International Journal of Statistical Sciences *ISSN 1683-5603*
Vol. 22(1), 2022, pp 79-98

OLS Bias and MSE in a Unit Root Model

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[Received September 12, 2021; Accepted March 7, 2022]

Abstract

We revisit the problem of deriving analytically tractable expressions for lower order moments of the ordinary least squares (OLS) estimator in autoregressive models with unit roots. Simple algebraic techniques are used to approximate the series sums of the first two moments derived by Tsui and Ali (1991). Compared to the exact moment values obtained by numerical methods, it is found that our approximate closed forms in simple functions are reasonably accurate for a wide range of sample sizes. We also validate the numerical accuracy of asymptotic mean and variance derived by Shenton and Vinod (1995).

Keywords: Unit-root process, closed-form approximations, OLS bias, MSE.

AMS Subject Classification: 62M10.

1. Introduction

Since the seminal work of Hurwicz (1950) and White (1961) on distributional properties of the OLS estimator $\hat{\phi}$ of the first-order autoregressive parameter ϕ, it has spawned a vast statistical and econometric literature on autoregressive models and the unit root models. However, without knowledge of the exact distributions, the distributional properties of $\hat{\phi}$ in finite samples have been extensively explored in both analytical and computational approaches. Among others, Ullah (2004) and Choi (2015) provide excellent surveys of the methodology and salient tools for finite sample econometrics and the unit root models. For a standard AR(1) model with zero initial value and NID disturbance terms, there is a long history of studies deriving series expansions to approximate lower-order moments of the OLS estimator. For example, Hurwicz (1950) obtains a closed form expression for the first moment of $\hat{\phi}$ for $|\phi| \leq 1$ when the sample size is 3. Later, White (1961) obtains the first three terms for each of the first and second moments of $\hat{\phi}$. Shenton and Johnson (1965) extend White (1961) to obtain the first five terms for each of the first and second moments of $\hat{\phi}$. In early 1990's, Tsui and Ali (1991) extend Shenton and Johnson (1965) to approximate the first four moments of $\hat{\phi}$ by series sums. And Abadir (1993) obtains an impressive but complicated closed-form expression of the first moment of $\hat{\phi}$, which needs evaluation of infinite sums based on generalized hyperbolic functions. Moreover, Shenton and Vinod (1995) re-examine the moment issue and obtain asymptotic expressions for the mean, mean squared error and variance of $\hat{\phi}$, but there has been little discussion of their numerical accuracy. More recently, Phillips (2012) obtains an integral representation of the finite sample bias of $\hat{\phi}$ and also provides asymptotic expansions for the bias. In this paper, we extend Tsui and Ali (1991) to sum their moment series by integrals and obtain tractable closed-form expressions in simple functions for the mean and variance of $\hat{\phi}$.

Although multivariate frameworks and more flexible structures of autoregressive models have been the subject of recent research, there is still a continuing interest in the finite-sample distributional properties of $\hat{\phi}$. For example, Phillips (2012) explores the application of the delta method and continuous mapping theorem to the indirect inference estimator in first order autoregressive estimation. Convenient expressions for $E(\hat{\phi})$ are in demand to correct finite sample bias and to price derivative securities (Phillips and Yu (2009). Tang and Chen (2009) indicate that the OLS bias and mean squared error of $\hat{\phi}$ have bearings on the estimation of continuous models with the mean-reversion parameter in interest rate processes.

The main contribution of this paper is to provide convenient expressions in simple functions for the OLS bias and mean squared error (MSE) of $\hat{\phi}$ in a standard unit root model. In addition, the tractable bias and MSE expressions are useful in correcting parameter estimation. Moreover, it avoids numerical inaccuracy of moment values obtained by conventional simulation estimation, as Hansen (2014) remarks that such simulated moments can be substantially inaccurate unless the simulation size is very large. As a by-product, we validate the numerical accuracy of the apparently neglected expressions for mean and variance of $\hat{\phi}$ derived by Shenton and Vinod (1995).

The remainder of this section highlights the gist of the approach by Shenton and Johnson (1965) and the extension by Tsui and Ali (1991). Section 2 introduces the algebraic techniques to sum the Tsui and Ali's moment series for the mean of $\hat{\phi}$. Sections 3 extends the approach to the second moment. Asymptotic moment values are also compared. Section 4 assesses the numerical accuracy of the moment formulas, which are compared with exact values obtained by numerical methods and with values obtained from other closed form approximations in the literature. Section 5 contains our final remarks.

Consider the following first order autoregressive (AR(1)) model,

$$y_t = \phi y_{t-1} + \epsilon_t, \;\; t = 1, 2, 3, \cdots, n \tag{1}$$

where the initial value $y_0 = 0$, $|\phi| = 1$, and $\epsilon_t \sim NID(0, \sigma^2)$. Without loss of generality, σ^2 can be set to 1. For a sample of n observations, $y = (y_1, y_2, \dots, y_n)'$, the OLS estimator of ϕ is

$$\hat{\phi} = \frac{\sum_{t=1}^{n} y_{t-1} y_t}{\sum_{t=1}^{n} y_{t-1}^2} = \frac{U}{V}. \tag{2}$$

White (1961) shows that $\hat{\phi}$ is expressible in terms of a ratio of two quadratic forms in normal variables U and V. The joint density function of y is

$$f(y) = (2\pi)^{-n/2} \exp\left\{-\frac{1}{2}\sum_{t=1}^{n} (y_t - \phi y_{t-1})^2\right\}$$

and the joint moment generating function of U and V is well-established that

$$\begin{aligned} M(p, q) &= E[\exp(pU - qV)] \\ &= D_n(p, q)^{-\frac{1}{2}}, \end{aligned} \tag{3}$$

where $D_n(p, q)$ is the determinant of a symmetric tri-diagonal matrix, $C = (c_{i,j})$ with the non-zero elements being: $c_{i,i} = 1 + \phi^2 + 2q, i = 1, 2, 3, \dots, (n-1)$; $c_{n,n} = 1$; and $c_{i,i+1} = c_{i+1,i} = -(\phi + p), i = 1, 2, 3, \dots, (n-1)$, respectively.

Following White (1961), Shenton and Johnson (1965) obtain expressions for the first two moments of $\hat{\phi}$ about ϕ, with

$$E(\hat{\phi} - \phi) = \int_0^{\infty} (\partial_{\phi} U_0) dq, \tag{4}$$

$$E(\hat{\phi} - \phi)^2 = \int_0^{\infty} (\partial_{\phi}^2 U_1 + U_0) dq, \tag{5}$$

where

$$U_r = \frac{q^r}{r!}(D_n(0,q))^{-1/2} \tag{6}$$

and $\partial_\phi^j = (\frac{\partial}{\partial\phi})^j$ denotes the j^{th} partial derivative with respect to ϕ, which is evaluated at $\phi = 1$ after differentiation. In order to evaluate these integrals explicitly, Shenton and Johnson (1965) show that

$$D_n(0,q) = \sum_{s=0}^{n-2} A_s^{(n)}\phi^{2s}, \tag{7}$$

where $D_0(0,q) = D_1(0,q) = 1$ and

$$\begin{gathered}
A_r^{(n)} = \frac{(1+2q)^{n-2r-1}}{r!}\sum_{s=0}^{r-1}\binom{r-1}{s} r^{(s)}(n-s-2)^{r-s}(2q)^{r-s}, \text{ for } 1 \le r \le n-2, \\
r^{(s)} = r(r-1)(r-2)\cdots(r-s+1), \\
A_0^{(n)} = (1+2q)^{n-1}, \text{ for } n > 0, \\
A_1^{(n)} = 2q(n-2)(1+2q)^{n-3}, \text{ for } n \ge 2.
\end{gathered} \tag{8}$$

The inverse of the square root of $D_0(0,q)$ can be expanded as an infinite series,

$$D_n^{-1/2}(0,q) = \sum_{s=0}^{\infty} \frac{B_s^{(n)}\phi^{2s}}{(1+2q)^{(n-1)/2}} \tag{9}$$

where $B_s^{(n)}$ is a function of n and q, which satisfies the recurrence relation

$$B_s^{(n)} = -\frac{1}{2sA_0^{(n)}}\sum_{r=0}^{s}(s+r)B_r^{(n)}A_{s-r}^{(n)}, \tag{10}$$

in which $A_s^{(n)} = 0$ for $s > (n-2)$ and $B_0^{(n)} = 1$. By Fubini's theorem, moment expressions in (4)-(5) involving integrals of sums can be evaluated as sums of integrals by using (6) and (9). As an example, the OLS bias of $\hat{\phi}$ can be obtained as

$$E(\hat{\phi} - \phi) = \int_0^\infty \sum_{s=0}^{\infty} \frac{2sB_s^{(n)}\phi^{2s-1}}{(1+2q)^{\frac{n-1}{2}}}dq = \sum_{s=0}^{\infty}\int_0^\infty \frac{2sB_s^{(n)}\phi^{2s-1}}{(1+2q)^{\frac{n-1}{2}}}dq \tag{11}$$

Theoretically, one can obtain many terms in series expansion of the moments of $\hat{\phi}$ about ϕ by term-wise integration. However, it is very burdensome as the number of integrals to be evaluated at the s^{th} terms is proportional to s^2. Tsui and Ali (1991) compute more than 15 terms for each of the first two moments of $\hat{\phi}$ about ϕ in (4)-(5), and have been able to use algebraic techniques to detect regularities among the terms for each of the moments. These regularities are useful in establishing general expressions in series expansions for the moments of $\hat{\phi}$ about ϕ, when $|\hat{\phi}| \leq 1$. However, Tsui and Ali's infinite series converges quite slowly for $|\phi| = 1$, particularly for large sample sizes. In this paper, we suggest to approximate the upper limit of summation of the infinite series by matching moment values of $\hat{\phi}$ with those computed numerically using the algorithm by Tsui and Ali (1994). Such upper limits will serve as the upper bounds of integrals when deriving closed form approximations to the moments of $\hat{\phi}$.

2. Approximation to the first moment

When $|\phi| \leq 1$, the OLS bias of $\hat{\phi}$ in series expansion derived by Tsui and Ali (1991) is given as follows:

$$E(\hat{\phi} - \phi) = \sum_{s=1}^{\infty} \alpha_s \phi^{2s-1}, \tag{12}$$

where

$$\alpha_1 = \frac{-2(n-2)}{(n+1)^{[2]}},\ \alpha_2 = \frac{12}{(n+5)^{[3]}},\ \alpha_s = 6sA_n(s), \text{ for } s \geq 3, \tag{13}$$

with $n^{[s]} = n(n-2)(n-4) \ldots (n-2s+2)$,

$$\begin{aligned} A_n(s) &= \frac{(n+4s-4)^{[s-2]}}{(n+4s-3)^{[s+1]}} \\ &= \frac{(n+4s-4)(n+4s-6)\cdots(n+4s-4-2(s-2)+2)}{(n+4s-3)(n+4s-5)\cdots(n+4s-3-2(s+1)+2)}. \end{aligned} \tag{14}$$

We note that the upper limit of summation for all four moment expressions of $\hat{\phi}$ about ϕ in Tsui and Ali (1991) should have been written as ∞ instead of $n-2$. In what follows we employ algebraic techniques to approximate the infinite sum of series in (12) and derive a closed form approximation for the bias of $\hat{\phi}$ in terms of simple functions. The idea is to express $A_n(s)$ as ratios of the gamma function and to exploit its well-known properties to obtain the approximation. Dividing each term in the numerator and denominator of $A_n(s)$ in (14), we get

$$A_n(s) = 2^{-3} \frac{\Gamma(n/2+2s-1)/\Gamma(n/2+s+1)}{\Gamma(n/2+2s-1/2)/\Gamma(n/2+s-3/2)} \tag{15}$$

where $\Gamma(\cdot)$ is the gamma function. Expression (12) can be rewritten as an infinite sum of gamma functions,

$$E(\hat{\phi}-\phi) \approx \frac{-2(n-2)}{(n+1)^{[2]}}\phi + \frac{12}{(n+5)^{[3]}}\phi^3$$
$$+\frac{3}{4}\sum_{s=3}^{\infty} s\frac{\Gamma(\mathrm{n}/2+2\mathrm{s}-1)/\Gamma(\mathrm{n}/2+s+1)}{\Gamma(n/2+2s-1/2)/\Gamma(n/2+s-3/2)}\phi^{2s-1}. \tag{16}$$

To simplify the ratios of gamma functions in (16), we apply the uniqueness theorem of the gamma function, which states that for a constant c, $\Gamma(x)/\Gamma(x+c)$ can be approximated by $x^{-\mathrm{c}}$ as $x \to \infty$ (see Titchmarsh (1939, p. 58)). For convenience, let

$$a_1 = n/2+2s, \text{ and } a_2 = n/2+s, \tag{17}$$

we have

$$\frac{\Gamma(n/2+2s-1)}{\Gamma\left(n/2+2s-\frac{1}{2}\right)} = \frac{\Gamma(a_1-1)}{\Gamma\left(a_1-\frac{1}{2}\right)} \approx \left\{1-\frac{1}{a_1}\right\}^{-1/2} a_1^{-1/2}$$
$$\approx \left\{1+\frac{1}{2a_1}+\frac{3}{8a_1^2}+O(n^{-3})\right\} a_1^{-1/2} \text{ and}$$
$$\frac{\Gamma(n/2+s-3/2)}{\Gamma(n/2+s+1)} = \frac{\Gamma(a_2-3/2)}{\Gamma(a_2+1)} \approx \left\{1-\frac{3}{2a_2}\right\}^{-5/2} a_2^{-5/2}$$
$$\approx \left\{1+\frac{15}{4a_2}+\frac{315}{32a_2^2}+O(n^{-3})\right\} a_2^{-5/2}. \tag{18}$$

Substituting results in (18) into (15), we have

$$A_n(s) \approx 2^{-3}B_n(s)a_1^{-1/2}a_2^{-5/2},$$

with

$$B_n(s) = 1+\left(\frac{1}{2a_1}+\frac{15}{4a_2}\right)+\left(\frac{3}{8a_1^2}+\frac{15}{8a_1a_2}+\frac{315}{32a_2^2}\right)+O(n^{-3}). \tag{19}$$

The approximate bias of $\hat{\phi}$ in (16) becomes

$$E(\hat{\phi}-\phi) \approx \frac{-2(n-2)}{(n+1)^{[2]}}\phi + \frac{12}{(n+5)^{[3]}}\phi^3 + \frac{3}{4}\sum_{s=3}^{\infty} sB_n(s)a_1^{-1/2}a_2^{-5/2}\phi^{2s-1}. \tag{20}$$

For large n, a_1 and a_2 will also be large. We may truncate $B_n(s)$ up to terms of $O(n^{-2})$ so as to confine $A_n(s)$ to $O(n^{-5})$. After some algebra, the following proposition is in order.

Proposition 1.1: When $\phi = 1$ and $n \geq 50$, the bias of $\hat{\phi}$ can be approximated by

$$\begin{aligned} E(\hat{\phi}-\phi) \approx\ & -\frac{1.78260}{n} + \frac{5.10887}{n^2} - \frac{12.97932}{n^3} + \frac{44.50000}{n^4} - \frac{443.37500}{n^5} + \frac{9662.12500}{n^6} \\ & - \frac{175557.21875}{n^7} + \frac{2.64222 \times 10^6}{n^8} - \frac{3.53654 \times 10^7}{n^9} \end{aligned} \tag{21}$$

Proof:

Based on the result in (20), for $\phi = 1$, we approximate the infinite sum in (12) by straightforward integration, with the change of variables of s/n to u. We have

$$\begin{aligned} \frac{3}{4}\sum_{s=3}^{\infty} sB_n(s)a_1^{-1/2}a_2^{-5/2} & \approx \frac{3}{4}\int_3^{n_1} sB_n^*(s)a_1^{-1/2}a_2^{-5/2}ds, \\ & = \frac{6}{n}\int_{3/n}^{n_1/n} B_n^*(nu)u(1+4u)^{-1/2}(1+2u)^{-5/2}du, \end{aligned} \tag{22}$$

where $B_n^*(s)$ denotes the truncated $B_n(s)$ in (19) containing six terms up to $O(n^{-2})$. Also, $n_1 = 0.7n$, which is the approximate upper limit of summation obtained by matching the infinite sum in (12) with exact values of $E(\hat{\phi}-\phi)$ computed numerically by Tsui and Ali (1994). For convenience, define

$$I_1(i,j) = 6\int_{3/n}^{n_1/n} B_n^{**}(i,j)2^{(i+j)}n^{(-i-j-1)}u(1+4u)^{-1/2-i}(1+2u)^{-5/2-j}du, \tag{23}$$

where $B_n^{**}(i,j)$ denotes the coefficient associated with the product of $(1/a_1)^i$ and $(1/a_2)^j$ in $B_n^{**}(nu)$. We have

$$\frac{6}{n}\int_{3/n}^{0.7} B_n^{*}(nu)u(1+4u)^{-1/2}(1+2u)^{-5/2}du = \sum_{i=0}^{2}\sum_{j=0}^{2-i} I_1(i,j) \tag{24}$$

For each $I_1(i,j)$ in (24), we obtain its Maclaurin series expansion up to $O(n^{-10})$. They include

$$I_1(0,0) = \frac{0.21741}{n^2} - \frac{27}{n^3} + \frac{378}{n^4} - \frac{4070.250}{n^5} + \frac{40095}{n^6} - \frac{382269.3750}{n^7} + \frac{3.620\times 10^6}{n^8} - \frac{3.45163\times 10^7}{n^9} + O(n^{-10}), \tag{25}$$

$$I_1(0,1) = \frac{0.26883}{n^2} - \frac{1.52079\times 10^{-14}}{n^3} - \frac{54}{n^4} + \frac{972}{n^5} - \frac{12514.50}{n^6} + \frac{140259.60}{n^7} - \frac{1.46584\times 10^6}{n^8} + \frac{1.47787\times 10^7}{n^9} + O(n^{-10}), \tag{26}$$

$$I_1(1,0) = \frac{0.20150}{n^2} + \frac{5.63788\times 10^{-14}}{n^3} - \frac{54}{n^4} + \frac{1188}{n^5} - \frac{18832.50}{n^6} + \frac{260982}{n^7} - \frac{3.37436\times 10^6}{n^8} + \frac{4.19478\times 10^7}{n^9} + O(n^{-10}), \tag{27}$$

$$I_1(2,0) = \frac{0.20871}{n^3} - \frac{1.18141\times 10^{-12}}{n^4} - \frac{108}{n^5} + \frac{3240}{n^6} - \frac{66825}{n^7} + \frac{1.16348\times 10^6}{n^8} - \frac{1.83836\times 10^7}{n^9} + O(n^{-10}), \tag{28}$$

$$I_1(1,1) = \frac{0.26834}{n^3} + \frac{4.54348 \times 10^{-13}}{n^4} - \frac{108}{n^5} + \frac{2808}{n^6} - \frac{50301}{n^7} + \frac{763409}{n^8} - \frac{1.05658 \times 10^7}{n^9} + O(n^{-10}), \tag{29}$$

$$I_1(0,2) = \frac{0.34939}{n^3} - \frac{4.25589 \times 10^{-14}}{n^4} - \frac{108}{n^5} + \frac{2376}{n^6} - \frac{35721}{n^7} + \frac{451980}{n^8} - \frac{5.19157 \times 10^6}{n^9} + O(n^{-10}). \tag{30}$$

We also obtain series expansion of α_1 and α_2 in (13) with terms up to $O(n^{-10})$, where

$$\alpha_1 = -\frac{2}{n} + \frac{4}{n^2} - \frac{2}{n^3} + \frac{4}{n^4} - \frac{2}{n^5} + \frac{4}{n^6} - \frac{2}{n^7} + \frac{4}{n^8} - \frac{2}{n^9} + O(n^{-10}), \tag{31}$$

$$\alpha_2 = \frac{12}{n^3} - \frac{108}{n^4} + \frac{696}{n^5} - \frac{3960}{n^6} + \frac{21252}{n^7} - \frac{110628}{n^8} + \frac{566256}{n^9} + O(n^{-10}). \tag{32}$$

Substituting the expansions in (25)-(32) into (20) and after some algebra, we obtain the result as shown in (21).

3. Approximation to the second moment

In this section, we derive approximate closed form expressions for the second moment of $\hat{\phi}$ about ϕ by a similar procedure used in Section 2. Details of the derivations are relegated to the Appendix.

3.1. Second moment

When $|\phi| \leq 1$, the second moment of $\hat{\phi}$ about ϕ (mean squared error) in series expansion is given by Tsui and Ali (1991) as follows:

$$E(\hat{\phi} - \phi)^2 = \sum_{s=0}^{\infty} \omega_s \phi^{2s}, \tag{33}$$

where

$$\omega_0 = \frac{n^2 - 4n + 7}{(n+1)^{[3]}}, \tag{34}$$

$$\omega_1 = \frac{-(n^3 - 6n^2 - 25n + 42)}{(n+5)^{[4]}}, \tag{35}$$

$$\omega_2 = \frac{3(n^2 + 16n - 57)}{(n+9)^{[5]}}, \tag{36}$$

$$\omega_s = 3H_n(s)C_n(s), \text{ for } s \geq 3 \tag{37}$$

and

$$C_n(s) = \frac{(n + 4s - 4)^{[s-3]}}{(n + 4s + 1)^{[s+3]}}, \tag{38}$$

$$H_n(s) = -20s^3 + (16n - 38)s^2 + (10n^2 - 8n - 16)s + (n^3 + 2n^2 - 9n + 2), \tag{39}$$

with $n^{[s]} = n(n-2)(n-4)\cdots(n-2s+2)$.

Proposition 2.1: When $\phi = 1$ and $n \geq 50$, the mean squared error of $\hat{\phi}$ can be approximated by

$$E(\hat{\phi} - \phi)^2 \approx \frac{13.28574}{n^2} - \frac{69.91775}{n^3} + \frac{260.85853}{n^4} + \frac{160.81440}{n^5} - \frac{10294.15388}{n^6} + \frac{33943.18750}{n^7} + \frac{2.27422 \times 10^6}{n^8} - \frac{7.25612 \times 10^7}{n^9}. \tag{40}$$

Proof: See the Appendix.

Proposition 2.2: When $\phi = 1$ and $n \geq 50$, the variance of $\hat{\phi}$ can be approximated by

$$\mu_2(\hat{\phi}) \approx \frac{10.10807}{n^2} - \frac{51.70360}{n^3} + \frac{188.48408}{n^4} + \frac{452.08518}{n^5} - \frac{12498.02670}{n^6} + \frac{74076.04414}{n^7} + \frac{1.53611 \times 10^6}{n^8} - \frac{6.10571 \times 10^7}{n^9} \tag{41}$$

Proof: See the Appendix.

Proposition 2.3: When $\phi = 1$ and $n \geq 50$, the standard deviation of $\hat{\phi}$ can be approximated by

$$\sigma(\hat{\phi}) \approx \frac{3.17932}{n} - \frac{8.13124}{n^2} + \frac{19.24422}{n^3} + \frac{120.31565}{n^4} - \frac{1716.04901}{n^5} + \frac{6532.54243}{n^6} + \frac{266396.60080}{n^7} - \frac{8.89550 \times 10^6}{n^8} - \frac{5.00310 \times 10^7}{n^9}. \tag{42}$$

Proof: See the Appendix.

3.2. Comparison with other results

The dominant terms in the first two moments of $\hat{\phi}$ can be obtained from expressions (21), (40) and (42). They are:

$$\begin{aligned} &\lim_{n\to\infty} nE(\hat{\phi} - 1) \approx -1.78260 \\ &\lim_{n\to\infty} n^2 E(\hat{\phi} - 1)^2 \approx 13.28574 \\ &\lim_{n\to\infty} n\sigma(\hat{\phi}) \approx 3.17932 \end{aligned} \tag{43}$$

The corresponding values obtained by Shenton and Johnson (1965), and Vinod and Shenton (1996) are -1.7814, 13.2857 and 3.17996, respectively. Our results match at least to the second decimal place with those numerically reported in the literature. The discrepancy may be due to the loss of partial information when applying the uniqueness theorem to convert ratios of gamma functions, when approximating the infinite upper limit of summation of the moment series by a finite upper bound, and when approximating the series sum by definite integration.

4. Accuracy checking

Performances of the closed form approximations to the first two moments of $\hat{\phi}$ are compared to those analytical expressions available in the literature, with the exact values computed numerically by using the algorithm of Tsui and Ali (1994). The following specifications are in order.

[a] TA-1 to TA-2 denote the first two moment expressions of $\hat{\phi}$ about ϕ by Tsui and Ali in (12) and (33), with the infinite upper limits of summation truncated to $0.7n$ and $0.16n$. MV-1 to MV-2 denote the approximate closed forms of the corresponding moments of $\hat{\phi}$ about ϕ in (21) and (40).

[b] MN-TA denotes Tsui and Ali's series sum of $E(\hat{\phi})$ in (12), with the infinite upper bound truncated to $0.7n$. MN-3 terms, MN-6 terms and MN-9 terms denote the first three, six and nine terms of $E(\hat{\phi})$ in (21). Also, MN-SV denotes the expression for $E(\hat{\phi})$ derived by Shenton and Vinod (1995), which is truncated to $O(n^{-4})$, with

$$E(\hat{\phi}) \approx 1 - \frac{3.562860}{2n-1} + \frac{22.251441370}{(2n-1)^2} - \frac{96.542980535}{(2n-1)^3}\ ; \tag{44}$$

[c] SD-TA denotes the standard deviation of $\hat{\phi}$ obtained indirectly from Tsui and Ali's infinite series for the first two moments of $\hat{\phi}$ about ϕ in (12) and (33), with the corresponding infinite upper bounds truncated to $0.7n$ and $0.16n$, respectively. SD-3 terms, SD-6 terms and SD-9 terms denote the first three, six and nine terms of $\sigma(\hat{\phi})$ in (42). Moreover, SD-SV denotes the asymptotic standard deviation of $\hat{\phi}$ by Shenton and Vinod (1995), which is obtained by taking the square root of the variance with terms truncated to $O(n^{-5})$. That is

$$\mathrm{Var}(\hat{\phi}) \approx \frac{40.448692}{(2n-1)^2} - \frac{736.46468}{(2n-1)^3} + \frac{2375.0884}{(2n-1)^4}\ ; \tag{45}$$

All approximations and exact values are computed by Mathematica and in most of the cases, we take various sample size *n=20, 25, 30, 40, 50, 60, 70, 75, 80, 90, 100, 150, 200, 250, 300, 350, 400, 450* and *500*. To save space, Tables 1-3 display some representative results for assessing the numerical accuracy of the approximations considered in [a]-[c]. All the exact and approximate moment values can be obtained upon request.

The exact and approximate values of the first two moments of $\hat{\phi}$ about ϕ are displayed in Panels A-B of Table 1. As can be gleaned from Panel A, TA-1 and MV-1approximate exceptionally well, matching the exact values of the OLS bias almost all the time to the fifth decimal place for $n \geq 150$. For $n \leq 100$, they are reasonably accurate at least to the third decimal place. The accuracy improves as sample size increases. In addition, as can be observed from Panel B, TA-2 and MV-2 match exact values of the second moment of $\hat{\phi}$ about ϕ at least to the sixth decimal point for $n \geq 200$, and at least to the fifth decimal place for $n \leq 150$. As such, our results provide some justifications for truncating the infinite upper limits of summation of Tsui and Ali series for the first two moments of $\hat{\phi}$ about ϕ to $0.7n$ and $0.16n$, respectively.

Table 2 displays the exact and approximate values of $E(\hat{\phi})$ computed by various approaches. As can be observed from Columns [2]-[5], values of all the five approximations are almost identical to each other for sample sizes ranging from 50 to 500. They match the exact values up to the fourth decimal place for $n \geq 75$. In addition, there are no differences among the values computed by MN-3 terms, MN-6 terms and MN-9 terms, which are also identical to those obtained by MN-SV. As such, MN-SV and MN-3 terms can be used to correct the OLS bias of $\hat{\phi}$ when $\phi = 1$.

Table 3 reports values of the standard deviation of $\hat{\phi}$ computed by various approximations. As can be observed, when $n \leq 100$, SD-SV performs relatively better than the other four approximations, matching the exact values at least to the fifth decimal place; whereas those by the four approximations match the exact values to the fourth decimal place in most of the cases. However, the relative discrepancy among SD-SV and SD-3 terms to SD-9 terms disappears for $n \geq 150$, as all five approximations match the exact values at least to the fifth decimal place for almost all of the cases. For example, when $n = 50$, the standard deviations of $\hat{\phi}$ are $(0.06405, 0.060402, 0.060488)$ for Exact, SD-SV and SD-3 terms, respectively; whereas for $n = 150$, the corresponding values are $(0.020825, 0.020825, 0.020828)$ for Exact, SD-SV and SD-3 terms, respectively. Hence, both SD-SV and SD-3 terms could be used as convenient approximations for $\sigma(\hat{\phi})$ when $n \geq 50$.

5. Concluding remarks

We have derived analytical expressions in simple functions for the OLS bias and mean squared error of $\hat{\phi}$ in a standard unit-root model. They are obtained by summing the hitherto unresolved series sum of moments of $\hat{\phi}$ derived by Tsui and Ali (1991). When compared to the exact moment values computed by numerical methods, our tractable expressions are reasonably accurate for a wide range of sample sizes. Hence, they may serve as benchmarks for comparison with approximations derived from other approaches. In addition, expressions for the bias and mean squared error of $\hat{\phi}$ could be useful for bias correction and variance reduction in parameter estimation. Moreover, the algebraic techniques introduced in this paper is applicable to summing those Tsui and Ali's series expansion in the stationary case, alongside setting appropriate bounds for the series sum to yield tractable expressions for the moments of $\hat{\phi}$, which will be taken up in future research.

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Table 1: Comparison of exact and approximate moment values of $\hat{\phi}$ when $\phi = 1$

n	50	75	100	150	200	300	400	500
Panel A: $E(\hat{\phi}-\phi)\times 10^{-2}$								
Exact	-3.3813	-2.2938	-1.7354	-1.1671	-0.8791	-0.5886	-0.4424	-0.3544
TA-1	-3.3337	-2.2917	-1.7358	-1.1675	-0.8795	-0.5887	-0.4427	-0.3546
MV-1	-3.3706	-2.2889	-1.7328	-1.1661	-0.8787	-0.5886	-0.4425	-0.3545
Panel B: $E(\hat{\phi}-\phi)^2\times 10^{-4}$								
Exact	47.9208	22.0232	12.6018	5.6989	3.2340	1.4500	0.8191	0.5258
TA-2	47.9819	22.0410	12.6093	5.7011	3.2348	1.4503	0.8194	0.5258
MV-2	47.9661	22.0444	12.6127	5.7028	3.2357	1.4506	0.8195	0.5259

Notes: Exact: values computed by using the algorithm of Tsui and Ali (1994);
TA-1-TA2: Series sum of Tsui and Ali (1991) for the first two moment of $\hat{\phi}$ about ϕ in (12) and (33), with upper limit of summation truncated to $0.7n$ and $0.16n$;
MV-1 to MV-2: Approximate closed form expressions in (21) and (40).

Table 2: Approximate mean of $\hat{\phi}$ when $\phi = 1$

n	Exact	MN-SV	MN-TA	MN-3 terms	MN-6 terms	MN-9 terms
50	0.9662	0.9667	0.9662	0.9663	0.9663	0.9663
75	0.9771	0.9771	0.9771	0.9771	0.9771	0.9771
100	0.9826	0.9826	0.9826	0.9827	0.9827	0.9827
150	0.9883	0.9883	0.9883	0.9883	0.9883	0.9883
200	0.9912	0.9912	0.9912	0.9912	0.9912	0.9912
300	0.9941	0.9941	0.9941	0.9941	0.9941	0.9941
400	0.9956	0.9956	0.9956	0.9956	0.9956	0.9956
500	0.9965	0.9965	0.9965	0.9965	0.9965	0.9965

Notes: Exact: values computed by using the algorithm of Tsui and Ali (1994);
MN-SV: values computed by using the series expansion of Shenton and Vinod (1995) in (44);
MN-TA: values computed by using the series sum of Tsui and Ali (1991) in (12), with the upper limit of summation truncated to $0.7n$;
MN-3 terms to MN-9 terms: values computed by using the first 3 terms, 6 terms and 9 terms of the closed form of $E(\hat{\phi}-\phi)$ in (21).

Table 3: Approximate standard deviation of $\hat{\phi}(\times 10^{-2})$ when $\phi = 1$

n	Exact	SD-SV	SD-TA	SD-3 terms	SD-6 terms	SD-9 terms
50	6.0405	6.0402	6.0720	6.0488	6.0502	6.0502
75	4.0941	4.0940	4.0974	4.0991	4.0994	4.0994
100	3.0968	3.0968	3.0978	3.0999	3.1000	3.1000
150	2.0825	2.0825	2.0828	2.0840	2.0840	2.0840
200	1.5688	1.5688	1.5688	1.5696	1.5696	1.5696
300	1.0505	1.0505	1.0506	1.0508	1.0508	1.0508
400	0.7896	0.7896	0.7896	0.7898	0.7898	0.7898
500	0.6326	0.6326	0.6325	0.6326	0.6326	0.6326

Notes: Exact: values computed by using the algorithm of Tsui and Ali (1994);
SD-SV: values obtained from taking the square root of $var(\hat{\phi})$ derived by Shenton and Vinod (1995) in (45);
SD-TA: values computed by using the series expansion of Tsui and Ali (1991) in (12) and (33), with the upper limit of summation truncated to $0.7n$ and $0.16n$;
SD-3 terms to SD-9 terms: values computed by taking the first 3 terms, 6 terms and 9 terms of the closed form of $\sigma(\hat{\phi})$ in (42).

Appendix

Proof of Proposition 2.1:

Applying the similar algebraic techniques to $C_n(s)$ of $E(\hat{\phi}-\phi)^2$ in (38), we have

$$C_n(s) \approx 2^{-6} P_n(s) a_1^{-5/2} a_2^{-7/2}, \tag{46}$$

where

$$P_n(s) \approx 1 + \left(\frac{5}{2a_1} + \frac{21}{4a_2}\right) + \left(\frac{35}{8a_1^2} + \frac{105}{8a_1 a_2} + \frac{567}{32a_2^2}\right) + O(n^{-3}). \tag{47}$$

We have

$$E(\hat{\phi}-\phi)^2 \approx \omega_0 + \omega_1 + \omega_2 + \frac{3}{64}\sum_{s=3}^{\infty} H_n(s) P_n(s) a_1^{-5/2} a_2^{-7/2}, \tag{48}$$

where ω_0, ω_1 and ω_2 are as defined in (34)-(36), and $H_n(s)$ is a polynomial of degree 3 in s as defined in (39).
For $\phi = 1$, we approximate the infinite sum in (48) by a finite sum, with the infinite upper limit of summation replaced by $n_2 = 0.16n$. Here, for a given sample n, n_2 is calibrated by matching values of $E(\hat{\phi}-\phi)^2$ in (33) with exact values numerically computed by the algorithm of Tsui and Ali (1994). See Panel B of Table 1 for the numerical comparison under various sample sizes.
We approximate the series sum by straightforward integration, with the change of variables from s/n to u to obtain

$$\sum_{s=3}^{\infty} H_n(s) P_n(s) a_1^{-5/2} a_2^{-7/2} \approx \frac{2^6}{n^5}\int_{3/n}^{n_2/n} H_n(nu) P_n^* (1+4u)^{-5/2}(1+2u)^{-7/2} du, \tag{49}$$

where $P_n^* = 1 + \frac{5}{2a_1} + \frac{21}{4a_2}$.
For convenience, we define the following function

$$I_2(i,j,k) = 2^{6+i+j} n^{k-5-i-j} H_n^*(k) \int_{3/n}^{0.16} u^k (1+4u)^{-5/2-i} (1+2u)^{-7/2-j} du \tag{50}$$

where $H_n^*(k)$ denotes the coefficient associated with u^k in the product of polynomials $H_n(\cdot)$ and P_n^*.

The integral in (49) can be decomposed into a sum of integrals,

$$\frac{2^6}{n^5} \int_{3/n}^{0.16} H_n(nu) P_n^* (1+4u)^{-5/2} (1+2u)^{-7/2} du = \sum_{i=0}^{1} \sum_{j=0}^{1-i} \sum_{k=0}^{3} I_2(i,j,k) \tag{51}$$

For each $I_2(i,j,k)$ in (51), we obtain its MacLaurin series expansion up to $O(n^{-10})$.

$$I_2(0,0,0) = \frac{3.85046}{n^2} - \frac{184.29909}{n^3} + \frac{4477.34590}{n^4} - \frac{87256.29909}{n^5} + \frac{1.49527 \times 10^6}{n^6} - \frac{2.35747 \times 10^7}{n^7} + \frac{3.51105 \times 10^8}{n^8} - \frac{5.02087 \times 10^9}{n^9} + O(n^{-10}) \tag{52}$$

$$I_2(0,0,1) = \frac{2.00710}{n^2} - \frac{1.60568}{n^3} - \frac{2883.2113}{n^4} + \frac{100224}{n^5} - \frac{2.29637 \times 10^6}{n^6} + \frac{4.33164 \times 10^7}{n^7} - \frac{7.28503 \times 10^8}{n^8} + \frac{1.13651 \times 10^{10}}{n^9} + O(n^{-10}) \tag{53}$$

$$I_2(0,0,2) = \frac{0.27423}{n^2} - \frac{0.65129}{n^3} + \frac{1.63891 \times 10^{-12}}{n^4} - \frac{9216}{n^5} + \frac{37440}{n^6} - \frac{9.37215 \times 10^6}{n^7} + \frac{1.87050 \times 10^8}{n^8} - \frac{3.27077 \times 10^9}{n^9} + O(n^{-10}) \tag{54}$$

$$I_2(0,0,3) = -\frac{0.03609}{n^2} - \frac{7.32258 \times 10^{-12}}{n^3} + \frac{8.21172 \times 10^{-11}}{n^4} - \frac{1.08112 \times 10^{-9}}{n^5} + \frac{25920}{n^6} - \frac{1.05754 \times 10^6}{n^7} + \frac{2.66717 \times 10^7}{n^8} - \frac{5.36077 \times 10^8}{n^9} + O(n^{-10}) \tag{55}$$

$$I_2(0,1,0) = \frac{7.01176}{n^3} - \frac{369.97649}{n^4} + \frac{10112.89421}{n^5} - \frac{215985.97649}{n^6} + \frac{3.97867 \times 10^6}{n^7} - \frac{6.64720 \times 10^7}{n^8} + \frac{1.03750 \times 10^9}{n^9} + O(n^{-10}) \quad (56)$$

$$I_2(0,1,1) = \frac{3.44579}{n^3} - \frac{2.75663}{n^4} - \frac{5765.51326}{n^5} + \frac{223488}{n^6} - \frac{5.59613 \times 10^6}{n^7} + \frac{1.13449 \times 10^8}{n^8} - \frac{2.02356 \times 10^8}{n^9} + O(n^{-10}) \quad (57)$$

$$I_2(0,1,2) = \frac{0.45473}{n^3} - \frac{1.07998}{n^4} - \frac{2.90059 \times 10^{-9}}{n^5} - \frac{18432}{n^6} + \frac{831744}{n^7} - \frac{2.27236 \times 10^7}{n^8} + \frac{4.87344 \times 10^8}{n^9} + O(n^{-10}) \quad (58)$$

$$I_2(0,1,3) = \frac{0.05859}{n^3} - \frac{3.35498 \times 10^{-11}}{n^4} + \frac{3.11125 \times 10^{-9}}{n^5} - \frac{9.79270 \times 10^{-8}}{n^6} + \frac{51840}{n^7} - \frac{2.36390 \times 10^6}{n^8} + \frac{6.51629 \times 10^7}{n^9} + O(n^{-10}) \quad (59)$$

$$I_2(1,0,0) = \frac{6.48543}{n^3} - \frac{371.02915}{n^4} + \frac{11269.63115}{n^5} - \frac{266675.02915}{n^6} + \frac{5.42530 \times 10^6}{n^7} - \frac{9.97624 \times 10^7}{n^8} + \frac{1.70743 \times 10^9}{n^9} + O(n^{-10}) \quad (60)$$

$$I_2(1,0,1) = \frac{3.03871}{n^3} - \frac{2.43097}{n^4} - \frac{5764.86194}{n^5} + \frac{246528}{n^6} - \frac{6.80688 \times 10^6}{n^7} + \frac{1.51877 \times 10^8}{n^8} - \frac{2.97405 \times 10^9}{n^9} + O(n^{-10}) \quad (61)$$

$$I_2(1,0,2) = \frac{0.39019}{n^3} - \frac{0.92671}{n^4} - \frac{1.49349 \times 10^{-8}}{n^5} - \frac{18432}{n^6} + \frac{914688}{n^7} - \frac{2.74991 \times 10^7}{n^8} + \frac{6.48263 \times 10^8}{n^9} + O(n^{-10}) \tag{62}$$

$$I_2(1,0,3) = -\frac{0.04946}{n^3} - \frac{8.13193 \times 10^{-11}}{n^4} + \frac{4.06596 \times 10^{-9}}{n^5} - \frac{1.16409 \times 10^{-7}}{n^6} + \frac{51840}{n^7} - \frac{2.61274 \times 10^6}{n^8} + \frac{7.94707 \times 10^7}{n^9} + O(n^{-10}) \tag{63}$$

We also obtain the Maclaurin series expansion of ω_0, ω_1 and ω_2 in (34)-(36) up to $O(n^{-10})$, with

$$\omega_0 = \frac{1}{n} - \frac{1}{n^2} + \frac{5}{n^3} + \frac{11}{n^4} + \frac{41}{n^5} + \frac{119}{n^6} + \frac{365}{n^7} + \frac{1091}{n^8} + \frac{3281}{n^9} + O(n^{-10}) \tag{64}$$

$$\omega_1 = -\frac{1}{n} + \frac{14}{n^2} - \frac{73}{n^3} + \frac{338}{n^4} - \frac{1585}{n^5} + \frac{7574}{n^6} - \frac{36793}{n^7} + \frac{180698}{n^8} - \frac{893665}{n^9} + O(n^{-10}) \tag{65}$$

$$\omega_2 = \frac{3}{n^3} - \frac{27}{n^4} - \frac{186}{n^5} + \frac{8010}{n^6} - \frac{136887}{n^7} + \frac{1799343}{n^8} - \frac{20769396}{n^9} + O(n^{-10}). \tag{66}$$

Putting expressions (52)-(66) back to (48) and after some algebra, we obtain the approximate closed form of $E(\hat{\phi} - \phi)^2$ in (40).

Proofs of Propositions 2.2 and 2.3:

The variance of $\hat{\phi}$ can be expressed in terms of the first two moments of $\hat{\phi}$ about ϕ, where

$$\mu_2(\hat{\phi}) = E(\hat{\phi} - \phi)^2 - (E(\hat{\phi}) - \phi)^2 \tag{67}$$

Plugging the approximate expressions for the first two moments of $\hat{\phi}$ in (21) and (40) into (67) and after some simplification, we obtain the closed form approximation as shown in (41). Also, the approximate standard deviation of $\hat{\phi}$ in (42) is obtained by taking the series expansion of the square root of $\mu_2(\hat{\phi})$ in (41) accordingly.