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Visualizing the Centre of Gravity using the (Empirical) Cumulative Distribution Functions

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Abstract

If a two-dimensional region is endowed with a uniform thickness, and the so constructed three-dimensional space is filled with a material of uniform density, then the centre of gravity (CG) can be found using the marginal (empirical) cumulative distribution functions (E)CDF of all dimensions; the same can be done if *any* three-dimensional region is filled with a material of uniform density. This simple method is justified by the results in Sarkar and Rashid (2016) which visualizes the mean of a single random variable based on its (E)CDF instead of the more traditional visualization of the mean as a fulcrum underneath a dot plot or a probability mass/density function. The method easily extends to several disjoint regions and/or regions having varying densities.

Keywords: Balance; Plumb Line; Projection; Linear Combination; Weighted Average.

AMS Subject Classification: 82M36.

Inspiration

Twin statisticians K and M love to eat thinly sliced tomatoes. For simplicity, we shall assume that their tomatoes are perfectly spherical with North Pole being the point where the tomato was once connected to the vine. The brothers bought an expensive, fancy slicing machine. To make all slices equally thick, they set the distance between the machine's circular blade and its base at 0.635 cm. You won't be surprised to learn that they made this decision after conducting an elaborately designed experiment to determine the optimal thickness that will keep the tomato slices intact and maximize the number of slices, would you?

The slicing machine manufacturer recommends cutting tomatoes in half with a straightedge knife and placing the circular face on the base so that the tomato will remain stable during slicing. The brothers happily complied with the manufacturer's recommendation. Furthermore, they agreed to discard the last slice for it typically has too much skin and not enough juice and it may not be as thick as the other slices. However, the brothers could not agree on the direction of the initial cut: Statistician K always cut the tomatoes along the equatorial plane (that is, along the plane through the centre orthogonal to the diameter through the North Pole), while Statistician M did so with a meridian plane containing that diameter. Despite their unresolved disagreement on this one highly contentious issue, on which the manufacturer was silent, they ate the same number of slices and the same volume of tomatoes (under the assumption the tomatoes are perfect spheres of a constant radius). And as the fables say, "They lived happily ever after."

0. Tribute to Sinha Brothers

The first author was fortunate to have been a student of the younger of the Twin Statisticians at the Indian Statistical Institute in early 1980's. Although not a student of the elder Twin Statistician, attesting to his outgoing generosity, the first author enjoyed a special invitation to celebrate his quarter century production of Ph.D. scholars. Through many turns in life's journey, he has been adopted into the Sinha family as a favorite nephew. During his long association and collaboration with the Sinha Brothers, he has benefitted from several positive impacts — inspiration, enthusiasm, and a zest for problem-solving.

The second author, while serving as a local host, was transformed into an unsuspecting student, and was infected with the Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) method promulgated by the younger Twin Statistician. Since then, the second author has applied the TOPSIS method in several real-world data sets and published three articles. He has also benefitted from his advice and guidance in many respects.

There are many stories of intellectual stimulations we have enjoyed in his presence: in front of the chalk board; over a cup of tea; at dinners; and during casual walks. During a car ride, the first author learned about the twelve-penny problem, which resulted in two papers. His explanation of the correct analysis of Sudoku as an experimental design, prompted the discovery and analysis of orthogonal Sudoku designs, again resulting in two papers. Academic accomplishments aside, the familial care, the constant encouragement, and the inspiration to reach lofty standard of excellence are the most cherished blessings we have received through our association with the Twin Statisticians.

1. Introduction

Inspired by the tomato slicing story, we who love to eat potatoes (*albeit* against the doctors' advice) wanted to make perfectly shaped French Fries. For simplicity, assume that our potatoes, after peeling, are convex solids of uniformly dense material. We cut potatoes into uniform slices of thickness a, and discarded the uneven end slices; then we cut each slice into strips of equal thickness b, and discarded the uneven end strips; finally, we trimmed off the ends of the strips to make them perfect rectangular parallelopipeds. (You may replicate our procedure without fear of a lawsuit for patent violation.)

If all the trimmed off portions together amount to a negligible proportion of the potato, which assumption is reasonable if you let the thicknesses a and b tend to zero, you can find the volume of the potato by adding up the lengths of the strips and then multiplying the sum by ab, the area of the cross-section. The Riemann inner sum interpretation of integration guarantees the aforesaid claim (see [1]).

Whereas volume measures the amount of space an object occupies, we are also interested in the object's centre of gravity (CG) because all objects, irrespective of their shapes, behave and act as though their mass is concentrated at the CG. In uniform gravity, the CG is the same as the centre of mass. Understanding CG is critical to designing an aircraft or playing many sports effectively (see [12]). Any sport that involves balance—from figure skating to surfing—and requires quick adjustments to maintain control without expending too much energy, can benefit from knowing where the CG is at any given moment. This is the reason tennis players plant their feet wide apart, high jumpers curl their bodies up and around the pole, and tight rope walkers carry long sticks or bend their knees.

Calculus informs us that the CG of any 3-D region endowed with an arbitrary probability density function (PDF) $f(x, y, z)$ (the physical density function

$\rho(x,y,z)$ divided by the mass $m=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\rho(x,y,z)\,dz\,dy\,dx)$ is a point whose coordinates $(\bar{x},\bar{y},\bar{z})$ are obtainable by integration. For example,

$$\bar{x}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x\,f(x,y,z)\,dz\,dy\,dx=\int_{-\infty}^{\infty}x\,f_X(x)\,dx$$

where $f_X(x)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y,z)\,dz\,dy$ is the marginal PDF of X obtained from $f(x,y,z)$ by integrating out y and z (see [4]). Likewise, one can obtain $\bar{y}$ and $\bar{z}$ using marginal PDF's of Y and Z respectively. However, until these integrals are evaluated, one cannot locate the CG.

Under the assumption the tomatoes are perfect spheres of a constant density everyone knows the CG of a tomato is at the geometric centre of the sphere. However, as curious observers, the twin statisticians K and M posed the questions: "Where is the CG of each hemisphere after the initial cut with a straightedge knife? When the machine has removed several slices, where is the CG of the remaining cap of the sphere?" Rumour has it that one of them used such questions to determine if he would admit a student to conduct research under his guidance.

In contrast, for our odd-shaped potatoes, the CG is not obtainable from geometric considerations alone, even though each potato strip has its CG at the geometric centre of the parallelopiped (where its three diagonals coincide). How should one aggregate these centres of gravity for the strips to find the overall CG of the potato (or all strips combined)?

In this paper, we visualize the CG of an arbitrarily shaped 3-D region filled with material of uniform density, or of arbitrary density. Interested readers may see, [2], for other visualization techniques.

In Section 2, we recall how to visualize the mean of a single random variable having an arbitrary mass/density function. Section 3 extends the notion to 2-D regions with uniform density. Section 4 finds the CG of 2-D regions with arbitrary density, and of 3-D regions with constant density. Section 5 documents some connections between the CG and the geometric centres of 2-D and 3-D regions of constant density. Section 6 extends the technique to 3-D regions filled with arbitrary density. Interested readers may doublecheck their understanding by solving exercise problems scattered throughout the paper, including the questions already posed by the twin statisticians and by us.

2. Visualizing the Mean of a Single Variable

The location of the mean is the CG of a one-dimensional object whose mass is concentrated at several discrete points (with equal or unequal weights) or over a continuum of points with an associated PDF. Here is a traditional way to visualize the mean: Imagine a rectangular rigid sheet of material of negligible weight (say, it is made of a polystyrene such as Styrofoam) on which a number line is drawn parallel to one side of the rectangle. Next, imagine that on this rectangular sheet are planted heavy metal (say, made of lead) balls of equal weight, or cylindrical rods of equal cross-section and equal density, but lengths proportional to the probability masses, or a lamina of uniform density in the shape of a region below the PDF (and above the x-axis). It is optional — but a good practice — to also plant a mirror image (about the number line) of the heavy metal pieces. Then mount this physical structure on a straight wedge orthogonal to the number line keeping it balanced. The location of the wedge (or fulcrum) in relation to the number line is the mean or the CG of the single variable, see [13], as illustrated in Figure 1.

Sarkar and Rashid [6] gives an alternative method to obtain the mean of a single variable requiring no physical structure, but only a 2-D picture of the (empirical) cumulative distribution function (ECDF) of the variable. The paper was translated into German in [8] by the Editor of *Stochastik in der Schule*; and it was extended by the authors in [7] to also include visualizing the median, mean deviation and standard deviation. Visualizing the mean involves imposing a vertical line on the ECDF and sliding it until the area to the left of the vertical line (below the ECDF and above the x-axis) equals the area to the right of the vertical line (above the ECDF and below the horizontal line $y = 1$), and is illustrated in Figure 2.

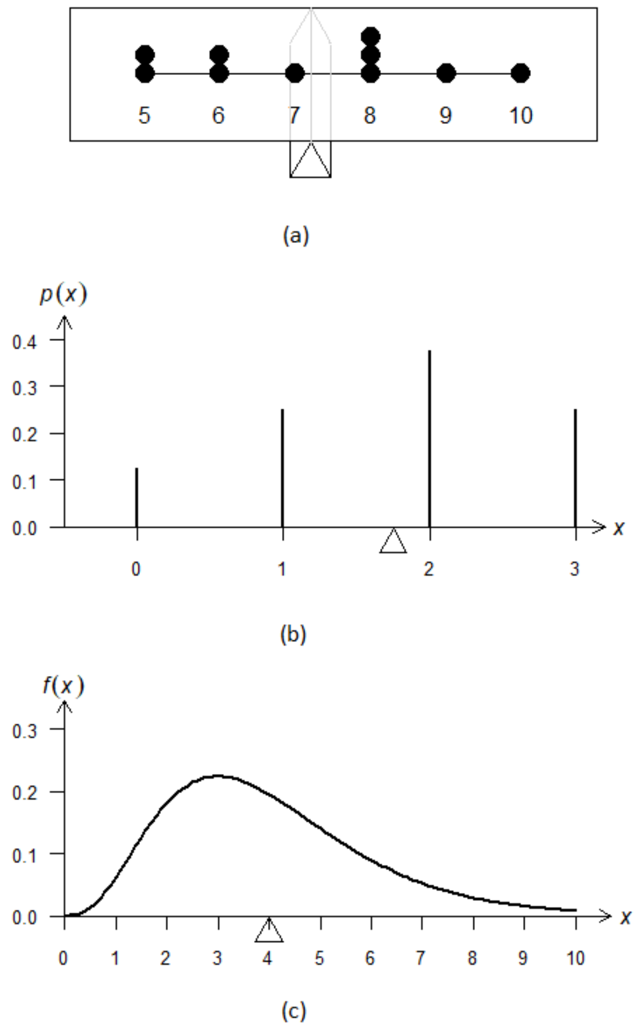


Figure 1: To find the location of the mean balance the physical structures on a straight wedge orthogonal to the number line. This can be done for (a) a dataset, (b) a probability mass function and (c) a probability density function.

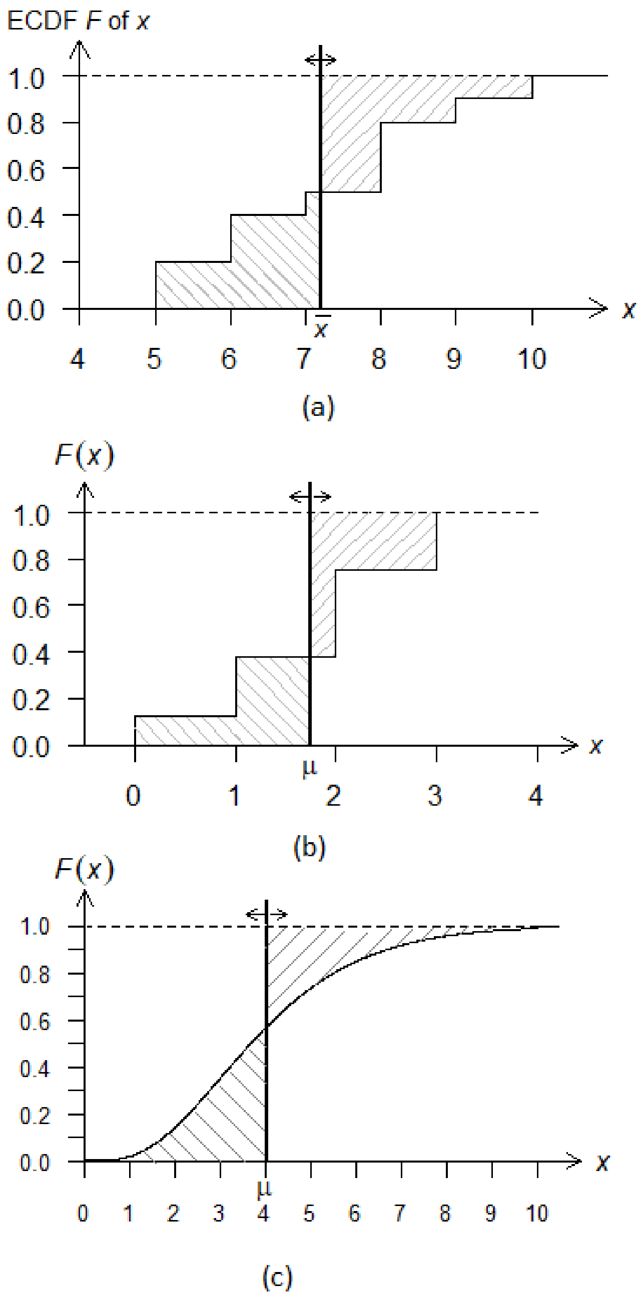


Figure 2: To find the location of the mean slide a vertical line until the left and the right shaded areas become equal. This can be done for (a) a dataset, (b) a probability mass function and (c) a probability density function.

Note that if the vertical scale of the ECDF is magnified or shrunk, the location of the mean (that is, the location of the area-equalizing vertical line) *does not change at all*! Therefore, the vertical axis can be eliminated without any loss. However, if the horizontal scale is linearly transformed, the location of the mean is similarly transformed, since $E[a + bX] = a + b\ E[X]$.

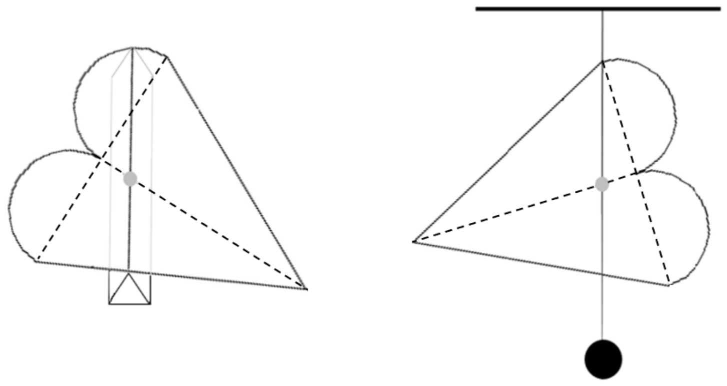
An experimental study (Sarkar and Rashid, [10]) has found that students are better at verifying the location of a vertical line that equalizes areas on its two sides than verifying the location of a fulcrum that balances a dot plot (for data), or a PMF plot (for discrete random variable), or a PDF plot (for a continuous random variable).

3. The CG of a 2-D Region

The centre of gravity of a scatterplot is the mean vector $(\bar{x}, \bar{y})$. See, [3]. It is also the point of intersection of the regression lines of *y* on *x* and *x* on *y*.

If a lamina (of uniform thickness and made of material of uniform physical density) is cut out in the shape of any 2-D region, then its CG can be found by balancing the lamina on a pinpoint. Alternatively, one may balance the lamina on a straight wedge and mark the line along the wedge and repeat the task after rotating the lamina roughly 90 degrees. (There is no need to be exact in measuring the rotation. However, a rotation close to 90 degrees ensures a smaller measurement error.) The point of intersection of the new balancing line along the wedge with the first balancing line is the CG of the lamina. See Figure 3(a). Conducting such balancing tasks may be time consuming and somewhat frustrating; but the concept as a thought experiment is not too difficult to grasp.

An alternative method (see [11]) works reasonably well: Hang the lamina from any one point close to the boundary until it comes to rest next to a plumb line (a thread carrying a heavy ball). Draw the vertical line of (the shadow of) the thread on the lamina. Then do the same task starting from another point. See Figure 3(b). The CG of the lamina is the point where the two drawn lines intersect. Sometimes, in view of symmetry, one of these lines can be anticipated without experimentation.



(a) (b)

Figure 3: The CG of a 2-D lamina using (a) a wedge, and (b) a plumb line. In view of symmetry, balancing a second time is unnecessary.

Without conducting the physical experiments mentioned above, we can find the CG of a 2-D region, by extending the concepts in Sarkar and Rashid [7] as follows: Superimpose the lamina on a set of many (say, 20 or 100) vertical grid lines at regular intervals. Measure the lengths of the intersections of these grid lines with the 2-D region. (These segments of intersection when bottom aligned and rescaled become a proxy PMF.) Calculate the cumulative total lengths going from left to right; and divide them by the grand total length of all segments of intersection. A plot of these relative cumulative lengths against the locations of the parallel grid lines becomes a proxy ECDF. Find the (possibly new) line parallel to the grid lines that equalizes the areas to the left and the right and draw it on the lamina. Next, repeat the process after rotating the grid lines by 90 degrees, and making them horizontal. (Since this is a thought experiment, we can insist on an exact 90 degree rotation; but exactness is *not* necessary. Sometimes a different rotation may be simpler to handle in view of symmetry.) Figure 4 illustrates the method, where we have intentionally eliminated the grid lines to avoid a lot of clutter.

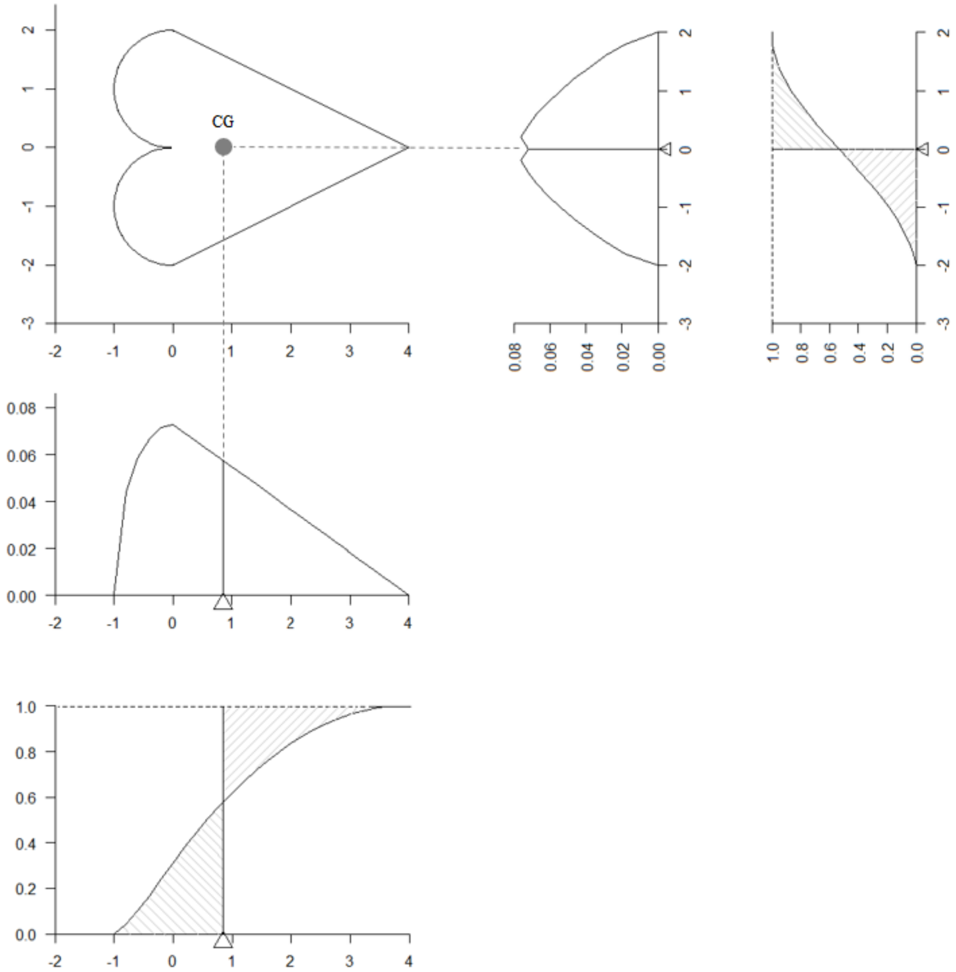


Figure 4: The CG of a 2-D lamina is the point of intersection of the area-bisecting lines corresponding to the ECDF of projections in two (orthogonal) directions.

If one drills a cylindrical hole through the CG of the 2-D lamina of uniform thickness and uniform density, insert a tight-fitting pin through the hole, and mounting the pin horizontally on a pair of grooves spins the lamina, it will come to rest at arbitrary positions. See Figure 5(c, d). If the hole is drilled anywhere else, the lamina will come to rest only when the CG is vertically below the hole. See Figure 5(a, b).

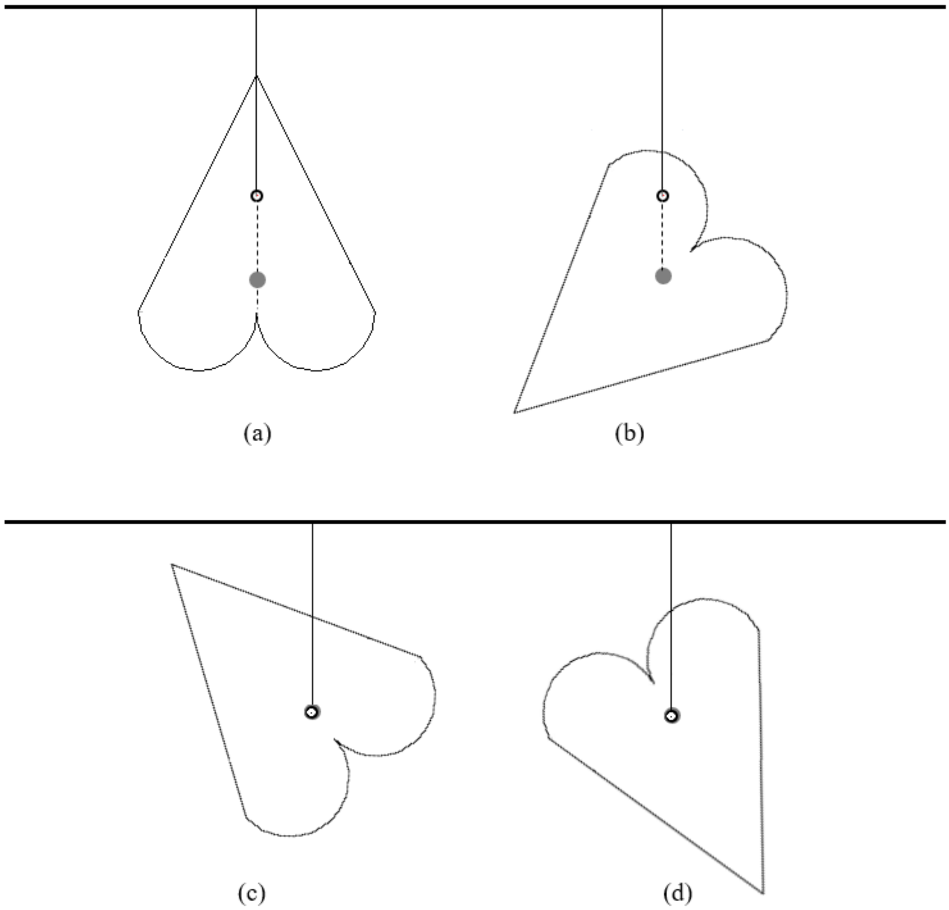


Figure 5: (a), (b) A 2D lamina, held vertically, after rotation stops when the CG is vertically below the centre of rotation. (c), (d) If an object is rotated about its CG, it can stop in *any* orientation, two of which are shown.

What happens if the 2-D lamina is uniformly thick, but is made up of unevenly dense material? We interchange the thickness and the physical density (which task is impossible to accomplish in the physical world, but it is a rather simple matter in our thought experiment). That is, we imagine a lamina whose physical density is the same everywhere, but the thickness at a point (x, y) is proportional to the original density $\rho(x, y)$. This takes us to a special case of a 3-D region filled with uniformly dense material and is solved in the next section.

4. The CG of a 3-D Region

Suppose that we are given a 3-D object made of material of uniform physical density. Conducting a physical experiment to discover the CG is almost impossible. At best, one can hang the 3-D object from a pin, pointing down, and when the object comes to rest, raise another pin vertically below the first pin to touch the object. Then vertical line joining the two pinpoints passes through the CG. This line passes through the object and as such cannot be drawn; only its end points can be marked on the object. Repeating the experiment, one can identify the CG as the point of intersection of the two lines designated by marking their endpoints. See Figure 6.

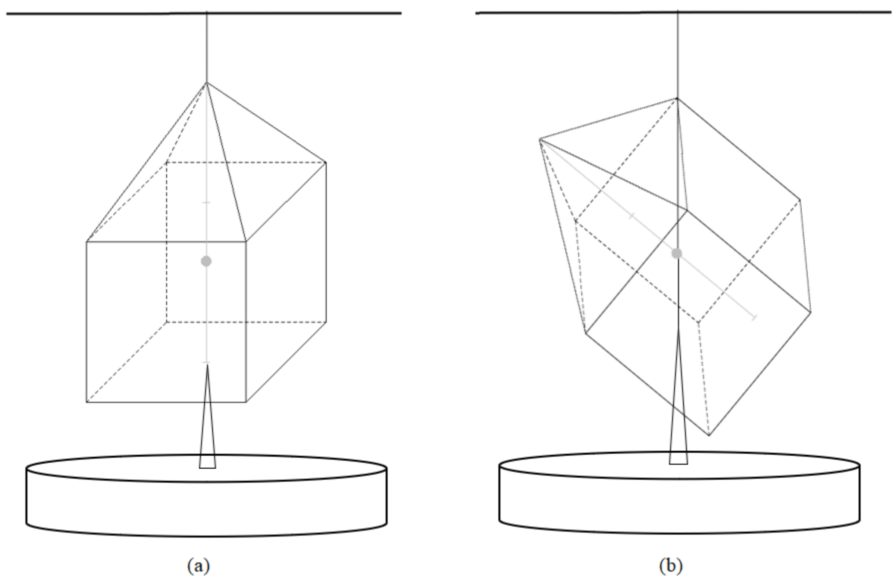


Figure 6: Identifying the CG of a 3-D object by hanging it. One can bypass hanging (a), in view of symmetry.

Extending our thought experiment described in the previous section, now we can superimpose the 3-D object onto a set of vertical lines located on a rectangular grid on the xy-plane, and then bottom align the line segments of intersection so that their lower ends touch the xy-plane. This will serve as a proxy of the marginal joint PDF of (X, Y) when the heights of the vertical line segments are normalized to have a total height of one. From this point on, obtaining the marginal ECDF of X and the mean of X is a straight-forward matter; likewise, obtain the marginal ECDF of Y and the mean of Y.

Next, one can project all vertical line segments of intersection (before they are projected to the xy-plane) to the z-axis and separated by a constant gap. This will endow the z-axis with a (non-uniform) marginal PDF. Convert that marginal PDF into a CDF and find the mean of Z using the area-equalizing line for a single random variable. See Figure 7.

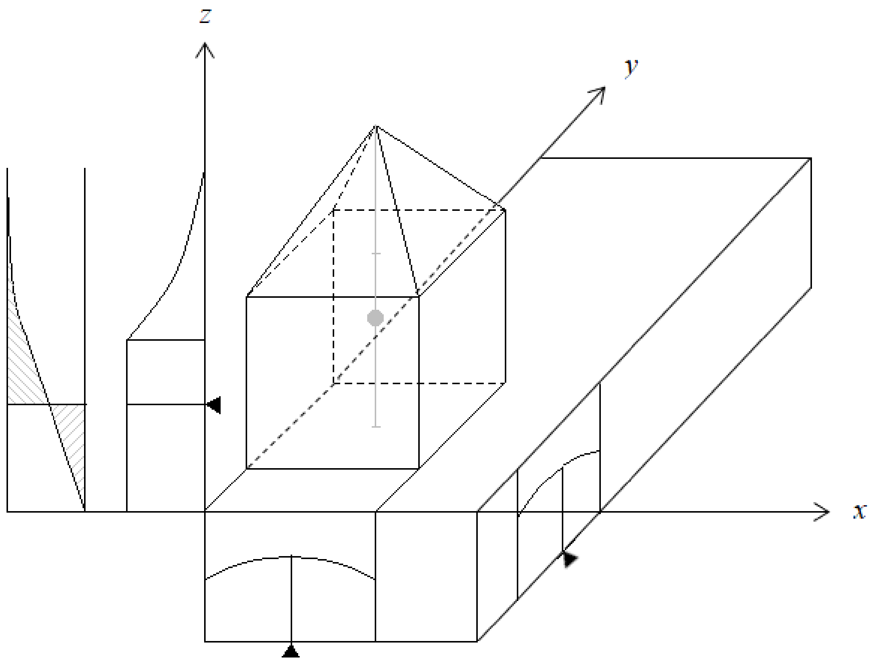


Figure 7: Project the 3-D object onto the xy-plane and then onto the x-axis and the y-axis. Separately, project the same 3-D object onto the z-axis. Then find $\bar{x}, \bar{y}, \bar{z}$ using the univariate ECDF's.

Alternatively, to obtain the mean of Z, one can turn the grid lines sideways making them orthogonal to the yz-plane and then project the line segments of intersection onto the yz-plane. The mean of Y obtained by the two different projections on the xy- and the yz-planes must be the same, providing a check in the visualization. A final confirmation can be designed by turning the grids to become orthogonal to the xz-plane and by obtaining the means of X and Y based on the line segments of intersection with the object. Such built-in redundancy serves to verify accuracy.

5. The CG and Other Geometric Centres of 2-D/3-D Objects

Oftentimes the CG may coincide with some other geometric centres. For example, for a triangular lamina of uniform density, the CG is at the centroid (the point of intersection of the medians) of the triangle. For a parallelogram, the CG is at the point of intersection of the diagonals. What about the CG of an arbitrary quadrilateral? Figures 8(c) and 9(d) show how to find the CG of convex and concave quadrilaterals, respectively.

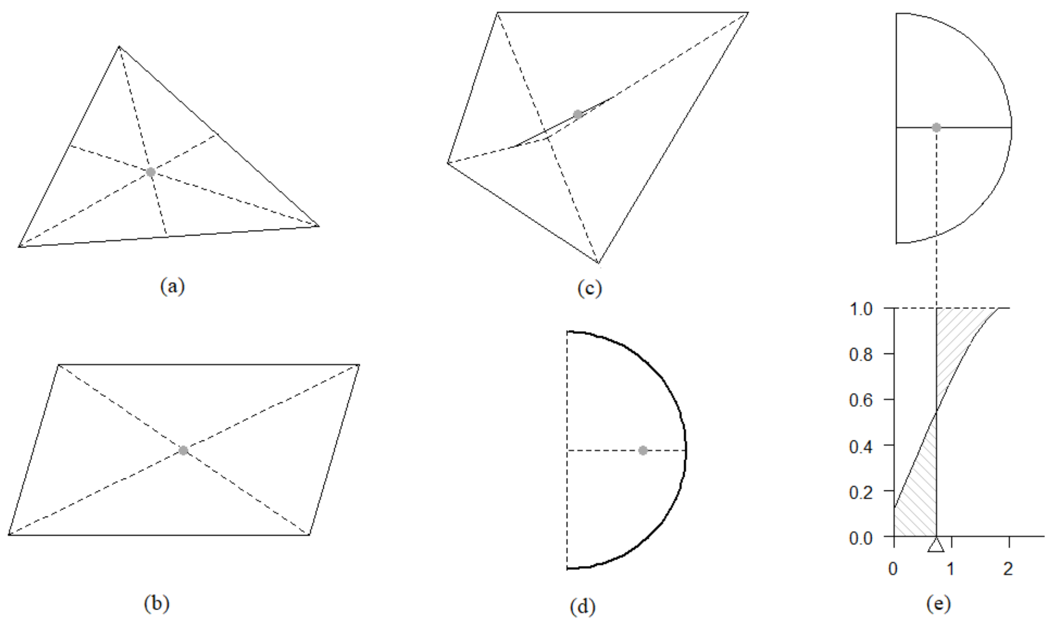


Figure 8: The CG's of (a) a triangle, (b) a parallelogram, (c) a convex quadrilateral, (d) a semi-circle, and (e) a semi-disk.

Figure 8(c) also illustrates how two CG's of two triangles are combined to form a common CG of the convex quadrilateral: Simply find the weighted average of the given two CG's with weights proportional to the thought-of mass concentrated at each CG. The CG of the right semi-disk $\{(x, y): x^2 + y^2 \leq 1, x > 0\}$ shown in

Figure 8(e) is at $\left(\frac{4}{3\pi}, 0\right)$. Consequently, the CG of the right semi-circle (boundary of disk) $\{(x, y): x^2 + y^2 = 1, x > 0\}$ shown in Figure 8(d) is at $\left(\frac{2}{\pi}, 0\right)$.

Thus, when several objects together form a composite object, the CG of the composite object is given by the weighted average of the CG's of the individual objects. To reiterate: When an object is dissected into several parts then the weighted average of the CG's of the parts gives the CG of the original object. What if we remove a portion of an object? How can we find the CG of the resultant remaining object based on the CG's of the original object and the removed object? We illustrate the results in Figure 9 and thereafter explain how to obtain them.

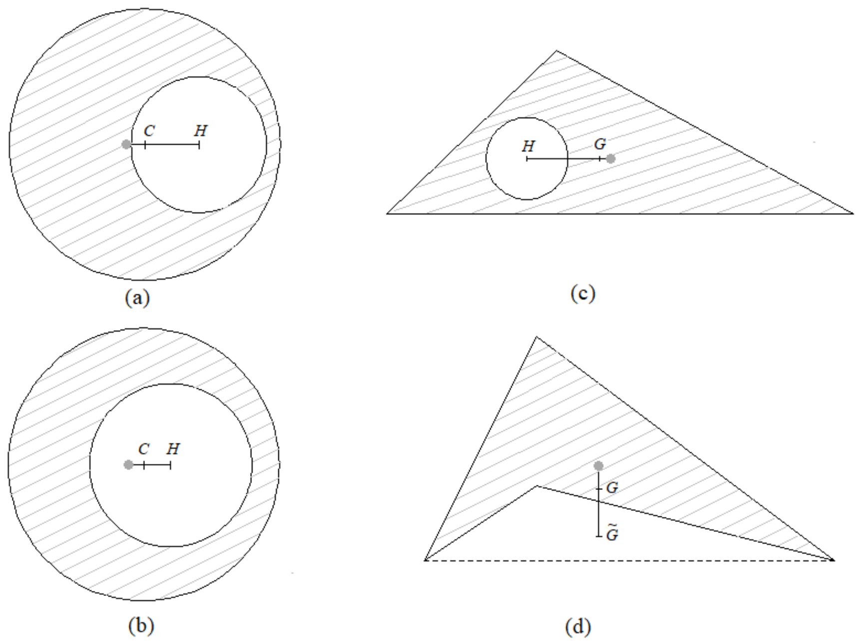


Figure 9: To find the CG of the leftover portion when (a, b) a circular hole is punched out of a bigger circle, (c) a circular hole is punched out of a triangle, and (d) a smaller triangle is removed from a larger triangle with the same base. Assume each object is made up of uniform thickness and uniform density.

To locate CG of the resultant object, in Figure 9(a, b) go from the centre H of the hole of radius r to centre C of the given circle of radius unity and continue another

$r^2/(1 - r^2)$ of the distance already covered. In (c) go from centre H of the hole of radius r to the centroid G of the given triangle and continue another $\frac{\pi r^2}{t - \pi r^2}$ of the distance so covered, where t is the area of the given triangle. In (d) go from centroid $\tilde{G}$ of the removed triangle to the centroid G of the given triangle and continue another $\frac{\tilde{t}}{t - \tilde{t}}$ of the distance so covered, where $\tilde{t}$ is the area of the removed triangle and t is the area of the given triangle.

Note that in Figure 9 we have simply reverse engineered the additive process, giving credence to the truism: "Subtraction is the inverse operation of addition." That is, to find $u - v$, we ask what must be added to v to obtain as sum u.

Using the above principle, a reader may solve the following 2-D problems:

P2.1 Find the CG's of the smaller sector of a unit circle with center C formed by radii CA and CB making an angle $2\theta < \pi$ between them. Answer: The CG is on the mid-radius of the sector at a distance $(2/3)\ sin\ \theta/\theta$ from C.

P2.2 Find the CG's of the smaller part of a circle with center C separated by a chord subtending an angle 2θ at the C. Answer: The CG is in the smaller part of the circle on the radius orthogonal to the chord and at a distance d from C given by $d = (2/3)\ sin\ \theta(1 - \cos\theta)/(\theta - \sin\theta\cos\theta)$.

P2.3 Find the CG of a composite object obtained by attaching external equilateral triangles to the three sides of a triangle measuring 5, 12, 13 cm.

P2.4 Find the CG of a composite object obtained by attaching external semicircles to the three sides of a triangle measuring 20, 48, 52 cm.

P2.5 Find the CG of a semi-disk with density proportional to the distance from the midpoint C of the diameter. Answer: The CG is in the disk on the line orthogonal to the diameter at a distance $3/(2\pi)$ from C.

P2.6 Find the CG of a semi-disk of radius one with density proportional to *one minus* the distance from the midpoint C of the diameter. Answer: The CG is in the disk on the line orthogonal to the diameter at a distance $1/\pi$ from C.

We also leave it to the reader to extend the ideas of this section to 3-D objects of uniform density by solving the following 3-D problems:

P3.1 Find the CG when to the top face of a unit cube a pyramid is augmented such that its base is a unit square matching with the cube's top face, and its slant faces are equilateral unit triangles. Answer: Volume ratio of pyramid to unit cube is $1{:}\,3\sqrt{2}$, and the CG of the pyramid is $\frac{\sqrt{2}}{4}$ above the square base (or the upper square face of the cube). The overall CG is at height $\frac{3.25+\sqrt{2}}{6+\sqrt{2}} = 0.6291$ from the bottom square face of the cube.

P3.2 Find the CG of the remainder solid when a unit octahedron is excavated out of the composite solid in Problem 3.1, or equivalently, a pyramid is excavated such that its base is a unit square matching with the cube's top face, and its slant faces are equilateral unit triangles. Answer: The CG of the remainder solid is at height $\frac{3.25-\sqrt{2}}{6-\sqrt{2}} = 0.4003$ from the bottom square face of the cube.

P3.3 If every point on the boundary of a plane base (of arbitrary shape) is joined to an apex using slant line segments, we obtain a cone. Find the CG of the frustum of such a cone when we remove another cone using a plane cut through the midpoints of slant lines. Answer: The original cone, the removed cone and the frustum all have their CG's on the line joining the CG of the 2-D base to the apex. If h is the height of the cone (the perpendicular distance between the apex and the base), then the CG of the cone is at height $h/4,$ and the CG of the frustum is at height $\frac{11h}{56}$ from the larger base.

P3.4 Find the CG of a hemisphere when the unit sphere is sliced by the xy-plane. Also find the CG of the cap of a sphere when the sphere is sliced off with a place cut at height $a \in [-1, 1]$. Answer: The CG of the hemisphere is at height $\frac{3}{8}$. We find it mildly surprising that whereas the CG of a half-disk shown in Figure 8(d) involves π, the CG of a hemisphere does not! More generally, the CG of the cap of the sphere is at height $\frac{3}{4}\left(a + \frac{1}{2+a}\right)$.

P3.5 Find the CG of a unit ball from which another ball of radius ½ and tangential to the unit ball has been removed. Answer: Rotate the ball so that the point of tangency is at the North Pole. Then the CG is at height $-\frac{1}{14}$.

P3.6 Find the CG of the northern hemisphere (with a spherical hollow) when the ball with a spherical hollow mentioned in Problem 3.5 is cut along the equatorial plane. Answer: The CG is at height $\frac{1}{3}$.

6. The CG of 3-D Objects with Uneven Density

We can extend the techniques of the previous sections to find the CG of a 3-D object with non-uniform density as follows: Slice the object thinly and treat each slice as a 2-D region endowed with uniform thickness, but uneven density. Next, think of each slice converted into a 3-D region of uneven thickness proportional to the original density but filled with material of uniform density, and find the CG of each slice as done previously. Finally, compute the weighted average of the CG's of all slices to find the overall CG of the 3-D object. Below we give an example.

Suppose that we have a unit cube $[0,1]^3$ filled with material having density at (x, y, z) proportional to $x + y + z$, with proportionality constant 2/3. The contours of equal density are shown in Figure 10 (a). These are planes orthogonal to the diagonal D joining $(0, 0, 0)$ to $(1, 1, 1)$. Where is the CG of this loaded cube?

The density being symmetric in (x, y, z), the CG is on D; hence, it is of the form $\bar{x} = \bar{y} = \bar{z}$. We shall show that $\bar{x} = 5/9$.

Let us obtain the marginal PDF of X. First, we take a thin slice through x parallel to the yz-plane. See Panel (a). This slice has uniform thickness, as small as you can imagine, but nonuniform density proportional to $x + y + z$. Next, in Panel (b), we replace the thin slice by a solid of uniform density on base $y, z \in [0,1] \times [0,1]$ shown on the left side and height $x + y + z$ shown horizontally. The volume of this new solid equals that of a rectangular parallelopiped $x \times 1 \times 1$ *plus* essentially a unit cube, or $x + 1$. Hence, $f_X(x)$ is proportional to $x + 1$, which we depict in Panel (c) by mounting a right isosceles triangle with legs 1 attached on top of a unit square. The CG of the triangle satisfies $x = 2/3$ with mass ½, while the CG of the square satisfies $x = 1/2$ with mass 1. Hence, the combined CG, obtained by taking a weighted average, satisfies

$$\bar{x} = \frac{1\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{2}{3}\right)}{1 + \frac{1}{2}} = \frac{5}{9}$$

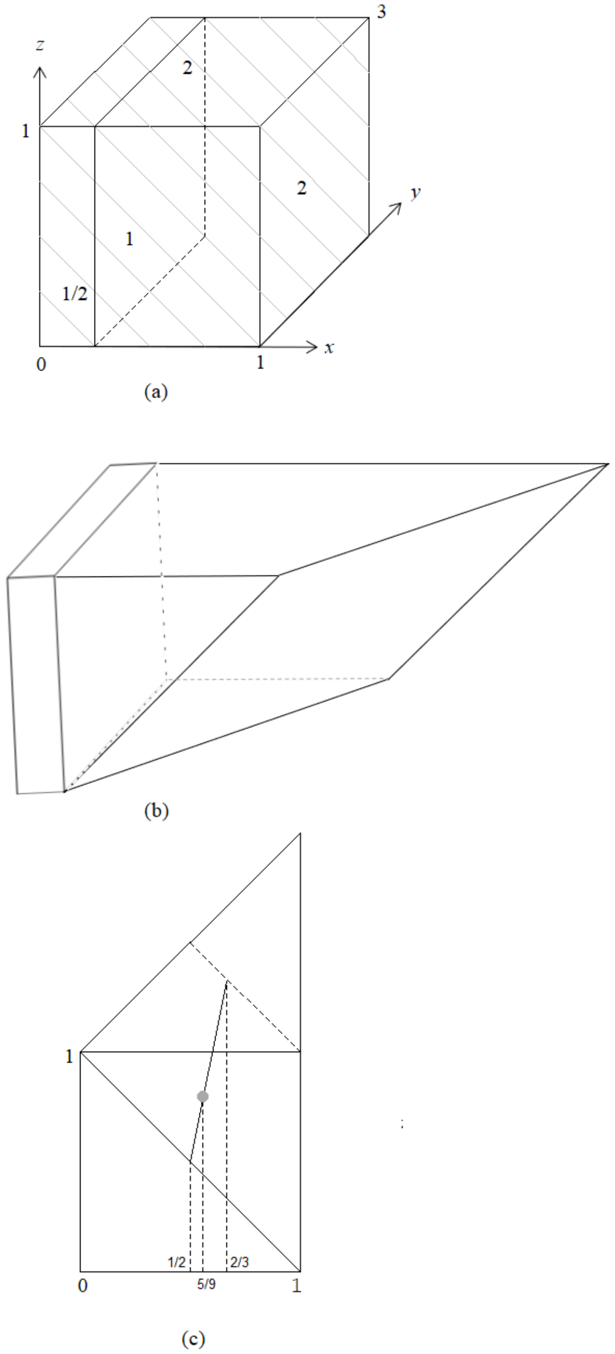


Figure 10: Finding the CG of a unit cube with density proportional to $x + y + z$

We leave it to the reader to show that a cube $[0,1]^3$ with density at (x, y, z) proportional to xyz has CG at $(2/3, 2/3, 2/3)$. Also, for a northern hemisphere of radius unity with density proportional to the distance from the centre $C = (0, 0, 0)$ of the base circle has a CG at $(0, 0, 2/5)$. Likewise, for a northern hemisphere of radius unity with density proportional to *one minus* the distance from the centre $C = (0, 0, 0)$ of the base circle has a CG at $(0, 0, 7/20)$.

Afterthoughts

We count it a privilege and a blessing to have been students and collaborators of one of the twin statisticians. There are many stories of intellectual stimulations we have enjoyed in his presence: in front of the chalk board; over a cup of tea; at dinners; and during casual walks. During a car ride, one of us learned about the twelve-penny problem, which resulted in papers [5] and [9]. For full disclosure, we admit the tomato slicing episode narrated in the introduction is entirely fictitious, though not unrealistic. Stories like this surround legendary figures, which our honourable twin statisticians surely are.

We sincerely hope any potential student wishing to write a PhD dissertation under either member of the twin statisticians will have read this paper and be prepared to answer any question involving CG. Of course, the twin statisticians have in their arsenal a multitude of other crazy problems to unnerve anyone. But their love for the profession and care for the deserving student can overcome any weakness in them and adopt them as beloved disciples. We speak from experience.

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Appendix

```
# Figure 1(a)
dev.new(width=5,height=3)
par(mai=c(1,1,1,1.2))
x=c(8,5,6,10,5,6,7,8,8,9)
n=length(x)
x1=min(x)
x2=max(x)
stripchart(x,method="stack", pch = 19, cex=2, xlim=c(x1,x2),
las=1,
xaxs="i",yaxs="i",frame.plot=FALSE,ylab="",xlab="",
xaxt="n",xpd=TRUE,
col="black")
rect(4,0,11,2, xpd=TRUE)
axis(1, pos=1, at=c(seq(0,10,1)),padj=0,cex=.8,tck=0)
y<-mean(x);y
points(y,-.38, pch=2, cex=3,xpd=T,col="black")
points(y,1.63, pch=2, cex=3,xpd=T,col="lightgray")
segments(y,-.05,y,1.95,col="lightgray",xpd=T)
segments(y+.27,-.6,y+.27,1.4,col="lightgray",xpd=T)
segments(y+.27,-.6,y+.27,.03,col="black",xpd=T)   # black
segments(y-.27,-.5,y-.27,1.4,col="lightgray",xpd=T)
segments(y-.27,-.5,y-.27,.03,col="black",xpd=T) # black
# Figure 1(b)
dev.new(width=5,height=3)
par(mfrow=c(1,1),mai = c(1,.60,.60,.60
ex1=c(3, 4, 7, 8, 9)
x<-c(0,1,2,3)
p_x<-c(1/8,1/4,3/8,1/4)
sum(p_x)
plot(x,p_x,type="h",xlim=c(-
.5,3),ylim=c(0,.4),lwd=2,xaxs="i",yaxs="i",
frame.plot=FALSE,xlab="",ylab="",yaxt="n",axes=FALSE,xpd=T)
axis(1, at=seq(0,3,by=1),cex.axis=.8,las=1)
axis(2, at=seq(0,.5,by=.1),cex.axis=.8,las=1)
arrows(-.5,0,3.2,0,code=2, xpd = TRUE, length=.10)
text(3.3,0,xpd=TRUE,expression(italic(x)))
arrows(-.5,0,-.5,.45,code=2, xpd = TRUE, length=.10)
text(-.5,.5,xpd=TRUE,expression(italic(p(x))))
mean_x=sum(x*p_x)
points(mean_x,-.038, pch=2, cex=2,xpd=TRUE)
# Figure 1(c)
dev.new(width=5,height=3)
par(mfrow=c(1,1),mai = c(1,.60,.60,.60))
```

```
x<-seq(0,10,.1)
f_x<-dgamma(x,shape=4)
plot(x,f_x,type="l",ylim=c(0,.3),las=1,xaxs="i",yaxs="i",fra
me.plot=FALSE,xlab="",
ylab="",axes=FALSE,lwd=2)
axis(1, at=seq(0, 10, by=1),cex.axis=.8,las=1)
axis(2, at=seq(0, .3, by=.1),cex.axis=.8,las=1)
arrows(0,0,10.7,0,code=2, xpd = TRUE, length=.10)
text(11,0,xpd=TRUE,expression(italic(x)))
arrows(0,0,0,.345,code=2, xpd = TRUE, length=.10)
text(0,.37,xpd=TRUE,expression(italic(f(x))))
mean_x=4
points(mean_x,-.028, pch=2, cex=2,xpd=TRUE)

# Figure 2(a)
dev.new(width=5,height=3)
par(mfrow=c(1,1),mai = c(1,.80,.60,1.5))
x=c(8,5,6,10,5,6,7,8,8,9)
n=length(x);n;summary(x)
x1=min(x);x2=max(x)
x<-sort(x); n=length(x)
f<-rep(1/n,n);F<-cumsum(f)
plot(x,F,xlim=c(min(x)-
1,max(x)),ylim=c(0,1),type="s",las=1,xaxs="i",yaxs="i",
frame.plot = FALSE,ylab="", xlab="",xaxt="n")
axis(1,at=c(seq(x1-1,x2,1)))  # CHNAGE VALUE
arrows(min(x)-1,0,  max(x)+1,0,  code  =  2,  xpd  =  TRUE,
length=.12)
arrows(min(x)-1,0,  min(x)-1,1.15,  code  =  2,  xpd  =  TRUE,
length=.12)
segments(min(x)-1,1,max(x),1,lty=2)
segments(max(x),1,max(x)+.3,1,lty=1,xpd=TRUE)
segments(min(x),0,min(x),F[1],lty=1)
m=mean(x)
for(i in 1:length(x)){
  if(x[i]<m){

rect(x[i],0,m,F[i],col="grey",angle=135,density=15,border=NA
)
  }
  else if(x[i]>m){
    rect(m,F[i-
1],x[i],F[i],col="grey",angle=45,density=15,border=NA)
  }
```

```
}
abline(v=m)
par(new=TRUE)
plot(x,F,xlim=c(min(x)-
1,max(x)),ylim=c(0,1),type="s",las=1,xaxs="i",yaxs="i",
frame.plot = FALSE,ylab="", xlab="",xaxt="n",yaxt="n")
segments(min(x)-1,1,max(x),1,lty=2)
segments(max(x),1,max(x)+1,1,lty=1,xpd=TRUE)
segments(min(x),0,min(x),F[1],lty=1)
segments(mean(x),0,mean(x),1.13,xpd=TRUE,lwd=2)
arrows(mean(x)+.04, 1.07, mean(x)+.2, 1.07, code = 2, xpd =
TRUE, length=.065)
arrows(mean(x)-.04, 1.07, mean(x)-.2, 1.07, code = 2, xpd =
TRUE, length=.065)
segments(x1,0,x2,0)
text(x1-1,1.25,
expression(ECDF~italic(F)~of~italic(x)),xpd=TRUE, cex=.9)
text(x2+1.3,0, expression(italic(x)),xpd=TRUE, cex=1)
text(mean(x),-.08,      expression(bar(italic(x))),xpd=TRUE,
cex=.9)
#Figure 2(b)
dev.new(width=5,height=3)
par(mfrow=c(1,1),mai = c(1,.80,.60,1.5))
x<-c(0,1,2,3)
p_x<-c(1/8,1/4,3/8,1/4)
F<-cumsum(p_x)
plot(x,F_x,xlim=c(-
.5,4),ylim=c(0,1),type="s",las=1,xaxs="i",
yaxs="i",frame.plot=FALSE,xlab="",ylab="",cex=.8)
abline(h=1,lty=2);segments(4,1,5,1)
m<-sum(x*p_x)
abline(v=m)
segments(0,0,0,F[1])
for(i in 1:length(x)){
  if(x[i]<m){

rect(x[i],0,m,F[i],col="grey",angle=135,density=15,border=NA
)
  }
  else if(x[i]>m){
    rect(m,F[i-
1],x[i],F[i],col="grey",angle=45,density=15,border=NA)
  }
}
```

```
abline(v=m)
par(new=TRUE)
plot(x,F_x,xlim=c(-
.5,4),ylim=c(0,1),type="s",las=1,xaxs="i",
yaxs="i",frame.plot=FALSE,xlab="",ylab="",cex=.8,xaxt="n",ya
xt="n")
segments(m,0,m,1.13,xpd=TRUE,lwd=2)
arrows(m+.04,  1.07,  m+.2,  1.07,  code  =  2,  xpd  =  TRUE,
length=.065)
arrows(m-.04,  1.07,  m-.2,  1.07,  code  =  2,  xpd  =  TRUE,
length=.065)
arrows(-.5,1,-.5,1.13, code=2,xpd=TRUE,length=.10)
text(-
.5,1.23,expression(CDF~italic(F)~of~italic(x)),cex=.9,xpd=TR
UE)
arrows(-.5,0,4.4,0, code=2,xpd=TRUE,length=.10)
text(4.6,0, expression(italic(x)),xpd=TRUE,cex=.9)
text(m,-.1, expression(italic(mu)),xpd=TRUE, cex=1)
# Figure 2(c)
dev.new(width=5,height=3)
par(mfrow=c(1,1),mai = c(1,.80,.60,1.5)) #bottom, left, top,
right

x<-seq(0,10,.01)
F_x<-pgamma(x, shape = 4)
plot(x,F_x,type="l",las=1,xaxs="i",yaxs="i",frame.plot=FALSE
,xlab="",ylab="",axes=FALSE)
axis(1, at=seq(0, 10, by=1),cex.axis=.8,las=1)
axis(2, at=seq(0, 1, by=.1),cex.axis=.8,las=1)
segments(0,1,10,1,lty=2,xpd=T)
segments(10,1,10.5,1,xpd=TRUE)
arrows(0,0,11,0,code=2, xpd = TRUE, length=.10)
text(11.5,0,xpd=TRUE,expression(italic(x)))
arrows(0,0,0,1.15,code=2, xpd = TRUE, length=.10)
text(0,1.25,xpd=TRUE,expression(italic(F(x))))
mean_x=4
abline(v=mean_x)
text(mean_x,-.12, expression(italic(mu)),xpd=TRUE,cex=1)
```