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Comparison on Some Modified Confidence Intervals for Estimating the Process Capability Index Cp: Simulation and Application

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Abstract

In this paper, we proposed some modified interval estimators, namely the modified adjusted degrees of freedom (ADJ*), modified large-sample (LS*), and the augmented-large-sample (ALS*) confidence intervals for estimating the population process capability index, Cp. A simulation study has been conducted to compare the performance of the proposed interval estimators with the existing Exact, ADJ, LS, ALS and the modified trimmed standard deviation confidence intervals. We consider both simulated coverage probability and average width as a performance criterion. Simulation results evident that the exact method performed the best under normal distribution, while the proposed confidence intervals performed well for most of cases for skewed distribution. For illustration purposes, two real-life data from industry are analyzed which supported the simulation results to some degree. The proposed methods can be recommended to be used by the practitioners in various fields of production and engineering.

Keywords: Average Width, Confidence Interval, Coverage Probability, Process Capability Index; Robust Estimator and Simulation study.

AMS Subject Classification: 62N02; 68U20.

0. Tribute to Sinha Brothers

It is my great honor and privilege to contribute this article in the special issue of International Journal of Statistical Sciences (IJSS) in honor of the twin statisticians, Professor Bikas K. Sinha and Professor Bimal K. Sinha. Due to their outstanding and invaluable contributions in statistics, both are very well known and respected statisticians in the world. I used to call them as “Bikas Da” and “Bimal Da’ for about twenty years. I meet first, when both have visited my department, Department of Statistics at Florida International University as invited speakers occasionally between 2004 to 2006. I was also fortunate to visit Bikas Da at the Indian Statistical Institute, Calcutta, India in 2006. Their love, moral support, and positive attitude towards my profession have been highly inspirational to me. I am thankful to the Department of Statistics, Rajshahi University for giving me the opportunity to publish this article in IJSS in honor of Bikas Da and Bimal Da. I wish them a long life filled with happiness and good health.

1. Introduction

The process capability analysis (PCA) is a set of calculations used to test whether the evaluation meets the specification requirements Process Capability Index (PCI) is a simple measure producer’s capability to produce a product within the customer’s tolerance range. The process capability index (PCI) is defined as the quotient between the length of the acceptance interval and six times the standard deviation obtained as a result of the design and production processes. The process capability index (also called Cp) tells us whether the result is between the specification limits (Maiti and Saha, 2012). A higher Cp value indicates a better the process. First, we need to set up a lower specification limit (LSL) and upper specification limit (USL) for the process given according to some specifications defined by some predefined standards not relating to the nature of the process itself.

Let us assume that the quality characteristic of interest follows a normal distribution with mean (μ) and standard deviation (σ). A capable process will have almost all the measurements being within the specification limits. Therefore, the quality characteristic outside the LSL and USL is to be considered as nonconforming. The population process capability index (Cp) is defined as a ratio in equation (1.1), where we would like to know the how much wider is the range of LSL to USL than the six-Sigma range where 99.73% of the items will fall within statistically. The higher the ratio of specification width (USL-LSL) over the process spread (6σ), the higher capability of the process to produce fewer nonconforming products. The population process capability index can be defined as follows:

$$C_p = \frac{USL-LSL}{6\sigma} \qquad (1.1)$$

where USL= upper specification limit and LSL=lower specification limit and σ is the process standard deviation. As we know when the Cp value is higher than 1, the process is considered capable while the opposite being not capable or poor in terms of quality measure. A Cp value higher than 1.67 is considered as an indicator of an excellent process and so on. In the real-world processes, we can replace σ with the sample standard deviation S to get a point estimation of the Cp. Moreover, A more useful interval estimate of the Cp, the confidence intervals are usually computed too which is called the exact CI of Cp. More on capability indices under both normal and non-normal distributions, we refer to Kane (1986), Pearn et al. (1992, 1994, 1995), Kotz and Lovelace (1998), Yeh and Bhattacharya (1998) and very recently Abu-Shawiesh et al (2020), among others.

The population Cp heavily depends on the assumption that the quality characteristic measurements are independent and normally distributed. However, there are many situations of violations of these assumptions and therefore the exact confidence interval (CI) for the process capability index (Cp) may not be accurate. Many authors proposed robust method which is better than the exact confidence interval (CI) in these situations. Panichkitkosolkul (2016) considered revisions of the confidence intervals of the variance term under non-normality and proposed several robust CIs for Cp. Abu-Shawiesh et al, (2020) proposed a robust confidence interval for the process capability index (Cp) by means of a robust modified trimmed standard deviation ($MTSD$=$ST*$). Based on these literatures, especially Abu-Shawiesh et al,. (2020) and Panichkitkosolkul (2016), we proposed modified confidence intervals by replacing the sample mean with the sample median in confidence intervals in Abu-Shawiesh et al (2020).

The rest of the paper is organized as follows: In section 2, we present some existing and proposed confidence intervals. A simulation study has been conducted in section 3. As applications, two real life data are analyzed in section 4. Finally, some concluding remarks are outlined in section 5.

2. Statistical Methodology

In this section, we will discuss about some confidence interval estimators. Since, in real life, most of the data do not follow the normality assumptions, we will consider some interval estimators for Cp when the data do not follow the normality assumption.

2.1. The Exact Confidence Interval

Suppose X represents the quality characteristics under study. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n drawn from a normal population with mean μ and standard deviation σ. If the process standard deviation is unknown, it can be estimate from the sample standard deviations, s

The exact (1-α)100% confidence interval (CI) for the population process capability index, Cp is obtained as follows:

$$CI_{Exact} = \left(\hat{C}_p \sqrt{\frac{\chi^2_{(\frac{\alpha}{2}, n-1)}}{n-1}},\ \hat{C}_p \sqrt{\frac{\chi^2_{(1-\frac{\alpha}{2}, n-1)}}{n-1}} \right) \tag{2.1}$$

where

$$\hat{C}_p = \frac{USL-LSL}{6s} \text{ and } s = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}}$$

2.2. The Robust Confidence Interval

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a normal distribution with process mean μ and process standard deviation σ. The r-times symmetrically trimmed random sample is obtained by dropping both lowest and highest r values from both end, using the following formula:

$$\bar{x}_T = \frac{1}{n-2r} \sum_{i=r+1}^{n-r} x_{(i)} \tag{2.2}$$

where r=[αn] represents the greatest integer and trimming is done for α% ($0 \le \alpha \le 0.5$) of the sample size n. The sample standard deviation can be calculated as follows.

$$s_T = \sqrt{\frac{\sum_{i=r+1}^{n}\left(x_{(i)} - \bar{x}_T\right)^2}{n-2r-1}} \tag{2.3}$$

Then the confidence intervals can be computed using Equations (2.1) and (2.3) with S replace by ST* in (2.4).

$$\widehat{\sigma_T} = MTSD = S_T^* = 1.4826\, s_T \tag{2.4}$$

Following, Abu-Shawiesh et al. (2020), the robust (1-α)100% confidence interval (CI) for the population process capability index, Cp is obtained as follows

$$CI_{Robust} = \left(\hat{C}_p^* \sqrt{\frac{\chi^2_{(\frac{\alpha}{2}, n-1)}}{n-1}}, \hat{C}_p^* \sqrt{\frac{\chi^2_{(1-\frac{\alpha}{2}, n-1)}}{n-1}} \right) \tag{2.5}$$

where $\hat{C}_p^* = \frac{USL-LSL}{6s*_T}$, $\chi^2_{(\frac{\alpha}{2}, n-1)}$ and $\chi^2_{(1-\frac{\alpha}{2}, n-1)}$ are the α/2 and 1-α/2 quantiles of the Chi-square distribution with n-1 degrees of freedom (DF). For more on robust confidence interval, we refer Yeh and Bhattacharya (1998) and Abu-Shawiesh et al. (2020) among others.

2.3. Confidence Intervals for Cp under Non-normality

Originally, the confidence intervals for the variance under non-normality was proposed by Hummel and Hettmansperger (2004) and Burch (2014) and are further described in detail in Abu-Shawiesh et al. (2020). In non-normal situations, the coverage probability of the confidence interval can be considerably below 1−α. Hummel and Hettmansperger (2004) presented a confidence interval for population variance by adjusting the degrees of freedom of chi-square distribution. Panichkitkosolkul (2020) proposed three confidence intervals for the process capability index Cp based on the confidence intervals for the variance proposed by Hummel and Hettmansperger (2004) and Burch (2014) and provided them in the following subsections.

2.3.1. The Adjusted Degrees of Freedom Confidence Interval (ADJ)

The (1 − α)100% confidence interval for the Cp based on the confidence interval for σ^2 by adjusting the degrees of freedom of chi-square distribution is given by

$$CI_{ADJ} = \left(\frac{USL-LSL}{6S} \sqrt{\frac{\chi^2{}_{\alpha/2, \hat{r}}}{\hat{r}}}, \frac{USL-LSL}{6S} \sqrt{\frac{\chi^2_{1-\alpha/2, \hat{r}}}{\hat{r}}} \right) \tag{2.6}$$

where the adjusting the degree of freedom of chi-squared distribution is given by

$$\hat{r} = \frac{2n}{\hat{\gamma} + 2n/(n-1)}$$

and

$$\hat{\gamma} = \frac{n(n+1)}{(n-1)(n-2)(n-3)} \frac{\sum_{i=1}^{n}(X_i - \bar{X})^4}{S^4} - \frac{3(n-1)^2}{(n-2)(n-3)}.$$

2.3.2. Large-Sample Confidence Interval for the Variance (LS)

The $(1-\alpha)100\%$ confidence interval for the Cp based on the large-sample confidence interval for σ^2 is given by

$$CI_{LS} = \left(\frac{USL-LSL}{6S\sqrt{exp(z_{1-\alpha/2}\sqrt{A})}}, \frac{USL-LSL}{6S\sqrt{exp(-z_{1-\alpha/2}\sqrt{A})}} \right) \tag{2.7}$$

where $A = \frac{G_2+2n(n-1)}{n}$, $\quad S^2 = (n-1)^{-1}\sum_{i=1}^{n}(X_i - \bar{X})^2, and$

$$G_2 = \frac{n-1}{(n-2)(n-3)}[(n-1)g_2 + 6]$$

with $g_2 = \frac{m_4}{m_2^2} - 3, m_4 = n^{-1}\sum_{i=1}^{n}(X_i - \bar{X})^4$ and $m_2 = n^{-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$.

2.3.3. Augmented-Large-Sample Confidence Interval for the Variance (ALS)

The $(1-\alpha)100\%$ confidence interval for the Cp based on the augmented-large-sample confidence interval for σ^2 is given by

$$CI_{ALS} = \left(\frac{USL-LSL}{6S\sqrt{exp(z_{1-\alpha/2}\sqrt{B}+C)}}, \frac{USL-LSL}{6S\sqrt{exp}(-z_{1-\alpha/2})} \right) \tag{2.8}$$

where $B = \widehat{var}(\log(S^2))$, $C = \frac{\hat{k}_{e,5}+2n/(n-1)}{2n}$, and $\hat{k}_{e,5} = \left(\frac{n+1}{n-1}\right) G_2 \left(1 + \frac{5G_2}{n}\right)$.

2.4. Modification of ADJ, LS and ALS confidence intervals

Motived by the robust truncated confidence interval in section 2.2, we would like to use the sample median which is more resistant to outliers and skewed distribution, to define the sample standard deviation. Combined with the proposed methods in 2.3, we will propose three new confidence intervals by modifying Equations (2.6), (2.7) and (2.8) just by replacing each S by S*

$$S^* = \sqrt{\frac{\sum_{i=1}^{n}(X_i - Md)^2}{n-1}} \tag{2.9}$$

where Md= median of the observations, $X_1, X_2, \ldots, X_n$

2.4.1. Modified Adjusted Degrees of Freedom Confidence Interval (ADJ)

The (1 − α)100% confidence interval for the Cp based on the confidence interval for σ^2 by adjusting the degrees of freedom of chi-square distribution is given by

$$CI_{ADJ*} = \left(\frac{USL-LSL}{6S^*}\sqrt{\frac{X_{\alpha/2,\hat{r}}}{\hat{r}}}, \frac{USL-LSL}{6S^*}\sqrt{\frac{X^2_{1-\alpha/2,\hat{r}}}{\hat{r}}}\right) \tag{2.10}$$

where the adjusting the degree of freedom of chi-squared distribution is given by

$$\hat{r}_1 = \frac{2n}{\hat{\gamma}_1 + 2n/(n-1)}$$

and

$$\hat{\gamma}_1 = \frac{n(n+1)}{(n-1)(n-2)(n-3)}\frac{\sum_{i=1}^{n}(X_i - Md)^4}{S^{*4}} - \frac{3(n-1)^2}{(n-2)(n-3)}.$$

2.4.2. Modified large-Sample Confidence Interval for the Variance (LS)

The (1 − α)100% confidence interval for the Cp based on the large-sample confidence interval for σ^2 is given by

$$CI_{LS*} = \left(\frac{USL-LSL}{6S^*\sqrt{exp(z_{1-\alpha/2}\sqrt{A})}}, \frac{USL-LSL}{6S^*\sqrt{exp(-z_{1-\alpha/2}\sqrt{A})}}\right) \tag{2.11}$$

where $A = \frac{G_{2*}+2n(n-1)}{n}$, $, and \quad G_{2*} = \frac{n-1}{(n-2)(n-3)}[(n-1)g_{2*} + 6]$

with $g_{2*} = \frac{m_{4*}}{m_{2*}^2} - 3, m_{4*} = n^{-1}\sum_{i=1}^{n}(X_i - Md)^4$ and $m_{2*} = n^{-1}\sum_{i=1}^{n}(X_i - Md)^2$.

2.4.3. Modified Augmented-Large-Sample Confidence Interval for the Variance (ALS)

The (1 − α)100% confidence interval for the Cp based on the augmented-large-sample confidence interval for σ^2 is given by

$$CI_{ALS*} = \left(\frac{USL-LSL}{6S^* \sqrt{exp(z_{1-\frac{\alpha}{2}}\sqrt{B*}+C*)}}, \frac{USL-LSL}{6S^* \sqrt{exp(-z_{1-\frac{\alpha}{2}}\sqrt{B*}+C*)}} \right) \quad (2.12)$$

where $B^* = \widehat{\text{var}}(\log(S^{*2}))$, $C^* = \frac{\hat{k}_{e,5}+2n/(n-1)}{2n}$, and $\hat{k}_{e,5} = \left(\frac{n+1}{n-1}\right) G_{2*} \left(1 + \frac{5G_{2*}}{n}\right)$ and S^* is defined in equation (2.9).

3. Simulation Study

Since a theoretical comparison among the interval estimators is difficult, a simulation study is conducted in this section. Statistical software R 4.1.1 is used to compare the performances of the proposed intervals for the following normal and non-normal distributions: **(i)** N (50, 1) (ii) Gamma (4, 2); (iii) Gamma (0.75, 0.867) and (iv) Gamma (0.25, 0.50) distributions. The number of simulation replications was 50000 for each case. Random samples were generated from each of the above mentioned distributions with Cp =1.0 and samples sizes n=10, 25, 50 and 100. Coverage probability (CP) and average width (AW) of selected CIs were measured for each case. The most common 95% confidence interval (α=0.05) is used for measuring confidence level. When (α=0.05), an interval has perfect performance in terms of CP that will capture the true Cp between the lower and upper limits 95% of the time. The estimated CP and AW for this simulation study are given respectively by:

$$\hat{C}_p = \frac{\#(L \le C_p \le U)}{M} \text{ and } \widehat{AW} = \frac{\sum_{i=1}^{M}(U_i - L_i)}{M}, \quad (3.1)$$

where $\#(L \le Cp \le U)$ denotes the number of simulation runs for which Cp lies within confidence interval. We generated the following distributions in Table 3.1 for simulation comparisons. The true values of the process capability index Cp, LSL and USL are set in the Table 3.2.

Table 3.1: Probability distributions and the coefficient of skewness for Monte Carlo simulation.

Probability Distributions	Coefficient of Skewness
N(50,1)	0.000
Gamma(4,2)+48	1.000
Gamma(0.75,0.867)+49.1340	2.309
Gamma(0.25,0.5)+49.5	4.000

Table 3.2: True values of Cp, LSL and USL.

True Values of C_p	LSL	USL
1.00	47.00	53.00
1.33	46.01	53.99
1.50	45.50	54.50
1.67	44.99	55.01
2.00	44.00	56.00

The sample sizes were set at n=30,50,75 and 100 and the number of simulation trials was 50,000. The nominal level was fixed at 0.95. The simulated CPs and AWs for each of the distributions described above are presented in Tables 3.3-3.4 for N(50,1), Gamma (4,2), Gamma(0.75, 0.867) and Gamma (0.25, 0.5) respectively. For clear understanding, CPs and AWs for considered *n* are presented in Figures 3.1-3.4 respectively. From these Tables we may observe that when data are generating from N(50,1) distribution, the exact method has estimated coverage probabilities close to nominal level 0.95 for all sample sizes. However, the proposed methods ALS*, LS* and ADJ* are performing better than the corresponding existing methods ALS, LS and ADJ in the sense of higher coverage probabilities. Robust method performed the worse is left for the further study. The expected length for all methods do not differ significantly. However, Robust methods have the shortest length, while LS* method has the widest length. It may be noted that when sample sizes increase the expected length decreases for all methods. When the data are generated from skewed distributions (Gamma), the proposed methods ALS*, LS* and ADJ* are performing better than the rest of the intervals in the sense of high coverage probablities. Both exact and robust methods performed poorly that need for further study.

Table 3.3: Coverage probability and average length for N(50,1) simulations.

		Coverage Probability					Average Length				
Method	n\Cp	1	1.33	1.5	1.67	2	1	1.33	1.5	1.67	2
EXACT	30	0.9501	0.9498	0.9490	0.9515	0.9483	0.5258	0.6991	0.7885	0.8774	1.0519
	50	0.9479	0.9504	0.9491	0.9497	0.9479	0.4006	0.5335	0.6017	0.6690	0.8020
	75	0.9510	0.9504	0.9494	0.9499	0.9493	0.3247	0.4319	0.4873	0.5425	0.6496
	100	0.9483	0.9503	0.9505	0.9510	0.9502	0.2804	0.3726	0.4205	0.4679	0.5605
TRUNC 5%	30	0.4798	0.4772	0.4787	0.4738	0.4792	0.4114	0.5468	0.6168	0.6866	0.8233
	50	0.4059	0.4132	0.4117	0.4078	0.4128	0.3244	0.4320	0.4873	0.5418	0.6495
	75	0.2752	0.2798	0.2818	0.2814	0.2770	0.2642	0.3515	0.3966	0.4415	0.5285
	100	0.3435	0.3385	0.3428	0.3399	0.3405	0.2376	0.3157	0.3564	0.3965	0.4751
TRUNC 10%	30	0.8820	0.8854	0.8834	0.8858	0.8843	0.5221	0.6942	0.7836	0.8715	1.0460
	50	0.8832	0.8845	0.8846	0.8849	0.8836	0.4020	0.5352	0.6036	0.6712	0.8048
	75	0.8810	0.8806	0.8802	0.8765	0.8803	0.3200	0.4257	0.4803	0.5346	0.6400
	100	0.8811	0.8847	0.8834	0.8859	0.8803	0.2835	0.3766	0.4253	0.4732	0.5669
ALS	30	0.9381	0.9388	0.9361	0.9396	0.9362	0.5354	0.7114	0.8029	0.8939	1.0731
	50	0.9395	0.9404	0.9378	0.9402	0.9378	0.4022	0.5354	0.6038	0.6714	0.8058
	75	0.9434	0.9435	0.9411	0.9427	0.9420	0.3247	0.4318	0.4872	0.5422	0.6488
	100	0.9413	0.9443	0.9447	0.9443	0.9428	0.2802	0.3718	0.4203	0.4672	0.5601
ALS*	30	0.9386	0.9390	0.9367	0.9391	0.9368	0.5442	0.7216	0.8154	0.9082	1.0907
	50	0.9391	0.9405	0.9387	0.9409	0.9381	0.4052	0.5394	0.6083	0.6767	0.8123
	75	0.9442	0.9434	0.9423	0.9434	0.9426	0.3264	0.4339	0.4897	0.5450	0.6521
	100	0.9416	0.9446	0.9448	0.9443	0.9435	0.2811	0.3731	0.4217	0.4689	0.5620
LS	30	0.9292	0.9288	0.9286	0.9306	0.9269	0.5284	0.7021	0.7924	0.8821	1.0589
	50	0.9346	0.9358	0.9337	0.9364	0.9338	0.4008	0.5336	0.6017	0.6692	0.8030
	75	0.9407	0.9402	0.9379	0.9403	0.9388	0.3246	0.4317	0.4870	0.5420	0.6486
	100	0.9399	0.9421	0.9423	0.9428	0.9408	0.2802	0.3719	0.4204	0.4674	0.5603
LS*	30	0.9288	0.9288	0.9275	0.9289	0.9278	0.9288	0.9288	0.9275	0.9289	0.9278
	50	0.9346	0.9363	0.9345	0.9370	0.9338	0.9346	0.9363	0.9345	0.9370	0.9338
	75	0.9417	0.9405	0.9393	0.9404	0.9399	0.9417	0.9405	0.9393	0.9404	0.9399
	100	0.9402	0.9425	0.9432	0.9431	0.9415	0.9402	0.9425	0.9432	0.9431	0.9415
ADJ*	30	0.9322	0.9322	0.9291	0.9333	0.9303	0.5158	0.6854	0.7736	0.8612	1.0338
	50	0.9371	0.9382	0.9357	0.9379	0.9357	0.3958	0.5269	0.5941	0.6609	0.7929
	75	0.9419	0.9425	0.9405	0.9413	0.9412	0.3221	0.4283	0.4832	0.5378	0.6436
	100	0.9404	0.9436	0.9437	0.9436	0.9419	0.2787	0.3699	0.4181	0.4648	0.5572
ADJ	30	0.9388	0.9385	0.9352	0.9397	0.9365	0.5287	0.7024	0.7929	0.8826	1.0595
	50	0.9402	0.9413	0.9389	0.9407	0.9390	0.4016	0.5346	0.6029	0.6705	0.8047
	75	0.9442	0.9443	0.9424	0.9437	0.9435	0.3251	0.4323	0.4878	0.5429	0.6496
	100	0.9423	0.9449	0.9455	0.9447	0.9435	0.2807	0.3725	0.4211	0.4682	0.5612

Table 3.4: Coverage probability and average length for Gamma (4,2) simulations.

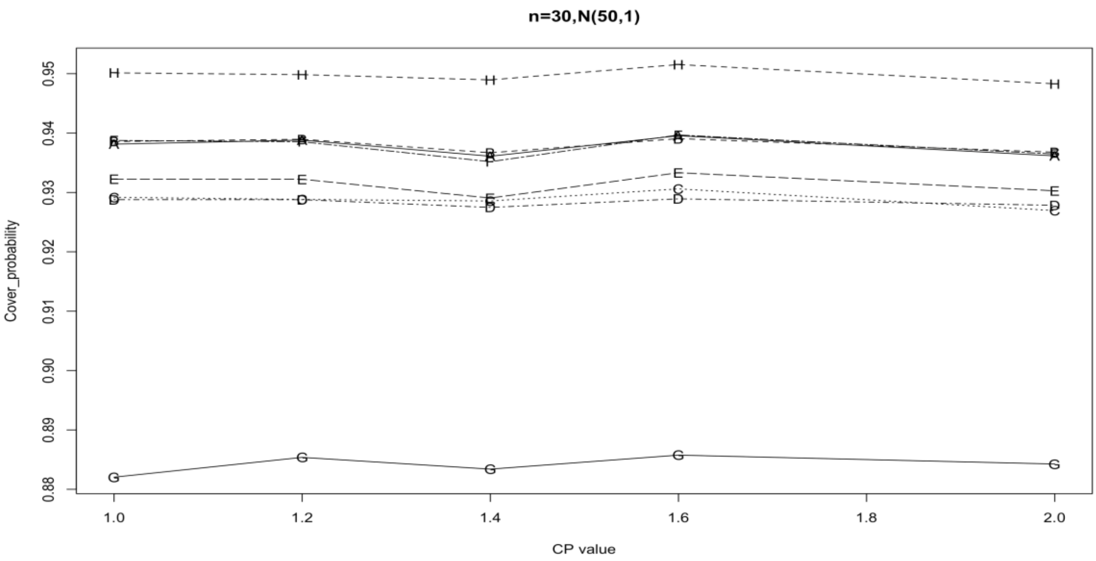
		Coverage Probability					Average Length				
Method	n\Cp	1	1.33	1.5	1.67	2	1	1.33	1.5	1.67	2
EXACT	30	0.881	0.878	0.879	0.877	0.879	0.534	0.711	0.802	0.892	1.067
	50	0.873	0.875	0.873	0.872	0.872	0.406	0.538	0.608	0.676	0.810
	75	0.869	0.870	0.870	0.867	0.868	0.327	0.435	0.491	0.546	0.654
	100	0.869	0.868	0.868	0.867	0.867	0.282	0.375	0.423	0.470	0.563
TRUNC 5%	30	0.548	0.549	0.548	0.547	0.544	0.424	0.564	0.636	0.708	0.847
	50	0.509	0.506	0.507	0.505	0.509	0.335	0.444	0.502	0.558	0.669
	75	0.404	0.403	0.402	0.400	0.400	0.272	0.362	0.408	0.454	0.544
	100	0.487	0.482	0.486	0.484	0.482	0.245	0.325	0.367	0.409	0.489
TRUNC 10%	30	0.850	0.850	0.850	0.848	0.851	0.544	0.724	0.817	0.910	1.087
	50	0.838	0.841	0.839	0.840	0.840	0.419	0.556	0.627	0.698	0.837
	75	0.848	0.846	0.847	0.845	0.848	0.332	0.442	0.498	0.555	0.665
	100	0.807	0.810	0.805	0.810	0.810	0.294	0.391	0.442	0.491	0.588
ALS	30	0.901	0.897	0.896	0.897	0.899	0.647	0.860	0.968	1.079	1.289
	50	0.905	0.906	0.905	0.905	0.905	0.493	0.656	0.741	0.825	0.987
	75	0.911	0.913	0.912	0.909	0.912	0.404	0.536	0.604	0.673	0.806
	100	0.917	0.917	0.916	0.916	0.917	0.350	0.465	0.526	0.584	0.701
ALS*	30	0.923	0.920	0.919	0.920	0.922	0.742	0.987	1.110	1.239	1.482
	50	0.930	0.934	0.932	0.931	0.931	0.559	0.743	0.839	0.934	1.118
	75	0.940	0.943	0.940	0.938	0.940	0.455	0.604	0.681	0.758	0.909
	100	0.947	0.945	0.944	0.946	0.945	0.394	0.522	0.591	0.657	0.789
LS	30	0.888	0.884	0.882	0.884	0.886	0.609	0.810	0.912	1.017	1.216
	50	0.897	0.901	0.901	0.899	0.900	0.478	0.635	0.717	0.798	0.956
	75	0.910	0.911	0.910	0.907	0.911	0.397	0.527	0.594	0.661	0.793
	100	0.918	0.917	0.915	0.916	0.917	0.347	0.460	0.521	0.579	0.694
LS*	30	0.906	0.902	0.901	0.902	0.905	0.906	0.902	0.901	0.902	0.905
	50	0.920	0.923	0.922	0.920	0.920	0.920	0.923	0.922	0.920	0.920
	75	0.932	0.935	0.932	0.930	0.933	0.932	0.935	0.932	0.930	0.933
	100	0.940	0.940	0.938	0.940	0.940	0.940	0.940	0.938	0.940	0.940
ADJ*	30	0.895	0.893	0.891	0.892	0.894	0.594	0.790	0.890	0.992	1.186
	50	0.906	0.907	0.907	0.906	0.906	0.472	0.627	0.708	0.788	0.944
	75	0.914	0.917	0.915	0.913	0.916	0.393	0.523	0.589	0.656	0.786
	100	0.922	0.922	0.920	0.920	0.922	0.345	0.458	0.518	0.575	0.690
ADJ	30	0.911	0.910	0.908	0.909	0.910	0.635	0.844	0.950	1.060	1.267
	50	0.919	0.921	0.920	0.920	0.920	0.503	0.668	0.755	0.840	1.006
	75	0.928	0.930	0.927	0.926	0.928	0.419	0.556	0.627	0.698	0.837
	100	0.933	0.932	0.932	0.932	0.933	0.367	0.487	0.551	0.612	0.734

Table 3.5: Coverage probability and average length for Gamma (0.75, .867) simulations.

		Coverage Probability					Average Length				
Method	n\Cp	1	1.33	1.5	1.67	2	1	1.33	1.5	1.67	2
EXACT	30	0.667	0.666	0.663	0.667	0.665	0.571	0.762	0.858	0.954	1.144
	50	0.654	0.652	0.649	0.655	0.655	0.423	0.563	0.633	0.705	0.845
	75	0.646	0.645	0.643	0.640	0.644	0.337	0.449	0.505	0.563	0.674
	100	0.637	0.643	0.635	0.637	0.640	0.288	0.384	0.433	0.481	0.577
TRUNC 5%	30	0.629	0.635	0.633	0.630	0.632	0.483	0.644	0.726	0.807	0.967
	50	0.653	0.654	0.655	0.651	0.651	0.381	0.506	0.570	0.635	0.760
	75	0.646	0.645	0.646	0.642	0.641	0.308	0.410	0.462	0.514	0.616
	100	0.687	0.690	0.686	0.686	0.688	0.280	0.372	0.421	0.468	0.561
TRUNC 10%	30	0.588	0.584	0.586	0.590	0.587	0.660	0.880	0.992	1.103	1.323
	50	0.485	0.486	0.484	0.489	0.491	0.504	0.670	0.754	0.839	1.007
	75	0.444	0.440	0.440	0.440	0.444	0.397	0.527	0.595	0.662	0.793
	100	0.297	0.297	0.292	0.296	0.295	0.352	0.468	0.529	0.588	0.704
ALS	30	0.863	0.864	0.863	0.866	0.865	1.112	1.477	1.670	1.858	2.217
	50	0.889	0.890	0.888	0.889	0.892	0.833	1.105	1.252	1.393	1.663
	75	0.903	0.904	0.906	0.903	0.904	0.672	0.894	1.011	1.122	1.346
	100	0.909	0.913	0.909	0.911	0.912	0.578	0.769	0.869	0.969	1.161
ALS*	30	0.937	0.938	0.937	0.939	0.939	1.717	2.278	2.577	2.874	3.419
	50	0.957	0.960	0.958	0.959	0.960	1.162	1.540	1.747	1.945	2.321
	75	0.967	0.967	0.968	0.968	0.966	0.896	1.192	1.348	1.497	1.795
	100	0.971	0.971	0.970	0.971	0.970	0.754	1.002	1.132	1.262	1.512
LS	30	0.809	0.807	0.808	0.811	0.809	0.889	1.183	1.335	1.485	1.778
	50	0.845	0.845	0.842	0.843	0.845	0.719	0.956	1.077	1.200	1.436
	75	0.866	0.865	0.868	0.866	0.864	0.608	0.808	0.913	1.015	1.217
	100	0.877	0.879	0.875	0.879	0.880	0.538	0.716	0.809	0.901	1.079
LS*	30	0.893	0.893	0.894	0.896	0.895	0.893	0.893	0.894	0.896	0.895
	50	0.924	0.926	0.924	0.924	0.926	0.924	0.926	0.924	0.924	0.926
	75	0.940	0.941	0.943	0.941	0.939	0.940	0.941	0.943	0.941	0.939
	100	0.948	0.950	0.948	0.950	0.950	0.948	0.950	0.948	0.950	0.950
ADJ*	30	0.833	0.832	0.829	0.825	0.838	0.850	1.141	1.298	1.440	1.730
	50	0.861	0.854	0.856	0.857	0.859	0.700	0.930	1.047	1.175	1.407
	75	0.875	0.873	0.878	0.873	0.882	0.599	0.797	0.893	1.001	1.196
	100	0.893	0.886	0.887	0.887	0.882	0.534	0.708	0.793	0.891	1.061
ADJ	30	0.873	0.879	0.874	0.870	0.876	0.914	1.225	1.392	1.544	1.856
	50	0.891	0.882	0.889	0.888	0.892	0.737	0.979	1.104	1.237	1.481
	75	0.902	0.902	0.901	0.898	0.909	0.622	0.828	0.929	1.039	1.244
	100	0.914	0.911	0.906	0.910	0.906	0.550	0.730	0.818	0.918	1.094

Table 3.6: Coverage probability and average length for Gamma (0.25, .5) simulations.

		Coverage Probability					Average Length				
Method	n\Cp	1	1.33	1.5	1.67	2	1	1.33	1.5	1.67	2
EXACT	30	0.460	0.455	0.454	0.455	0.456	0.666	0.892	1.002	1.116	1.333
	50	0.445	0.448	0.450	0.443	0.445	0.465	0.620	0.699	0.776	0.931
	75	0.442	0.437	0.444	0.436	0.442	0.360	0.479	0.539	0.602	0.719
	100	0.435	0.436	0.436	0.434	0.434	0.303	0.403	0.455	0.507	0.608
TRUNC 5%	30	0.476	0.473	0.472	0.476	0.474	0.649	0.867	0.975	1.086	1.299
	50	0.437	0.437	0.436	0.440	0.436	0.502	0.670	0.755	0.839	1.005
	75	0.405	0.404	0.406	0.403	0.405	0.401	0.534	0.601	0.670	0.801
	100	0.271	0.272	0.275	0.266	0.268	0.375	0.498	0.562	0.626	0.751
TRUNC 10%	30	0.205	0.203	0.203	0.202	0.206	1.084	1.445	1.632	1.813	2.167
	50	0.096	0.095	0.097	0.099	0.097	0.792	1.058	1.192	1.325	1.587
	75	0.054	0.055	0.056	0.055	0.055	0.605	0.804	0.906	1.011	1.208
	100	0.014	0.015	0.015	0.016	0.014	0.542	0.721	0.813	0.905	1.086
ALS	30	0.857	0.856	0.857	0.859	0.860	2.577	3.436	3.876	4.321	5.177
	50	0.883	0.884	0.884	0.884	0.882	1.733	2.316	2.609	2.910	3.470
	75	0.897	0.897	0.898	0.898	0.900	1.320	1.763	1.977	2.214	2.642
	100	0.906	0.904	0.906	0.906	0.904	1.109	1.466	1.651	1.840	2.213
ALS*	30	0.972	0.972	0.971	0.971	0.974	5.082	6.794	7.656	8.566	10.235
	50	0.980	0.980	0.980	0.980	0.979	2.575	3.443	3.877	4.330	5.159
	75	0.983	0.982	0.983	0.982	0.983	1.749	2.339	2.623	2.938	3.502
	100	0.983	0.983	0.983	0.984	0.982	1.396	1.846	2.079	2.319	2.788
LS	30	0.742	0.737	0.739	0.741	0.743	1.400	1.871	2.106	2.344	2.804
	50	0.787	0.790	0.790	0.791	0.788	1.096	1.465	1.650	1.834	2.193
	75	0.822	0.822	0.823	0.820	0.822	0.920	1.225	1.378	1.540	1.839
	100	0.839	0.841	0.840	0.837	0.838	0.817	1.082	1.220	1.360	1.633
LS*	30	0.870	0.868	0.867	0.870	0.870	0.870	0.868	0.867	0.870	0.870
	50	0.905	0.906	0.906	0.905	0.905	0.905	0.906	0.906	0.905	0.905
	75	0.926	0.927	0.927	0.925	0.927	0.926	0.927	0.927	0.925	0.927
	100	0.938	0.936	0.936	0.938	0.935	0.938	0.936	0.936	0.938	0.935
ADJ*	30	0.785	0.782	0.782	0.784	0.786	1.312	1.754	1.973	2.196	2.627
	50	0.819	0.822	0.820	0.820	0.817	1.042	1.393	1.568	1.743	2.085
	75	0.844	0.843	0.845	0.843	0.844	0.883	1.176	1.323	1.478	1.765
	100	0.859	0.858	0.859	0.858	0.856	0.789	1.046	1.179	1.314	1.577
ADJ	30	0.833	0.833	0.830	0.834	0.834	1.305	1.744	1.963	2.184	2.612
	50	0.851	0.855	0.853	0.852	0.849	1.010	1.348	1.519	1.688	2.020
	75	0.863	0.864	0.864	0.864	0.865	0.842	1.121	1.261	1.408	1.682
	100	0.875	0.872	0.872	0.873	0.871	0.745	0.988	1.113	1.241	1.490



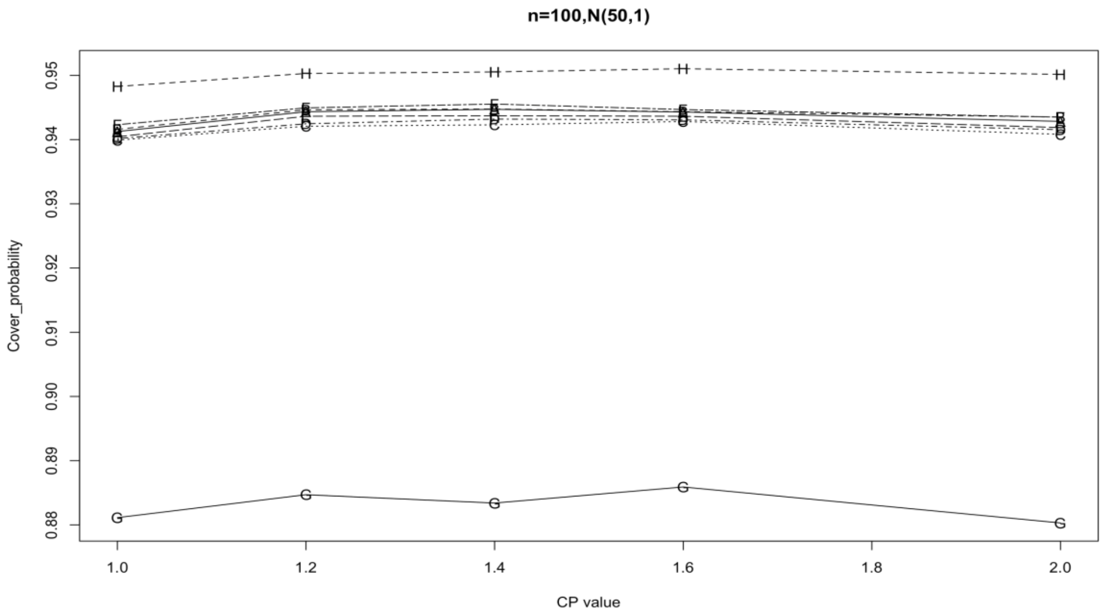


Figure 3.1: Estimated coverage probabilities of 95% confidence intervals for different n (50, 100) and Cp for N (50,1) distribution. Note that A=ALS, B=ALS*, C=LS, D=LS*, E=ADJ, F=ADJ*, G=Trunc (10%), H=exact

It is clear from Figure 3.1 that the exact CI is the best of all for normal distributions while the variance methods are roughly similarly well. The truncated CI is not as good as the other methods.

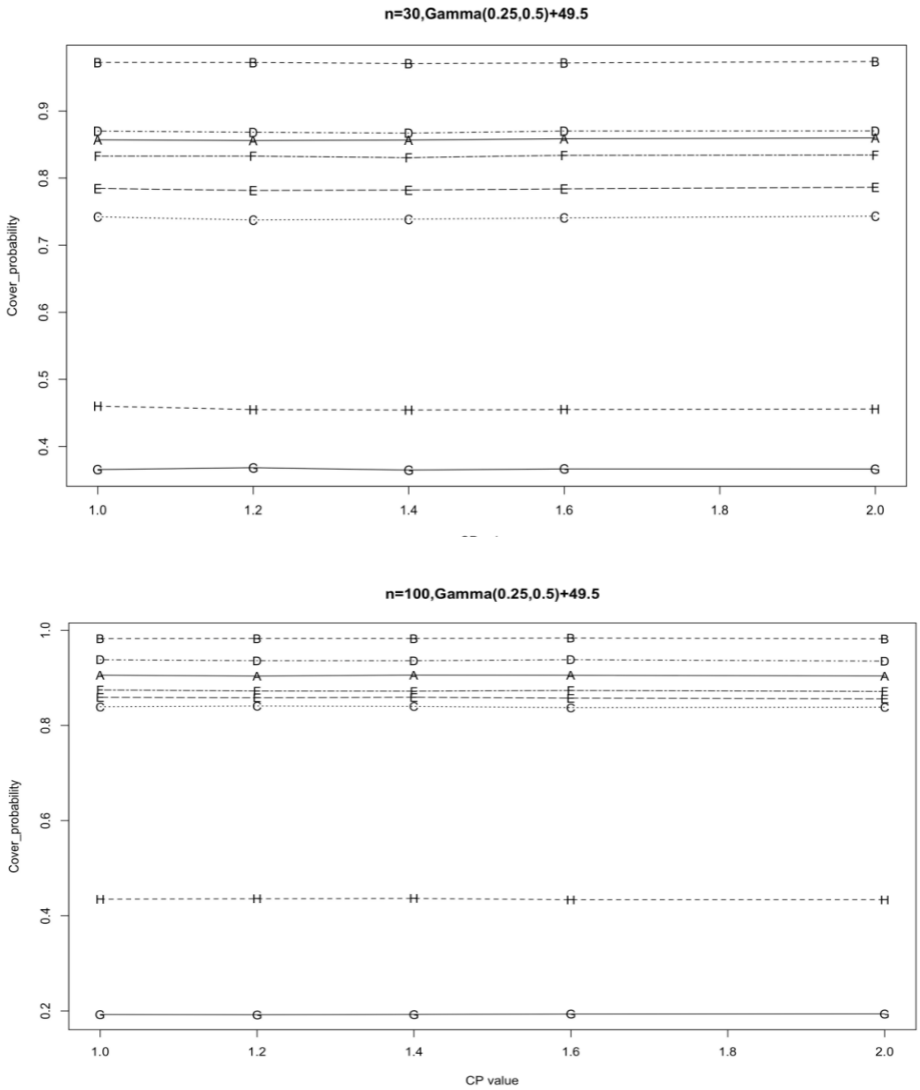


Figure3.2: Estimated coverage probabilities of 95% confidence intervals for different n (50, 100) and Cp for Gamma (0.25, 0.5) distribution. Note that A=ALS, B=ALS*, C=LS, D=LS*, E=ADJ, F=ADJ*, G=Trunc (10%), H=exact

From Figure 3.2, we can see that for skewed distributions, ALS* performing the best followed by LS*, ADJ* ALS, LS and ADJ and then Exact and robust truncated CIs.

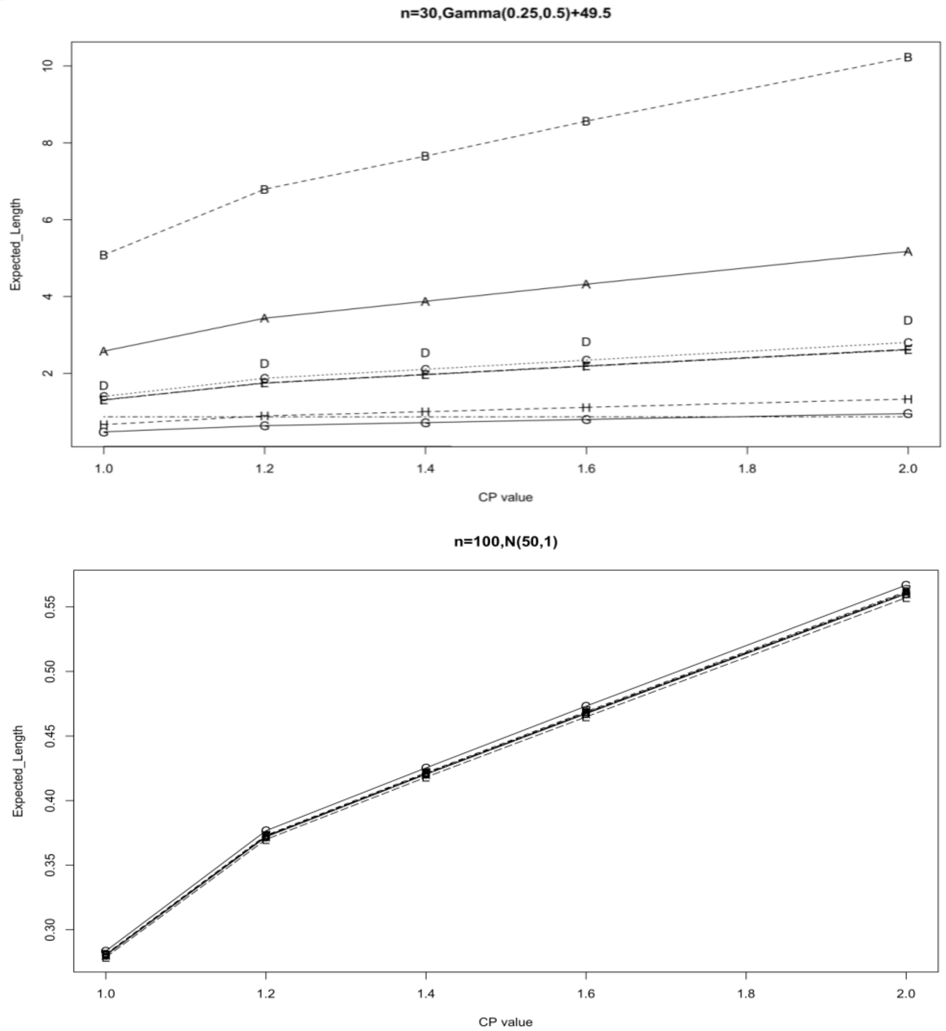


Figure 3.3: Estimated average widths of 95% confidence intervals for different n (50, 100) and Cp for N(50, 1) distribution. Note that A=ALS, B=ALS*, C=LS, D=LS*, E=ADJ, F=ADJ*,G=Trunc (10%), H=exact.

It is clear from Figure 3.3 that for normal case, the expected length are roughly the same for all methods. The higher the Cp value, the higher the lengths.

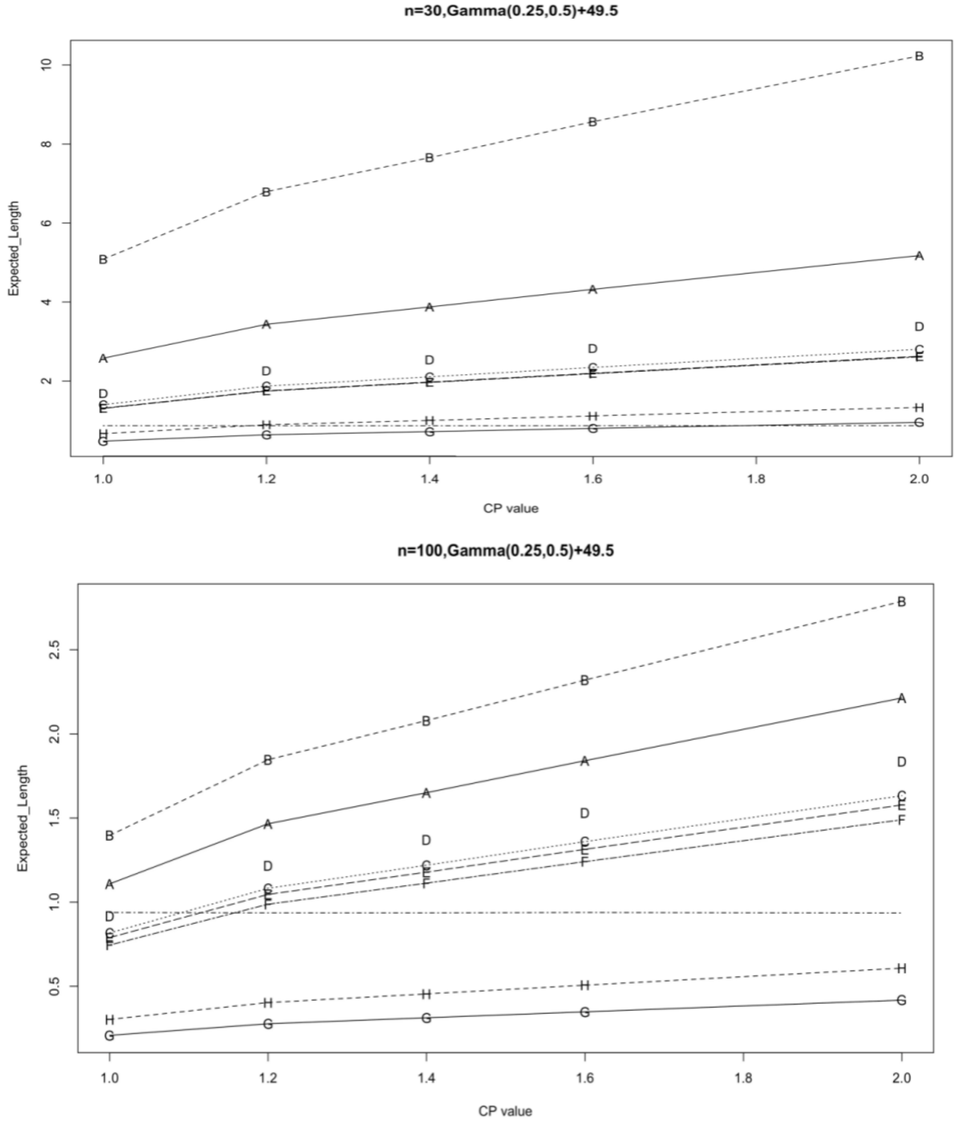


Figure 3.4: Estimated average widths of 95% confidence intervals for different n (50, 100) and Cp for Gamma(0.25, 0.50) distribution. Note that A=ALS, B=ALS*, C=LS, D=LS*, E=ADJ, F=ADJ*, G=Trunc (10%), H=exact.

It is clear from Figure 3.4 that the proposed ALS*, LS* and ADJ* CIs generally have reasonably larger length than the rest of the methods while the robust truncated CI and exact CI have the shortest length.

Applications

To illustrate the findings of the paper, we will analyze two real life data in this section.

4.1. Weight of the rubber edge data

In this example, we consider the weight of the rubber edge data (in gm), which is an important component that reflect the sound quality of the drive unit. The data in Table 4.1 was analyzed by Rezaie et al. (2006) and Abu-Shawiesh (2020) among others.

Table 4.1: Weight of the rubber edge data

8.63	8.65	8.57	8.57	8.54	8.69	8.63	8.64	8.59	8.61
8.6	8.66	8.65	8.5	8.61	8.61	8.63	8.67	8.54	8.62
8.65	8.58	8.65	8.67	8.67	8.65	8.69	8.66	8.62	8.63
8.59	8.65	8.64	8.64	8.52	8.69	8.66	8.66	8.61	8.55
8.57	8.64	8.63	8.57	8.61	8.59	8.56	8.71	8.53	8.51
8.72	8.58	8.64	8.69	8.64	8.75	8.59	8.61	8.58	8.65
8.73	8.7	8.65	8.56	8.66	8.65	8.66	8.68	8.62	8.54
8.67	8.62	8.54	8.62	8.66	8.56	8.6	8.62	8.61	8.66

Some descriptive statistics for weight measurements of rubber edge data are presented in Table 4.2.

Table 4.2: Descriptive Statistics of the rubber edge data

Statistics	Value
Sample mean	8.623
Sample median	8.630
Sample SD	0.052
Skewness	-0.194
Modified SD	0.060

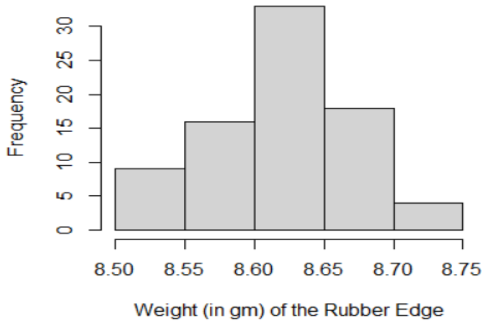


Figure 4.1: Histogram of the rubber edge data

The histogram of the data is provided in Figure 4.1. Shapiro test gives p-value 0.4657, which is more than 0.05. Both histogram and p-value indicated that the data are from a normal population. We will consider the lower and upper specification limits are 8.30 and 8.90 respectively. If the data falls outside of the specification limits will be considered as unacceptable. The 95% CIs for the weight measurements of rubber edge data for the exact and all proposed methods are computed and reported in Table 4.3.

Table 4.3: The 95% confidence intervals and widths for the rubber edge data

Methods	PCI	LCL	UCL	Width
Exact	1.91	1.62	2.21	0.60
ALS	1.91	1.63	2.19	0.57
ALS*	1.91	1.64	2.18	0.54
LS	1.94	1.65	2.22	0.57
LS*	1.92	1.65	2.19	0.54
ADJ	1.91	1.63	2.20	0.57
ADJ*	1.90	1.61	2.19	0.58
Truncate (10%)	1.93	1.63	2.24	0.60
Truncate (5%)	1.62	1.36	1.87	0.50

We can see from the Table 4.3 that the point estimate of Cp of the estimators, ADJ, ADJ*, LS, LS*, ALS and ALS* and 10% Truncated method all have values close to the exact Cp value of 1.91. However, 5% Truncated method has lower Cp value than the exact method. The width of the proposed intervals are shorter than the rest of the interval but the 5% Truncated method. Looking at these values basically support our simulation study results.

4.2. Polarizer manufacturing process data

The single hue value b (measured in NBS) is an important quality characteristic in the polarizer manufacturing process. To monitor the performance of this process, 25, each of sample size 2, were taken when the process is in control (Li et al. (2014). The resulting data are shown in Table 4.4 and summary statistics of the data are given in Table 4.5.

Table 4.4: Polarizer manufacturing process data

#	X	#	X	#	X	#	X	#	X
1	4.41,4.41	6	4.62,4.38	11	4.50,4.41	16	4.50,4.47	21	4.60,4.50
2	4.42,4.47	7	4.35,4.43	12	4.39,4.54	17	4.56,4.44	22	4.45,4.55
3	4.38,4.40	8	4.61,4.51	13	4.43,4.45	18	4.42,4.40	23	4.61,4.33
4	4.47,4.47	9	4.41,4.60	14	4.43,4.44	19	4.44,4.46	24	4.40,4.38
5	4.51,4.48	10	4.44,4.38	15	4.47,4.46	20	4.44,4.52	25	4.37,4.50

Table 4.5: Descriptive Statistics

Statistics	Value
Sample mean	4.460
Sample median	4.445
Sample SD	0.071
Skewness	0.641
Modified SD	0.109

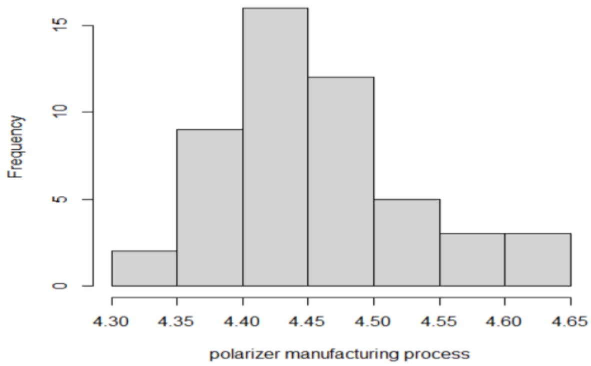


Figure 4.2: Histogram of the Polarizer manufacturing process data

The p-value for Shapiro test is 0.0282, which indicated data are not from a normal population. The histogram in Figure 4.2 clearly showed that data are from a right skewed distribution. We consider USL=4.7 and LSL=4.1 and the 95% CIs for the polarizer manufacturing process data for the exact and all proposed methods are computed and reported in Table 4.6.

Table 4.6: The 95% confidence intervals and widths for the rubber edge data

Methods	PCI	LCL	UCL	Width
Exact	1.41	1.13	1.69	0.56
ALS	1.39	1.12	1.66	0.54
ALS*	1.39	1.12	1.67	0.55
LS	1.44	1.16	1.71	0.55
LS*	1.40	1.13	1.68	0.55
ADJ	1.41	1.14	1.68	0.55
ADJ*	1.38	1.07	1.68	0.61
Truncate (10%)	1.44	1.16	1.72	0.57
Truncate (5%)	1.10	0.88	1.32	0.43

From Table 4.6 it appears that the widths of all estimators except 5% truncated interval and ADJ* are very close to each other's.

5. Some Concluding Remarks

This paper studied eight confidence intervals for estimating the process capability index Cp and three of them, namely ALS*, LS* and ADJ* are proposed based on the modified standard deviation. A simulation study has been conducted to compare the performance of the interval estimators under both normal and nonnormal distributions. Both coverage probability and average width were considered as a performance criterion. Our proposed interval estimators performed better in the sense of high coverage probability and average width for skewed distribution. While the exact method performed the best under normal distribution. Two real-world data are analyzed which supported the simulation study to some extent.

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