174	4.44944	4.60517
10	2	4.37041	4.38375	4.48440	4.60517
10	3	6.01153	6.07054	6.20401	6.25139
20	3	6.12558	6.12017	6.35778	6.25139

TABLE 3 : 95^{th} Percentiles of $T'(m,n)$

n	m	Simulated	Laguerre	Gamma	Asymp. χ^2
5	3	6.72258	6.70685	6.79204	7.81473
6	2	5.51856	5.48595	5.57098	5.99146
6	3	6.95853	6.98256	7.20240	7.81473
7	2	5.59186	5.55383	5.65383	5.99146
7	3	7.28998	7.33155	7.52045	7.81473
8	1	3.63472	3.61985	3.49434	3.84146
8	2	5.63245	5.59109	5.70560	5.99146
10	2	5.67897	5.63575	5.76900	5.99146
10	3	7.60121	7.62830	7.75374	7.81473
20	3	7.82557	7.71695	7.99824	7.81473

TABLE 4 : 99^{th} Percentiles of $T'(m,n)$

n	m	Simulated	Laguerre	Gamma	Asymp. χ^2
5	3	9.70612	9.76050	9.33530	11.34490
6	2	8.67075	8.62950	8.30870	9.21304
6	3	10.74810	10.68420	10.12620	11.34490
7	2	8.80420	8.83770	8.48350	9.21304
7	3	11.25610	11.18960	10.75330	11.34490
8	1	5.50168	5.44330	5.75490	6.63490
8	2	8.85718	8.91670	8.59350	9.21304
10	2	8.90635	9.00280	8.72870	9.21304
10	3	11.56150	11.59800	11.25260	11.34490
20	3	12.01090	11.69140	11.71930	11.34490

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