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Minimax Estimation of the Scale Parameters of the Laplace Double Exponential Distribution

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Abstract

In this paper, minimax estimators of the scale parameter of the Laplace double exponential distribution have been developed under modified linear exponential (MLINEX) loss function applying the theorem of Lehmann (1950). The efficiency of the estimators are also been studied.

Keywords: Minimax estimator, Laplace double exponential distribution, Modified linear exponential loss function.

AMS Classification: 62H10.

1. Introduction

In statistical inference, the minimax estimation is an upgraded non classical approach which was introduced by Wald (1950) exploiting the concept of the theory of game. It opens a new dimension in the theory of estimation and enriched the method of point estimation. Roy (1982, 1983) studied minimax estimation of the mean of normal distribution for convex and 0-1 type of loss functions and variance of rectangular distribution for quadratic loss function. Roy et al (2002) and Podder et al (2004) also studied minimax estimation of the parameter of Weibull distribution and Pareto distribution for quadratic and MLINEX loss functions. The most important elements in the minimax approach are the specification of the prior distribution and the use of the loss functions.

Laplace double exponential distribution is a very popular continuous distribution. It is a special case of Gamma distribution which is widely used and has considerable importance in statistical procedures.

The probability density function of the Laplace double exponential distribution is given by

$$f(x \mid \lambda,\theta) = \frac{1}{2\theta} e^{-\frac{|x-\lambda|}{\theta}}; \quad -\infty < x < \infty,\ \lambda > 0,\ -\infty < \theta < \infty \tag{1.1}$$

$$= 0 \qquad ; \quad \text{otherwise.}$$

where θ and λ are the scale and location parameters respectively of the distribution. Practically location parameter has limited use. Here, only the scale parameter is considered to estimate.

In this paper we have derived, the minimax estimator of the scale parameter θ of the Laplace double exponential distribution by using MLINEX loss function. The derivation depends primarily on the theorem due to Lehmann (1950) stated as follows:

Lehmann's Theorem **:** Let $\{F_\theta ; \theta \in \Theta\}$ be a family of distribution functions and D a class of estimators of θ Suppose that $d^* \in D$ is a Bayes estimator against a prior distribution $\xi^*(\theta)$ on the parameter space Θ and risk function $R(d^*, \theta)$ =constant of Θ; then d^* is a minimax estimator of θ.

2. Preliminaries

Modified Linear Exponential (MLINEX) loss function

MLINEX loss function was proposed by Wahed and BorhanUddin (1998) which is asymmetric and convex loss function for estimating θ by $\hat{\theta}$ and is given by

$$L(\hat{\theta},\theta) = W\left[\left(\frac{\hat{\theta}}{\theta}\right)^c - c\ln\left(\frac{\hat{\theta}}{\theta}\right) - 1\right] \quad ; c \neq 0,\ W > 0 \tag{2.1}$$

where W and c are two known parameters of loss function.

Bayes Estimator

The Bayes estimator of θ under MLINEX loss function provided by Wahed and Borhan Uddin (1998) is given by

$$\hat{\theta} = \left[E_{\theta}(\theta^{-c})\right]^{-\frac{1}{c}} \tag{2.2}$$

Risk function

Risk function is the expected value of loss function with respect to the given sample observations. Let $L(\hat{\theta},\theta)$ be the loss function for estimating θ by $\hat{\theta}$, then the risk function denoted by $R(\hat{\theta},\theta)$ is defined as

$$R(\hat{\theta},\theta) = E\left[L(\hat{\theta},\theta)\right] \tag{2.3}$$

3. Main Results

Bayes Estimator of θ

Let $X = (X_1, X_2, \cdots, X_n)$ be a random sample of size *n* drawn from the density (1.1), then the joint density of θ for given simple random sample is

$$\phi(\theta \mid X) = \prod_{i=1}^{n} f(x_i \mid \lambda, \theta)$$

$$= \left(\frac{1}{2\theta}\right)^{n} e^{-\frac{\sum |x_i - \lambda|}{\theta}} \tag{3.1}$$

Let us assume that θ has Jeffre's non-informative prior density defined as

$$g(\theta) \propto \frac{1}{\theta} \quad ; \quad \theta > 0 \tag{3.2}$$

Therefore the posterior density of θ for given random sample X obtained by combining (3.1) and (3.2) is

$$f(\theta \mid X) = \frac{\prod_{i=1}^{n} f(x_i \mid \lambda, \theta) g(\theta)}{\int_{\theta} f(x_i \mid \lambda, \theta) g(\theta) d\theta}$$

$$= \frac{\frac{1}{\theta^{n+1}} e^{-\frac{\sum_{i=1}^{n}|x_i-\lambda|}{\theta}}}{\int_0^{\infty} \frac{1}{\theta^{n+1}} e^{-\frac{\sum_{i=1}^{n}|x_i-\lambda|}{\theta}}}$$

$$= \frac{\left(\sum_{i=1}^{n}|x_i-\lambda|\right)^n e^{-\frac{\sum_{i=1}^{n}|x_i-\lambda|}{\theta}}}{\Gamma n \; \theta^{n+1}} \quad ; x, \theta, \lambda > 0$$

which implies that θ is distributed as inverse Gamma distribution with parameters n and $\sum_{i=1}^{n}|x_i - \lambda|$ i.e., $\theta \approx G(n, \sum_{i=1}^{n}|x_i - \lambda|)$.

Hence the Bayes estimator of θ under MLINEX loss function (2.1) is obtainable as

$$\hat{\theta}_M = \left[E_\theta(\theta^{-c})\right]^{-\frac{1}{c}}$$

Where $E_\theta(\theta^{-c}) = \int_{\theta} \theta^{-c} f(\theta \mid X) d\theta$

$$= \frac{\left(\sum_{i=1}^{n}|x_i-\lambda|\right)^n}{\Gamma n} \int_0^{\infty} \frac{1}{\theta^{n+c-1}} e^{-\frac{\sum_{i=1}^{n}|x_i-\lambda|}{\theta}} d\theta$$

$$= \frac{\left(\sum_{i=1}^{n}|x_i-\lambda|\right)^n}{\Gamma n} \frac{\Gamma(n+c)}{\left(\sum |x_i-\lambda|\right)^{n+c}}$$

$$= \frac{\Gamma(n+c)}{\Gamma n \left(\sum |x_i-\lambda|\right)^{c}}$$

Therefore

$$\hat{\theta}_M = \left(\frac{\Gamma n}{\Gamma(n+c)}\right)^{\frac{1}{c}} \sum_{i=1}^{n} | x_i - \lambda | = \text{KT}$$

where T = $\sum_{i=1}^{n} | x_i - \lambda |$ is a complete sufficient statistics for θ and K= $\left(\frac{\Gamma n}{\Gamma(n+c)}\right)^{\frac{1}{c}}$.

Risk function

The risk function under the MLINEX loss function (2.1) is given by

$$R_M(\hat{\theta}) = E\left\lfloor L(\hat{\theta},\theta)\right\rfloor$$

$$= WE\left[\left(\frac{\hat{\theta}_M}{\theta}\right)^C - c\ln\left(\frac{\hat{\theta}_M}{\theta}\right) - 1\right]$$

$$= W\left[\frac{1}{\theta^c} E(\hat{\theta}_M^c) - cE(\ln \hat{\theta}_M) + c\ln\theta - 1\right] \tag{3.3}$$

here $E(\hat{\theta}_M^c) = E(KT)^c = K^c E(T^c)$.

Since X is a Laplace double exponential variable so T = $\sum_{i=1}^{n} | x_i - \lambda |$ is a gamma variate with parameters *n* and *θ.*

Therefore

$$h(t) = \frac{1}{\theta^n \Gamma n} e^{-\frac{t}{\theta}} t^{n-1}$$

Here $E(T) = \frac{1}{\theta^n \Gamma n} \int_0^\infty e^{-\frac{t}{\theta}} t^{n+c-1} dt = \frac{\Gamma(n+c)}{\Gamma n} \theta^c$

Hence $E(\hat{\theta}_M^c) = K^c \dfrac{\Gamma(n+c)}{\Gamma \text{n}} \theta^c$

Therefore $E(\hat{\theta}_M^c) = \theta^c$ (3.4)

Since K=$\left(\dfrac{\Gamma \text{n}}{\Gamma(n+c)}\right)^{\frac{1}{c}}$

Again $E(\ln \hat{\theta}_M) = E[\ln(KT)] = \ln K + E(\ln T)$ (3.5)

$$E(\ln T) = \frac{1}{\theta^n \Gamma n} \int_0^\infty \ln t e^{-\frac{t}{\theta}} t^{n-1} dt; \text{ t>0 },\theta>0 \qquad \text{Let } \frac{t}{\theta} = y$$

$dt = \theta\, dy$; (limit remain unchanged)

hence we obtain

$$E(\ln T) = \frac{1}{\theta^n \Gamma n} \int_0^\infty \ln(\theta y) e^{-y} (\theta y)^{n-1} \theta\, dy$$

$$= \frac{\ln \theta}{\Gamma n} \int_0^\infty e^{-y} y^{n-1} dy + \frac{1}{\Gamma n} \int_0^\infty \ln y e^{-y} y^{n-1} dy \qquad = \ln \theta + \frac{\Gamma n'}{\Gamma n}$$

where $\int_0^\infty \ln y e^{-y} y^{n-1} dy$ is the first derivative of Γn with respect to *y*.

Using the results in (3.5) we obtain

$$E(\ln \hat{\theta}_M) = \ln K + \ln \theta + \frac{\Gamma n'}{\Gamma n}$$

Using the above results, the risk function becomes

$$R_M(\hat{\theta}) = W\left[\frac{\theta^c}{\theta^c} - c \ln K - c \ln \theta - c \frac{\Gamma n'}{\Gamma n} + c \ln \theta - 1\right]$$

$$= W\left[\ln K^{-c} - c\frac{\Gamma n'}{\Gamma n}\right] \qquad = W\left[\ln \frac{\Gamma(n+c)}{\Gamma n} - c\frac{\Gamma n'}{\Gamma n}\right]$$

which is constant with respect to θ as n and c are known and independent of θ.So from the Lehmann's theorem stated in section 1, it follows that

$\hat{\theta}_M = \left(\frac{\Gamma n}{\Gamma(n+c)}\right)^{\frac{1}{c}} \sum_{i=1}^{n} | x_i - \lambda |$ is the minimax estimator of the scale parameter θ of the Laplace double exponential distribution under the MLINEX loss function.

4. Efficiency of the Estimator

In this section, we are interested to find the efficiency of minimax estimator $\hat{\theta}_M$ with respect to the classical maximum likelihood estimator (MLE).

The log of the joint density of θ for the given sample $X = (x_1, x_2, \cdots x_n)$ is obtained from (3.1) as

$$\log \phi(\theta \mid x) = \log\left(\frac{1}{2}\right)^n - n\log\theta - \frac{\sum_{i=1}^{n} | x_i - \lambda |}{\theta}$$

MLE of θ can be obtained from the solution of the equation

$$\frac{\partial \log \phi(\theta \mid x)}{\partial \theta} = 0$$

which gives MLE of θ given by

$$\hat{\theta}_{ML} = \frac{\sum_{i=1}^{n} | x_i - \lambda |}{n}$$

and the variance of $\hat{\theta}_{ML}$ is obtained from

$$V(\hat{\theta}_{ML}) = \frac{1}{-E\left(\frac{\partial^2 \log \phi(\theta \mid x)}{\partial \theta^2}\right)}$$

Again $T=\sum_{i=1}^{n}|x_i-\lambda|$ is gamma variate with parameter *n* and *θ,* we can easily obtained

$$E(T)=n\theta$$

$$E(T^2)=n(n+1)\,\theta^2$$

and $V(T)=E(T^2)-\{E(T)\}^2=n\theta^2$

Therefore the variance of the minimax estimator $\hat{\theta}_M$ is given by

$V(\hat{\theta}_M)=V(KT)=K^2V(T)=K^2n\theta^2$ Hence the efficiency of $\hat{\theta}_M$ with respect to $\hat{\theta}_{ML}$ is given by

$$E=\frac{V(\hat{\theta}_{ML})}{V(\hat{\theta}_M)}=\frac{\theta^2/n}{K^2n\theta^2}=\frac{1}{n^2K^2}=\frac{1}{n^2\left(\frac{\Gamma n}{\Gamma(n+c)}\right)^{2/c}}$$

Note that

If c=1, E=1 implies that method of minimax estimation under MLINEX loss function and the classical MLE procedure are equally efficient.

If $c\geq 2$, *E*>1 indicates that method of minimax estimation under MLINEX loss function is always more efficient than the classical MLE procedure.

We have also compared the efficiency with the help of simulation study for the different values of the parameters (*θ* and *λ*) and sample sizes and found similar results. The results obtained in the simulation study is mentioned in the supplementary file.

Appendix

In this regards the supplementary file is provided for the simulation study.

Supplementary file

In the following simulation study we obtained that if c=1 then E=1 implies that method of minimax estimation under MLINEX loss function and the classical MLE procedure are equally efficient.

if c $\geq$ 2, *E*>1 indicates that method of minimax estimation under MLINEX loss function is always more efficient than the classical MLE procedure.

Table 1: Efficiency of the scale parameter for n=100, θ=1, λ=2 and different values of c

θ=1; λ=2; n=100	Variance of the classical estimator	Variance of the Bayes estimator	Efficiency (E)
c=1	0.03991225	0.03991225	1.00000000
c=2	0.03997131	0.03918372	1.02010000
c=3	0.03999236	0.03843689	1.04046800
c=4	0.03987563	0.03757930	1.06110600
c=5	0.03978658	0.03677079	1.08201601
c=6	0.03961665	0.03591067	1.10320005
c=7	0.03992280	0.03549765	1.12466013
c=8	0.03992460	0.03482612	1.14639828
c=9	0.03992525	0.03417039	1.16841653
c=10	0.03999245	0.03358687	1.19071691
c=11	0.03985876	0.03285149	1.21330147
c=12	0.03996161	0.03232690	1.23617225

Table 2: Efficiency of the scale parameter for n=120, θ=1, λ=2 and different values of c

θ=1; λ=2; n=120	Variance of the classical estimator	Variance of the Bayes estimator	Efficiency (E)
c=1	0.03332971	0.03332971	1.00000000
c=2	0.03331101	0.03276269	1.01673611
c=3	0.03325574	0.03217286	1.03365818
c=4	0.03320437	0.03160011	1.05076736
c=5	0.03324178	0.03112337	1.06806482
c=6	0.03315418	0.03054132	1.08555172
c=7	0.03332403	0.03020590	1.10322923
c=8	0.03325387	0.02966186	1.12109851
c=9	0.03332120	0.02925065	1.13916075
c=10	0.03327957	0.02875331	1.15741711
c=11	0.03326112	0.02828642	1.17586876
c=12	0.03326219	0.02784573	1.19451690

Table 3: Efficiency of the scale parameter for n=120, θ=-3, λ=3 and different values of c

$\boldsymbol{\theta}$= -3; $\boldsymbol{\lambda}$= 3; n=120	Variance of the classical estimator	Variance of the Bayes estimator	Efficiency (E)
c=1	1.981864e+51	1.981864e+51	1.000000e+00
c=2	3.200265e+49	3.147587e+49	1.016736e+00
c=3	1.104908e+51	1.068930e+51	1.033658e+00
c=4	3.120147e+51	2.969399e+51	1.050767e+00
c=5	2.258619e+51	2.114683e+51	1.068065e+00
c=6	1.530015e+51	1.409435e+51	1.085552e+00
c=7	2.827962e+51	2.563350e+51	1.103229e+00
c=8	8.595773e+50	7.667277e+50	1.121099e+00
c=9	1.189067e+51	1.043810e+51	1.139161e+00
c=10	1.837504e+51	1.587590e+51	1.157417e+00
c=11	2.594455e+51	2.206416e+51	1.175869e+00
c=12	3.423745e+50	2.866218e+50	1.194517e+00

Table 4: Efficiency of the scale parameter for n=120, $\boldsymbol{\theta}$=-5, $\boldsymbol{\lambda}$=15 and different values of c

c=1	9.884562e+29	9.884562e+29	1.000000e+00
c=2	9.247772e+29	9.095548e+29	1.016736e+00
c=3	1.844227e+29	1.784175e+29	1.033658e+00
c=4	3.777587e+29	3.595074e+29	1.050767e+00
c=5	3.794191e+29	3.552398e+29	1.068065e+00
c=6	1.814949e+28	1.671914e+28	1.085552e+00
c=7	1.224958e+29	1.110339e+29	1.103229e+00
c=8	2.676623e+29	2.387500e+29	1.121099e+00
c=9	2.497978e+29	2.192823e+29	1.139161e+00
c=10	1.631916e+29	1.409964e+29	1.157417e+00
c=11	1.013299e+26	8.617452e+25	1.175869e+00
c=12	7.341203e+28	6.145751e+28	1.194517e+00

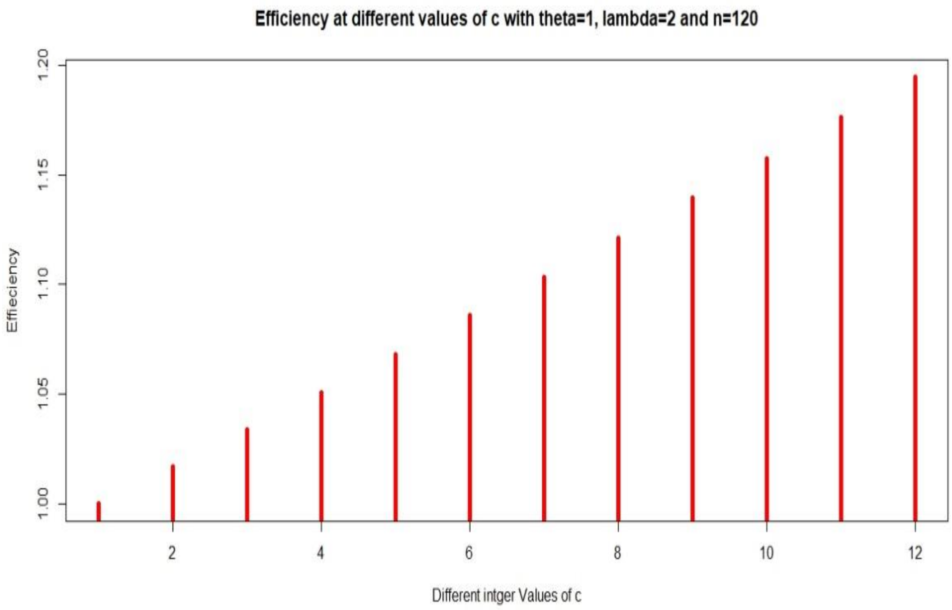

Figure 1: Efficiency of the scale parameter for different values of c

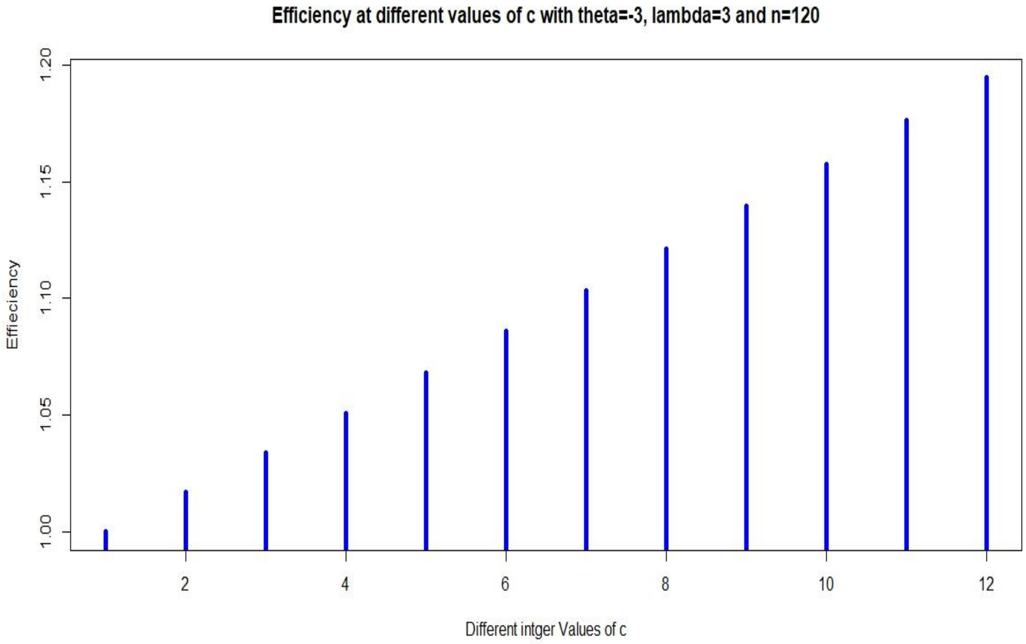

Figure 2: Efficiency of the scale parameter for different values of c

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