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Some Skew-Symmetric Double Inverted Distributions

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Abstract

We define skew-symmetric distributions based on the double inverted gamma, double inverted Weibull and double inverted compound gamma distributions, all of which have symmetric density about zero. Expressions are derived for the probability density function (pdf), cumulative distribution function (cdf) and the moments of these distributions. However, some of these quantities could not be evaluated in closed forms and we used special functions to express them.

Keywords and Phrases: Double inverted gamma distribution; double inverted Weibull distribution; double inverted compound gamma distribution; skew-symmetric distributions; density function; distribution function; moments.

AMS Classification: 62E15.

1 Introduction

Double inverted gamma, double inverted Weibull and double inverted compound gamma distributions have symmetric densities, obtained by extending the support of the densities of regular gamma, Weibull and compound gamma distributions respectively to the negative half of the real line. The density on the negative half of the real line is just a mirror image of the density on the positive half of the real line. In most cases these distributions are either unimodal (mode at zero) or bimodal with two modes having the same absolute value. Thus these distributions are suitable for modeling bimodal distributions having two modes which are at equal distance from the point of symmetry. However, these distributions may have certain limitations. For instance, even though there may be two distinct modes of the density function, yet the two local maxima may be identical. If m_1 and m_2 are the two modes of the symmetric double inverted Weibull distributions, then $m_1 = -m_2$ and $f(m_1) = f(m_2)$, where $f(.)$ is the density of a double inverted Weibull distribution. In practice, bimodal frequency distributions generally have distinct local maxima and therefore these symmetric bimodal densities with unique local maxima (but non-unique modes) may have limited applications.

In this article, we propose skew-symmetric versions of these double inverted distributions using the concept described in the seminal article by Azzalini (1985). This particular article defined the family of univariate skewed-normal distribution, an extension of symmetric normal distribution to a general class of asymmetrical distributions. An extension of the skewed-normal distribution to the multivariate case was studied by Azzalini and Dalia Valle (1996). Its application in statistics has been considered by Azzalini and Capitanio (1999). As a general result, Azzalini (1985) showed that any symmetric distribution could be viewed as a member of more general class of skewed distributions. In recent years, many authors have studied similar distributions. For example, see Ali and Woo (2006), Aryal and Nadarajah (2005), Nadarajah and Kotz (2003), Wahed and Ali (2001), Arnold and Beaver (2000), and Azzalini and Capitanio (1999).

We explicitly define skewed double inverted gamma, double inverted Weibull and double inverted compound gamma distributions. These distributions reduce to the corresponding symmetric versions when the skewness parameter is set to zero. They allow distinct local maxima for bimodal distributions and therefore provides greater flexibility for modeling bimodal frequency distributions. We also derive the expressions for the probability density function (pdf), cumulative distribution function (cdf) and the moments of these distributions. However, some of these quantities could not be evaluated in closed forms and we used special functions to express them.

The content of the article proceeds as follows. We give a brief review of the literature pertaining to the general method of constructing skew-symmetric distributions such as skew-normal, skew-t, skew-logistic distributions and discuss some of the useful properties in section 2. In section 3 we introduce skewed double inverted gamma distribution and derive some of its properties. Section 4 introduces skewed version of the double inverted Weibull model whereas skewed double inverted compound gamma model is discussed in section 5. We wrap up the article with some remarks in Section 6.

2 Some properties of a skew-symmetric distribution

Let X_1 and X_2 be two independent continuous random variables having identical pdf $g(.)$ which is symmetric about zero. Based on the result of Wahed and Ali (2001), for any real number c, X_1 and cX_2 are independent and have symmetric density about zero. By property of symmetry,

$$\begin{aligned} \frac{1}{2} &= P(X_1 - cX_2 \le 0) \\ \Leftrightarrow \frac{1}{2} &= \int_{-\infty}^{\infty} P(X_1 - cX_2 \le 0 | X_2 = x) g(x) dx \\ \Leftrightarrow \frac{1}{2} &= \int_{-\infty}^{\infty} G(cx) g(x) dx, \end{aligned}$$

where $G(.)$ is the cdf corresponding to the pdf $g(.)$. This leads to the following result:

Fact 2.1. If $g(x)$ and $G(x)$ are respectively the pdf and cdf of a continuous random variable with R^1 as the support of g, and $g(x) = g(-x)$ for all real x, then for any real number c, $f(z; c) \equiv 2g(z)G(cz)$is a skewed pdf of a random variable Z, which will be denoted by $SD(c)$.

Let us denote the cdf of a $SD(c)$ random variable Z by

$$F(z; c) = \int_{-\infty}^{z} f(t; c) dt.$$

The following properties hold as a consequence of the definitions of $f(z; c)$ and $F(z; c)$:

Fact 2.2. (Azzalini, 1985)

(a) $Z \sim SD(c) \quad \Leftrightarrow -Z \sim SD(-c)$, for any real number c.

(b) $F(z; c) = 1 - F(-z; c)$.

(c) $F(z; 1) = G^2(z)$, where $G(x)$is the original cdf of X.

(d) For positive z, $\lim_{c \to \infty} f(z; c)$ becomes a half-distribution of $g(x)$.

From Fact 2.2 (b) and (c) we note that for $c = -1$, $F(z; -1) = 1 - G^2(-z)$. The cdf of $SD(c)$ can be expressed as follows:

$$\begin{aligned} F(z;c) &= \int_{-\infty}^{z} f(t;c)dt = 2\int_{-\infty}^{z} g(t)G(ct)dt \\ &= 2\int_{-\infty}^{z}\int_{-\infty}^{ct} g(t)g(s)dsdt \\ &= G(z) - 2\int_{z}^{\infty}\int_{0}^{ct} g(t)g(s)dsdt. \end{aligned}$$

Defining $J(z;c) = \int_{z}^{\infty}\int_{0}^{ct} g(t)g(s)dsdt$, for every real number z and c, we have the following:

Fact 2.3. (Azzalini, 1985)

1. $J(z;c)$ is a decreasing function of z.
2. $J(z;c) = -\ J(z;-c)$.
3. $J(-z;c) = J(z;c)$.
4. $2\ J(z;1) = G(z)G(-z)$.

3 A double inverted gamma model

An inverted gamma variable with shape parameter α and scale parameter β has the pdf

$$h(x) = \frac{1}{\beta^{\alpha}\Gamma(\alpha)} e^{-\frac{1}{\beta x}} x^{-\alpha-1},\ x > 0,\ \alpha, \beta > 0. \tag{1}$$

The cumulative distribution function (cdf) and moments of the distribution are, therefore given by

$$H(x) = 1 - \frac{\Gamma(\alpha, 1/\beta x)}{\Gamma(\alpha)}$$

$E(X^k) = \frac{\Gamma(\alpha-k)}{\Gamma(\alpha)}\beta^{-k}$, for $\alpha > k$ where $\Gamma(\alpha, 1/\beta x) = \int_{1/\beta x}^{\infty} e^{-z} z^{\alpha-1} dz$.

We can define a double inverted gamma pdf, which is symmetric about zero as

$$g(x) = \frac{1}{2\beta^{\alpha}\Gamma(\alpha)} e^{-\frac{1}{\beta|x|}} |x|^{-\alpha-1},\ x \in R^1,\ \alpha, \beta > 0. \tag{2}$$

The cumulative distribution function (cdf) and moments of the double inverted gamma variable Y with pdf (2) are, therefore, given by

$$G(y) = \frac{1}{2}\left[1 + sgn(y)\frac{\Gamma(\alpha, 1/\beta|y|)}{\Gamma(\alpha)}\right],\ y \in R^1,$$

and

$$E(Y^k) = \frac{1+(-1)^k}{2}.\frac{\Gamma(\alpha-k)}{\Gamma(\alpha)}\beta^{-k},$$

for $\alpha > k$, respectively, where

$$\begin{aligned} sgn(x) &= 1 \text{ if } x \geq 0 \\ &= -1 \text{ if } x < 0. \end{aligned}$$

From Fact 2.1, the pdf of a skewed double inverted gamma distribution is defined as

$$f(z;c) = \frac{1}{2\beta^\alpha\Gamma(\alpha)}e^{-\frac{1}{\beta|z|}}|z|^{-\alpha-1}\left[1 + sgn(cz)\frac{\Gamma(\alpha, 1/\beta|cz|)}{\Gamma(\alpha)}\right],\ z \in R^1,\ c \in R^1. \quad (3)$$

The cdf of the distribution is given by

$$F(z;c) = G(z) - 2\int_z^\infty\int_0^{ct} g(t)g(s)dsdt = G(z) - 2J(z;c),\ c > 0.$$

Using formulas 3.381(3), 8.352(2) and 8.354(2) in Gradshteyn and Ryzhik (1965) and the density (1) of inverted gamma variate we get $J(z;c)$ as follows:
For $z > 0$ and $c > 0$,

$$\begin{aligned} J(z;c) &= \frac{1}{4\Gamma(\alpha)}\left(\frac{c}{c+1}\right)^\alpha \sum_{i=0}^{\alpha-1}\frac{1}{i!(c+1)^i}\gamma\left(\alpha+i, \frac{c+1}{c\beta z}\right),\ \text{if } \alpha \in N \\ &= \frac{\gamma\left(\alpha, \frac{1}{\beta z}\right)}{4\Gamma(\alpha)} - \frac{1}{4\Gamma^2(\alpha)}\sum_{i=0}^{\infty}\frac{(-1)^i c^{-\alpha-i}}{i!(\alpha+i)}\gamma\left(2\alpha, \frac{1}{\beta z}\right),\ \text{if } \alpha \notin N, \end{aligned}$$

where

$$\gamma(a,x) = \int_0^x e^{-z}z^{a-1}dz.$$

From formulas 3.381(4) and 6.455(1) in Gradshteyn and Ryzhik (1965) we obtain the k-th moment of the skewed distribution as:

$E(Z^k;c) = \frac{1+(-1)^k}{2\beta^k\Gamma(\alpha)}\Gamma(\alpha-k)+\frac{1-(-1)^k}{2\beta^k(\alpha-k)\Gamma^2(\alpha)}\Gamma(2\alpha-k)(\frac{c}{1+c})^{\alpha-k}F(1,2\alpha-k;\alpha-k+1;\frac{c}{1+c})$
if $\alpha > k$, $c > 0$, where $F(a,b;c;x) = \sum_{i=0}^{\infty}\frac{(a)_i(b)_i}{(c)_i}.\frac{x^i}{i!}$ is the Gauss hypergeometric function with pochammer symbol $(a)_i = \frac{\Gamma(a+i)}{\Gamma(a)} = a(a+1)(a+2)...(a+i-1)$.
Remark : For $c < 0$,

$$E(Z^k;c) = (-1)^k E((-Z)^k;-c). \tag{4}$$

The mean of the distribution is, therefore, given by

$$E(Z;c) = \frac{\Gamma(2\alpha-1)}{\beta(\alpha-1)\Gamma^2(\alpha)}\left(\frac{c}{1+c}\right)^{\alpha-1} F\left(1,2\alpha-1;\alpha;\frac{c}{1+c}\right), \tag{5}$$

when $\alpha > 1$, $c > 0$.

4 A double inverted Weibull model

The pdf of an inverted Weibull random variable X is given by

$$h(x) = \frac{\alpha}{\beta^\alpha}x^{-\alpha-1}e^{-(\frac{1}{\beta x})^\alpha},\ x > 0,\ \alpha,\beta > 0. \tag{6}$$

The cdf of the distribution is therefore

$$H(x) = e^{-(\frac{1}{\beta x})^\alpha},\ x > 0,$$

and the k-th moment of X is

$$E(X^k) = \beta^{-k}\Gamma\left(1-\frac{k}{\alpha}\right),\ \text{for } \alpha > k.$$

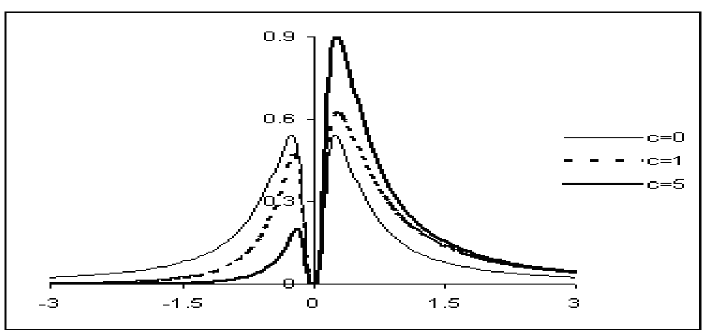

Figure 1: The pdf of skewed double inverted Weibull distribution for c = 0, 1, and 5, and α=1, β=2.

From the pdf (6) of X we can define the pdf of a double inverted Weibull random variable Y as

$$g(y) = \frac{\alpha}{2\beta^\alpha}|y|^{-\alpha-1}e^{-\left(\frac{1}{\beta|y|}\right)^\alpha}, \; y \in R^1 \tag{7}$$

The distribution is symmetric about zero and has the cdf

$$G(y) = \frac{1}{2}\left[1 + sgn(y)e^{-\left(\frac{1}{\beta|y|}\right)^\alpha}\right], \; y \in R^1.$$

The k-th moment of Y comes out to be

$$E(Y^k) = \frac{1+(-1)^k}{2}\beta^{-k}\Gamma\left(1 - \frac{k}{\alpha}\right), \text{ for } \alpha > k.$$

From the Fact 2.1, a skewed inverted double Weibull distribution is, therefore, defined by the pdf

$$f(z;c) = \frac{\alpha}{2\beta^\alpha}|z|^{-\alpha-1}e^{-\left(\frac{1}{\beta|z|}\right)^\alpha}\left[1 + sgn(cz)e^{-\left(\frac{1}{\beta|cz|}\right)^\alpha}\right], \; z \in R^1, \; c \in R^1.$$

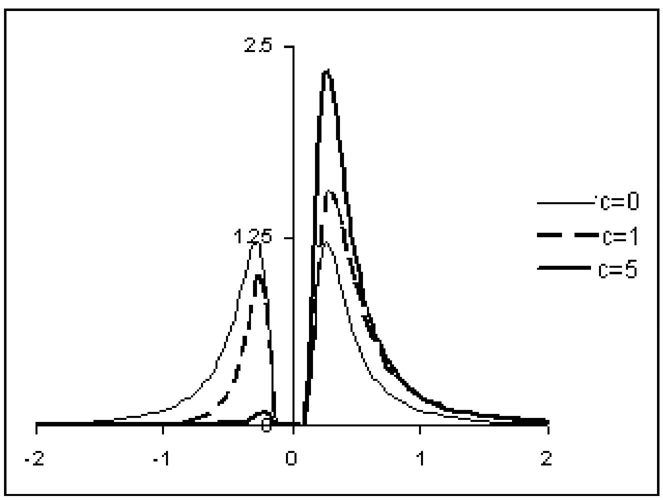

Figure 2: The pdf of skewed double inverted Weibull distribution for c = 0, 1, and 5, and α=2, β=3.

Figures 1-3 show the pdfs of skewed double inverted Weibull models for different values of c, α, and β. The densities are clearly bimodal. When $c = 0$, the distribution reduces to the standard double inverted Weibull and same local maxima of the density is achieved at two different points of the support. When $c > 0$, the density is positively skewed and the maxima on the positive side of the support is larger than that on the negative side of the support. As c varies, there is an obvious shift in skewness, however, the modes do not appear to change.

The cdf of the distribution is given by

$$F(z;c) = G(z) - \frac{1}{2(1+c^\alpha)}\left(1 - e^{-\frac{1+c^\alpha}{\beta^\alpha c^\alpha z^\alpha}}\right), \; z > 0, \; c > 0.$$

Using formulas 3.15 in Oberhettinger (1974), we obtain the k-th moment of the distribution as

$$E(Z^k;c) = \frac{1+(-1)^k}{2\beta^k}\Gamma\left(1 - \frac{k}{\alpha}\right) + \frac{1-(-1)^k}{2\beta^k}\Gamma\left(1 - \frac{k}{\alpha}\right)\left(\frac{c^\alpha}{1+c^\alpha}\right)^{1-\frac{k}{\alpha}},$$

for $\alpha > k, \; c > 0$. Hence the mean of the distribution is

$$E(Z;c) = \frac{1}{\beta}\Gamma\left(1 - \frac{1}{\alpha}\right)\left(\frac{c^\alpha}{1+c^\alpha}\right)^{1-\frac{1}{\alpha}}, \; \alpha > 1, \; c > 0.$$

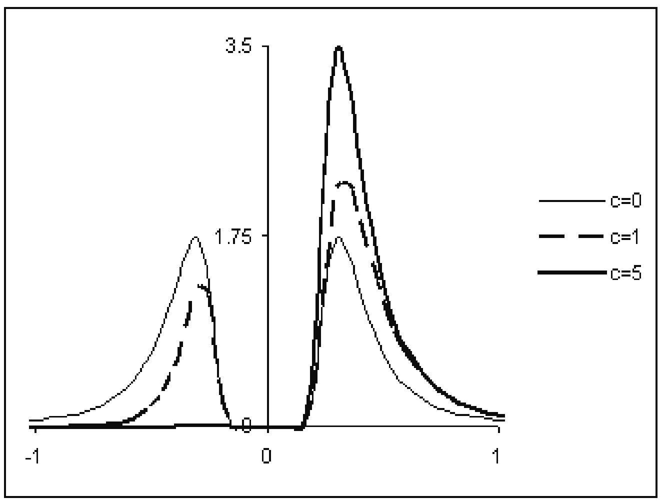

Figure 3: The pdf of skewed double inverted Weibull distribution for c = 0, 1, 5, and α=3, β=3.

5 A double inverted compound gamma model

The pdf of a compound gamma distribution is given by

$$f(x) = \frac{1}{B(\alpha,\beta)}x^{\alpha-1}(1+x)^{\alpha+\beta}, \; x > 0, \; \alpha, \beta > 0. \tag{8}$$

Hence the pdf of an inverted compound gamma variable X will be

$$h(x) = \frac{1}{B(\alpha, \beta)} x^{\beta-1}(1+x)^{\alpha+\beta}, \; x > 0, \; \alpha, \beta > 0, \tag{9}$$

which also defines a compound gamma distribution.

Noting that $U = \frac{X}{1+X}$ follows a beta distribution with parameters (β,α), it is easily seen that the cdf of X is given by

$$H(x) = \frac{B_{\frac{x}{1+x}}(\beta, \alpha)}{B(\beta, \alpha)}, \; x > 0,$$

where $B_t(a, b) = \int_0^t z^{a-1}(1-z)^{b-1}$. Also,

$$E(X^k) = \frac{\Gamma(\beta+k)\Gamma(\alpha-k)}{\Gamma(\alpha)\Gamma(\beta)}, \text{ if } \alpha > k.$$

From (9) the pdf of a double inverted compound gamma variable is defined by

$$g(x) = \frac{1}{2B(\alpha, \beta)} |x|^{\beta-1}(1+|x|)^{-(\alpha+\beta)}, \; x \in R^1, \; \alpha, \beta > 0 \tag{10}$$

and its cdf and k-th moment are as follows :

$$G(x) = \frac{1}{2}[1 + sgn(x) \frac{B_{\frac{|x|}{1+|x|}}(\beta, \alpha)}{B(\beta, \alpha)}], \; x \in R^1$$

$$E(X^k) = \frac{1+(-1)^k}{2} . \frac{\Gamma(\beta+k)\Gamma(\alpha-k)}{\Gamma(\alpha)\Gamma(\beta)}, \text{ if } \alpha > k.$$

The skewed double inverted compound gamma is then defined by the following pdf :

$$f(z; c) = \frac{1}{2B(\alpha, \beta)} |z|^{\beta-1}(1+|z|)^{-(\alpha+\beta)} \left[1 + sgn(z) \frac{B_{\frac{|cx|}{1+|cx|}}(\beta, \alpha)}{B(\beta, \alpha)} \right], \; z \in R^1, \; c \in R^1. \tag{11}$$

Figure 4 shows the pdfs of skewed double inverted compound gamma models for different values of c, α, and β. The densities are clearly unimodal. When $c = 0$, the distribution reduces to the standard double inverted compound gamma and have unique local maximum at $Z = 0$. When $c > 0$, the density is positively skewed but the maximum is still achieved at $Z = 0$.

The corresponding cdf can be expressed as

$$F(z;c) = G(z) - \frac{c^{\beta}}{2\beta B^2(\alpha,\beta)} \sum_{i=0}^{\infty} \frac{(\beta)_i(1-\alpha)_i c^i}{(1+\beta)_i i!} \int_z^{\infty} \frac{t^{2\beta+i-1}}{(1+t)^{\alpha+\beta}(1+ct)^{\beta+i}} dt,$$

for for $z > 0$, $c > 0$.

The cdf for negative values of z and c can be obtained from the relations $J(z;c) = -J(z;-c)$ and $J(-z;c) = J(z;c)$.

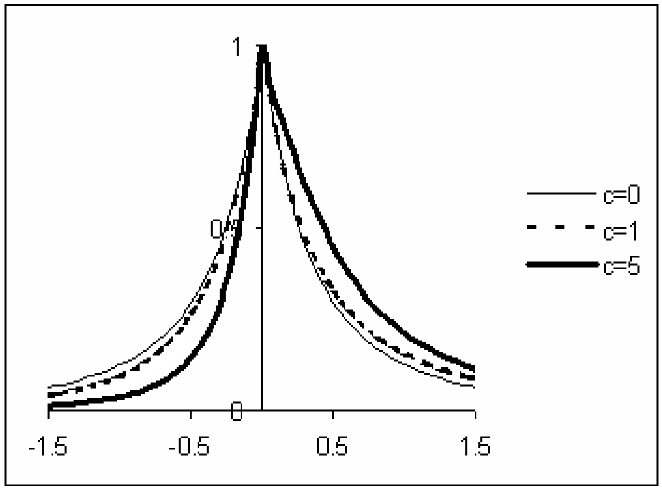

Figure 4: The pdf of skewed double inverted compound gamma distribution for c = 0, 1, 5, and α=1, β=2.

From (11) and the formulas 8.391 and 8.392 in Gradshteyn and Ryzhik (1965) and formula 2.29 in Oberhettinger (1974), we obtain the k-th moment of the distribution as follows:

For $0 < c < 1$,

$$E(Z^k;c) = \frac{1+(-1)^k}{2}\frac{B(\beta+k,\alpha-k)}{B(\alpha,\beta)} + \frac{1-(-1)^k}{2\alpha B(\alpha,\beta)} \sum_{i=0}^{\infty} \frac{(\beta)_i(1-\alpha)_i}{(\beta+1)_i i!} \frac{B(2\beta+k+i,\alpha-k)}{B(\alpha,\beta)}$$

$\times c^{\beta+i} F(\beta+i, 2\beta+k+i; 2\beta+\alpha+i; 1-c)$, if $k < \alpha$.

For $c \geq 1$,

$$E(Z^k;c) = \frac{1+(-1)^k}{2}\frac{B(\beta+k,\alpha-k)}{B(\alpha,\beta)} + \frac{1-(-1)^k}{2\beta B(\alpha,\beta)} c^{-(\beta+k)} \sum_{i=0}^{\infty} \frac{(\beta)_i(1-\alpha)_i}{(\beta+1)_i i!} \frac{B(2\beta+k+i,\alpha-k)}{B(\alpha,\beta)}$$

$\times F(\alpha+\beta, 2\beta+k+i; 2\beta+\alpha+i; 1-\frac{1}{c})$, if $k < \alpha$.

In particular, for $k = 1$ we have

$$E(Z;c) = \frac{1}{\beta}\left(\frac{c^{\alpha}}{1+c^{\alpha}}\right)^{1-\frac{1}{\alpha}}\Gamma\left(1-\frac{1}{\alpha}\right), \text{ if } \alpha > 1.$$

6 Concluding Remarks

In recent years, a fair amount of research has focused on extending families of symmetric densities such as normal, logistic, and t- to incorporate skewed distributions. This resulted in a vast literature of skew-symmetric families of distributions such as skew-normal, skew-logistic, skew-t giving a broader flexibility to statistical modeling of natural phenomena. Standard double inverted distributions such as double inverted gamma, double inverted Weibull and double inverted compound gamma distributions are symmetric and in many cases allow for the distribution to be bi-modal. However, one limitation of these distributions is that the peak of the density is same for both the modal values, making the distribution less appealing. In practice, bimodal distributions do not always have same maxima at different modal values. The distributions proposed in this article allows the maxima of the bimodal density to differ. Moreover, the distributions incorporate the standard regular double inverted distributions, providing more options for statistical modeling.

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