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Unbiased Sequential Binomial/Trinomial Parameter(s) Estimation: An Informative Review

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Abstract

In repeated Bernoulli trials, unified theory was developed almost 70 years back to examine the scope of 'sampling plans' providing unbiased estimation of the Bernoulli parameter 'p' and its functions. We review the literature and provide extension to trinomial distributions. Whereas Wald's SPRT is discussed in inference courses, rarely we find discussions on sequential estimation of the parameter 'p' and its functions. Some illustrative examples have been provided to demonstrate elaborately the underlying concepts and computations.

Keywords: Binary response, Bernoulli trials, Random walk, Accessible point, Continuation point, Boundary point, Inaccessible point, Path counting, Sampling plan, Closed sampling plan, Binomial sampling plan, Inverse binomial sampling plan, Pull-down sampling plan, Push-up sampling plan, Trinomial sampling plan, Unbiased sequential estimation.

AMS Classification: 62N02.

Footnote: The 3rd author [BKS] has been an avid admirer of Professor Basher Mian in whose honour this Felicitation Volume is published. They have known each other for more than 20 years and hold enormous amount of friendly mutual bondage. The 2nd author [PS] had been a student-cum-colleague of Professor Basher Mian in the Department of Statistics, University of Rajshahi. She has always upheld due respect to her Teacher. The 1st author [PT] had undertaken a summer project with BKS in 2016 and worked on this topic with tremendous energy and sincerity. All three of them have been fascinated by the level of intellect the researchers had to develop from time to time to appreciate the difficulty level and the solution associated with almost a trivial formulation involving the Bernoulli parameter 'p' in *Repeated Bernoulli Trials.* BKS himself was motivated after attending a series of lectures by the celebrated Probabilist Late Professor D Basu in the 1980's.
The authors have great pleasure in presenting this study in the present compilation in honour of an excellent teacher-cum-mentor Professor Basher Mian.

1. Introduction

1.1. A Historical Account

We will deal with the simplest and elementary experiment of coin tossing with $P[H] = p$ and $P[T] = q = 1 - p, 0 < p < 1.$ Further, we will work in the usual framework of "Repeated Bernoulli Trials [RBTs]". According to [Late] Professor D Basu, whatever we have usually learnt/taught in the framework of RBTs may be labelled as those based on "Direct Enumeration/Computation" of probability. The *real* challenge altogether skips attention of the learners.

His landmark question reads as: "Generate an event with probability $(p) = \sqrt{p}$."

We refer the readers to a paper by Sinha and Banerjee (1979) dealing with $f(p) = p^{\alpha},\ \alpha$[rational] > 0. There is a vast literature on this topic, spanning over a long period - since 1950's, or even earlier.

Several key references are: Girshick, Mosteller and Savage (1946), Wolfowitz (1946), Lehmann and Stein (1950). Blackwell and Girshick (1954), DeGroot (1959), Gupta (1967), Sinha and Sinha (1975, 1992), Sinha and Bose (1985), Bose and Sinha (1984), Sinha (1991), We hasten to add an informative Project Report [Tarafdar (2016)] in the list of references at the end as an unpublished document. To a beginner in this area, some of the references may serve as review articles – laying the foundation.

Before we turn to the basic notations and nomenclature, following Basu, we are tempted to cite two examples of $f(p)$ as (i) $\mathrm{f}(p) = 3.5pq$ or, as (ii) $f(p) = 12\ p^2q^2$ with not-so-immediate-solutions towards generating respective underlying events. In each case, we are required to find a solution to the twin pair $(n,\ E_n)$ where 'n' denotes the required number of Bernoulli trials and E_n is the underlying event for which $P[E_n] = f(p),\ 0 < p < 1$. Another related problem is to arrive at the UMVUE of $f(p)$, say $\hat{f}(p)$, which is also ***proper***, i.e., $0 \leq \hat{f}(p) \leq 1$ [since $f(p)$ is itself proper i.e., $0 \leq f(p) \leq 1$ for $0 \leq p \leq 1$]. A little reflection shows that $\hat{f}(p)$ is not necessarily so, irrespective of the choice of $f(p)$ and the underlying *data-generating scheme* so adopted!

1.2. Notations and Nomenclature

Let $Z_i, i = 1,2, \cdots$ be an i.i.d sequence of Bernoulli variates with $P[Z_i = 1] = p$ and $P[Z_i = 0] = 1 - p = q$ (say). We assume $p \in (0,1)$. Any realization of this process can be exhibited as a lattice path in the (X, Y)-*plane*, where a particle moves from the origin one step to the right along the X-axis if the incoming observation is 0 [Tail] and one step above along Y-axis if it is 1 [Head]. A *stopping rule* can be viewed as a sequence of functions φ_k, where φ_k is a function of $(Z_1, Z_2, \cdots, Z_k)$. Each φ_k take values 0 or 1. Given an integer 'k' and the set $(z_1, z_2, \cdots, z_k)$, $\varphi_k(z_1, z_2, \cdots, z_k) = 1$ indicates that we take one more observation i.e., (k+1)th observation and $\varphi_k(z_1, z_2, \cdots, z_k) = 0$ indicates that we stop at this kth stage . A point $\alpha = (x, y)$ is a *continuation point* if there exists one sequence of realizations $(z_1, z_2, \cdots, z_{x+y})$ leading to α such that $\varphi_j(z_1, z_2, \cdots, z_j) = 1 \; \forall j \leq x + y$. A point $\alpha = (x, y)$ is a *boundary point* if there exists one sequence of realizations $(z_1, z_2, \cdots, z_{x+y})$ leading to α such that $\varphi_j(z_1, z_2, \cdots, z_j) = 1 \; \forall j < x + y$ and $\varphi_{x+y}(z_1, z_2, \cdots, z_{x+y}) = 0$. A point may be a boundary point or a continuation point depending on the path. A point is an *accessible point* if it is either a boundary point or a continuation point. Points which are not accessible are regarded as inaccessible points. For any boundary point $\alpha = (x, y)$, $P(\alpha)$ denotes the probability of stopping at α and is given by

$$P(\alpha) = K(\alpha) p^y q^x, \text{ say}$$

where $K(\alpha)$ is the number of *accessible* paths from the origin to the point α.

A stopping rule yielding the boundary points together with their probabilities $P(\alpha)$ shall be called a sampling plan P.

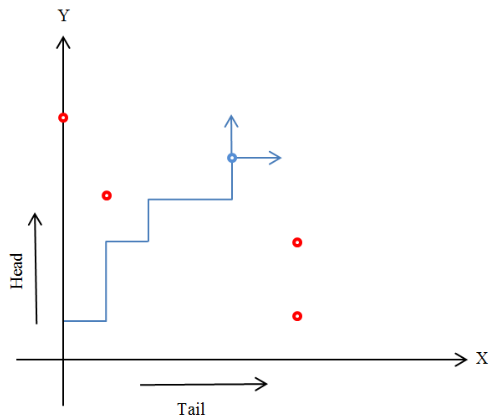

Figure 1: Boundary points

In Figure 1, red coloured points are boundary points and a motion of the random particle is shown along a free path. Accessible and inaccessible points depend on the sampling plan; hence they are not shown in the above figure.

1.2.1. Description of a Bernoulli Sampling Plan

We consider a coin tossing experiment with constant success probability $'p'$. Suppose the coin is tossed a given number, say n, of trials and then stopped, irrespective of the results obtained. We will denote the number of heads observed in the experiment by $'y'$ and the number of tails observed in the experiment by $'x'$ so that $x + y = n$. This is referred to as Binomial (n) sampling plan. In Figure 2, we exhibit such a plan.

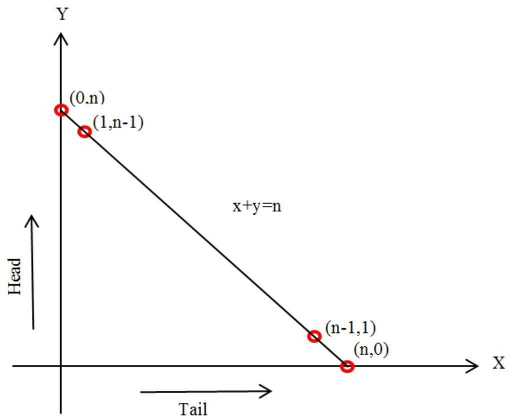

Figure 2: Binomial (n) Sampling Plan

Here the line $x + y = n$ is the absorbing boundary and the red circled points $[(n, 0)$ to $(0, n)]$ on the absorbing boundary are called the boundary points. The points falling inside the boundary are called free points or the accessible points and those falling beyond the absorbing boundary are known as inaccessible points. Note that such descriptions are being attributed to the points in the *(X, Y)*-plane with respect to the specific sampling plan

Now consider a different sampling plan. Suppose the coin tossing experiment is performed until 'k' heads appear. Then all the points lying on the line $y = k$ will be the boundary points for this plan. The plan diagram is given in Figure 3. This plan is referred to as Inverse Binomial Sampling (k) Plan.

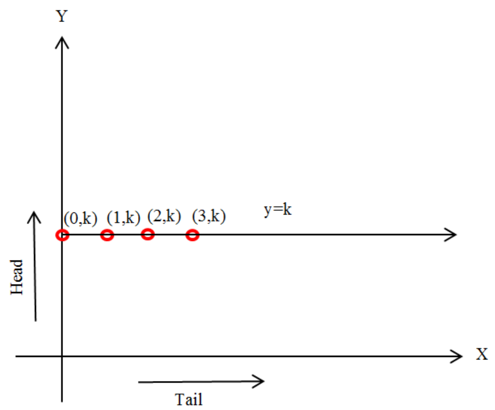

Figure 3: Inverse Binomial Sampling (k) Plan

In the above figure the line *y=k* is the absorbing boundary and the red circled points are the boundary points. There are an infinite number of boundary points on this line.

Now, we go through some definitions that are related to the notions of open and closed sampling plans.

1.2.2. Definition of closure of a sampling plan

Let $\alpha = (x, y) \in B$ be a typical boundary point where B = the collection of all the boundary points of the plan P. Let $P(\alpha)$ be the probability of reaching α starting from the point (0,0).

Then, $P(\alpha) = P(reaching\ \ \alpha) = \sum p^y q^x$, where the summation is over all the *accessible* paths from (0,0) to $\alpha = (x, y)$. We denote by $K(\alpha)$the *number* of such accessible paths from (0,0) to $\alpha = (x, y)$. Note that $K(\alpha) \leq \binom{x+y}{x}$ for every pair (x, y).

A sampling plan is said to be closed iff $\sum_{\alpha \in B} P(\alpha) = 1$ for all p. So, the two plans discussed above are simple and well-known examples of closed sampling plans. In a closed sampling plan, an experiment terminates with probability 1. Here the boundary point α and the route that the particle has traversed to reach α comprise the data for the experiment. It is to be noted that only closed sampling plans are of interest to an experimenter. A sampling plan which is not closed, is said to be ‘open’. Open sampling plans are not of any practical interest. We will focus on the closed sampling plans in the sequel.

1.2.3. Concept of path counting

Recall $K(\alpha) =$ the number of accessible paths from $(0,0)$to α. If all the paths from $(0,0)$ to α are *free / accessible* paths, then $K(\alpha) = \binom{x+y}{x}$. In case of binomial (n) sampling plan, all paths from $(0,0)$ to α=(x, y) are free paths when $x + y = n$, the binomial plan parameter. But in inverse binomial sampling (k) plan with 'parameter k', all paths from $(0,0)$ to α=(x, k) are *not* free. Also in many cases, the scenario may not be so easy. All the paths may not be free paths. That means, for a boundary point α=(x, y), the number of free paths may not be equal to $\binom{x+y}{x}$; it may be much less than $\binom{x+y}{x}$ in general terms. Intuitively, it would be of interest to get the value of $K(\alpha)$ for a given sampling plan and for a given boundary point α.

The calculation of the value of $K(\alpha)$ may be done by path-counting formula which is deeply combinatorial in nature. For non-standard sampling plans, we cannot avoid making a count of $K(\alpha)$ since closure of the plan is to be verified, to start with. Here is an excellent book in this direction. Mohanty (1979).

Towards construction of UMVUEs of parameters of interest, one unified approach has been suggested in the literature. We skip the verification of the fact that in the context of closed sampling plans arising out of Repeated Bernoulli Trials, the boundary point α and the path-counting number $K(\alpha)$ jointly define minimal sufficient statistics. Vide Sinha and Sinha (1992).

1.2.4. Concept of a Pull-Down plan

In 1-Step Pull-Down plan, we consider the number heads to be one unit less. That means, we pull down each boundary point of the plan P one unit along Y-axis towards X-axis. For example, the diagram of the Binomial Sampling (n) Plan would be transformed to the one in Figure 4 after 1-step pull-down as mentioned above.

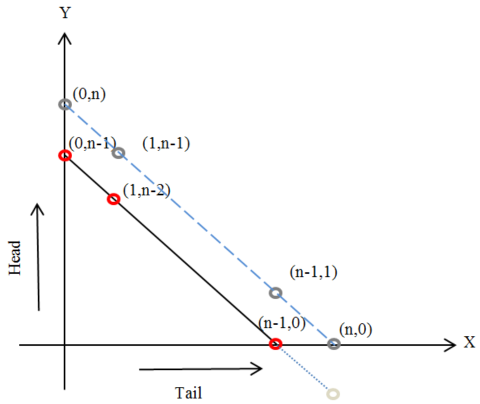

Figure 4: 1-Step Pull-Down Binomial Sampling (n) Plan

Here the *new* absorbing boundary (solid line) is $x + y = n - 1$ and the red circle points $[(n-1,0)$ to $(0, n-1)]$ on the *new* absorbing boundary are the *new* boundary points. We denote this sampling plan by $P_{-1}(\alpha_{-1})$, where $\alpha_{-1} = (x, y-1)$ are the new boundary points. It is readily seen that P_{-1} is closed. It is to be noted that due to 1-step pull down, the point $(n, 0)$ sank down below the X-axis. This concept generalizes naturally to Pull-Down Plans of 2- or more Steps.

In case of 1-Step Pull-Down plan for the Inverse Binomial Sampling (k) Plan, the coin tossing experiment is performed until $k - 1$ heads appear. Here we pull down the boundary line one unit along Y-axis towards X-axis. Therefore, all points lying on the line $y = k - 1$ are the *new* boundary points (Figure 5). Here P_{-1} is closed, too.

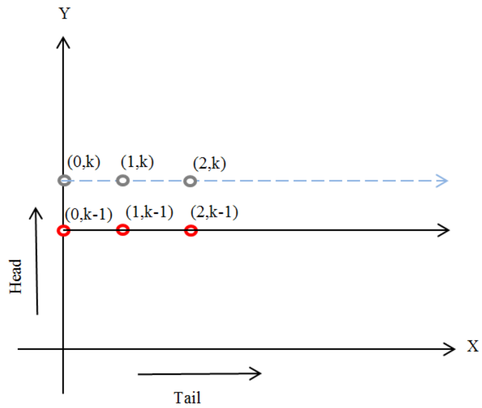

Figure 5: 1-Step Pull-Down Inverse Binomial Sampling (k) Plan

In Tarafdar (2016), it is demonstrated that for any *arbitrary* closed sampling plan P, the corresponding 1-Step Pull-Down Plan P_{-1} is also necessarily closed. Intuitively, this seems to be evident.

1.2.5. Concept of a Push-up plan

In this situation we consider the number heads to be one unit more. In other words, the plan is pushed-up one step in an analogous fashion. Then the diagram of the Binomial Sampling (n) Plan will look like as indicated in Figure 6.

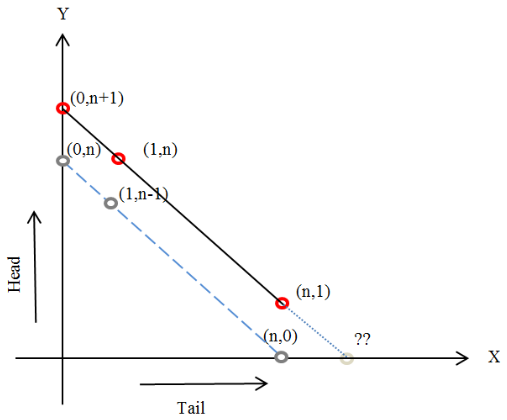

Figure 6:1-step Push-Up Binomial Sampling (n) Plan

Here the absorbing boundary is on the line $x + y = n + 1$ and the red circle points $[(n, 1)$ to $(0, n + 1)]$ on the absorbing boundary are the boundary points. That means, the point (n+1, 0) is excluded from B_{+1}. We denote this sampling plan by $P_{+1}(\alpha_{+1})$, where $\alpha_{+1} = (x, y + 1)$ are the new boundary points. We readily observe that P_{+1} is not closed, because, in the push-up plan, the moving point can reach $(n, 0)$ with probability p^n and thereafter it can move one more step along X-axis with probability p. Hence, under P_{+1}, the particle will not stop with probability 1.

In case of Push-Up plan for Inverse Binomial Sampling (k) Plan, the coin tossing experiment is performed until $k + 1$ heads appear. Therefore, all the points lying on the line $y = k + 1$are the boundary points (Figure 7). In this case, the plan P_{+1} is indeed closed.

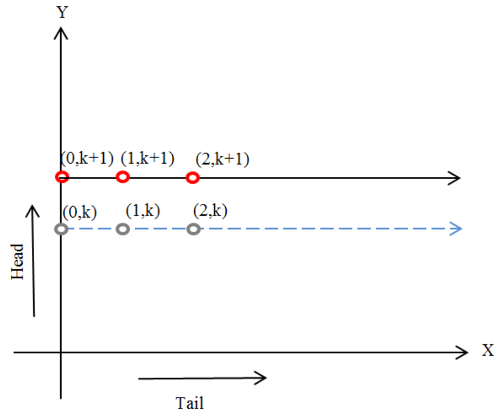

Figure 7: 1-step Push-Up Inverse Binomial Sampling (k) Plan

Therefore, it is found that P_{+1} is not always necessarily closed, although P is closed to start with.

However, it can be argued that once P_{+1} is closed, so is any plan of the type P_{+c} , $c = 2, 3, 4, \cdots$. On the other hand, though P_{-1} is closed (for $n > 1$), closure may not hold for P_{-2} or P_{-3} etc. i.e., for higher negative integer values of n.

2. Unbiased Estimation

2.1. Unbiased Estimation of p

For an arbitrary closed sampling plan, it is not very obvious to suggest the nature of an unbiased estimate p with the usual procedure of estimation. Here pull-down plan concept can effectively be used for unbiased estimation of p. Pull-down plan says: pull the whole set of boundary points of the plan by one step along Y-axis towards X-axis.

Now since a pull down sampling plan P_{-1} is closed, we can write

$\sum_{\alpha_{-1}} P_{-1}(\alpha_{-1})$=1.

Again, $P_{-1}(\alpha_{-1}) = K(x, y-1)p^{y-1}q^x$.

So, $\sum_{\alpha_{-1}} K(\alpha_{-1})p^{y-1}q^x = 1$.

We rewrite this as

$\sum_{\alpha_{-1}}[K(\alpha_{-1})/K(\alpha)]K(\alpha)\, p^y q^x = p$.

Since $\sum_{\alpha_{-1}}$and $\sum_{\alpha}$ are 1-1 functions, we can replace $\sum_{\alpha_{-1}}$by $\sum_{\alpha}$ in the above.

This gives an unbiased estimate of p as $\hat{p} = K(\alpha_{-1})/K(\alpha)$.

So, an estimate of 'p' is given by the ratio of the two path counts.

For instance, in case of binomial sampling (n) plan where $x + y = n, K(\alpha) = \binom{n}{x}$ and $K(\alpha_{-1}) = \binom{n-1}{x}$ and $\hat{p} = y/n$, which is the usual estimator of p. Note that 'y' denotes the head count in the plan.

Similarly, in case of inverse binomial sampling (k) plan, we apply 1-step pull-down approach and derive: $K(\alpha) = \binom{k+x-1}{x}$ and $K(\alpha_{-1}) = \binom{k+x-2}{x}$, which gives $\hat{p} = (k-1)/(x+k-1)$.

2.2. Unbiased Estimation of p^{-1}

Let P denote the original closed sampling plan. After 1-step push-up, the new plan is denoted by P_{+1} with α_{+1} as a typical boundary point for $P_{+1}(\alpha \to \alpha_{+1})$ such that $\alpha_{+1} = (x, y+1)$ and $K(\alpha_{+1})$ as the number of accessible paths from $(0,0)$ to α_{+1}. Suppose the original plan P is so chosen that the 1-step push-up plan P_{+1} is *also closed*. Then $\frac{1}{p}$ can be estimated as $K(\alpha_{+1})/K(\alpha)$. The proof follows readily.

Since $\sum_{\alpha_{+1} \in B_{+1}} K(\alpha_{+1})\, p^{y+1} q^x = 1$ (as P_{+1} is assumed to be closed)

$$\Rightarrow \sum_{\alpha_{+1} \in B_{+1}} [K(\alpha_{+1})/K(\alpha)] K(\alpha)\, p^y q^x = \frac{1}{p}$$

$$\Rightarrow \widehat{\frac{1}{p}} = K(\alpha_{+1})/K(\alpha).$$

As was mentioned before, $K(\alpha)$ values are usually calculated by path-counting.

Further by 2-step push-up plan we can find estimate of $\frac{1}{p^2}$ as $k(\alpha_{+2})/k(\alpha)$. Proceeding in this way unbiased estimate of any negative integer power of p can be derived.

It turns out that while binomial sampling (n) plans do not provide unbiased estimates of $1/p$ etc, all inverse binomial sampling (k) plans do provide the estimates.

However, the inverse binomial sampling (k) plan does not allow us to estimate $\frac{1}{q}, \frac{1}{q^2}, \cdots$ directly.

Remark 1. Bose and Sinha (1984) proved that closure of a push-up plan P_{+1} is necessary for unbiased estimation of 1/p.

2.3. An illustration of two sampling plans other than the traditional ones: Unbiased Estimation of p, q and pq

Plan 1: A coin tossing experiment is performed until 3 heads *or* 5 tails appear.

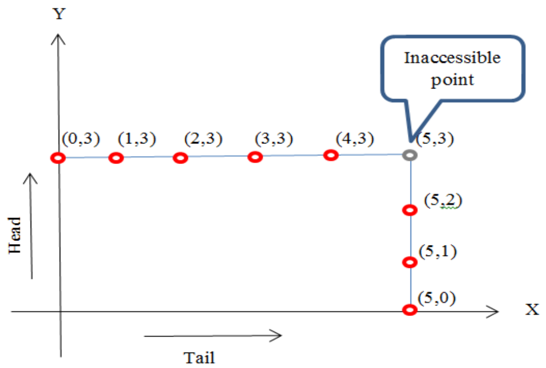

Figure 8: Boundary points for 3 Heads or 5 Tails

For such a sampling plan (Figure 8) the boundary points α, α_{-1}, $k(\alpha)$, $k(\alpha_{-1})$ and $\hat{p}$ for different values of α are given in Table 1.

Table 1: Estimation of p

α	α_{-1}	$k(\alpha)$	$k(\alpha_{-1})$	$\hat{p} = k(\alpha_{-1})/k(\alpha)$
(5,0)	-	1	0	0
(5,1)	(5,0)	$\binom{6}{1} - 1 = 5$	1	1/5
(5,2)	(5,1)	$\binom{6}{2} = 15$	$\binom{6}{1} - 1 = 5$	1/3
(4,3)	(5,2)	$\binom{6}{4} = 15$	$\binom{6}{2} = 15$	1
(3, 3)	(3,2)	$\binom{5}{3} = 10$	$\binom{4}{2} = 6$	3/5
(2, 3)	(2,2)	$\binom{4}{2} = 6$	$\binom{3}{2} = 3$	1/2
(1, 3)	(1, 2)	$\binom{4}{1} - 1 = 3$	$\binom{3}{1} - 1 = 2$	2/3
(0, 3)	(1, 2)	1	1	1

The closure properties are readily verified.

Analogously, we can provide an unbiased estimator of q. We skip this part. Below we take up the case of pq and straightaway furnish the results by referring to doubly pulled plan $P_{[-1,-1]}$ with boundary points denoted by (α_{-1}^{**}).

Table 2: Estimation of pq

Boundary points	$k(\alpha)$	$k(\alpha_{-1}^{**})$	$\widehat{pq} = k(\alpha_{-1}^{**})/k(\alpha)$
(5,0)	1	0	0
(5,1)	$\binom{6}{1} - 1 = 5$	1	1/5
(5,2)	$\binom{6}{2} = 15$	$\binom{5}{1} - 1 = 4$	4/15
(4,3)	$\binom{6}{4} = 15$	$\binom{4}{2} = 6$	2/5
(3, 3)	$\binom{5}{3} = 10$	$\binom{3}{2} = 3$	3/10
(2, 3)	$\binom{4}{2} = 6$	$\binom{3}{1} - 1 = 2$	1/3
(1, 3)	$\binom{4}{1} - 1 = 3$	1	1/3
(0, 3)	1	0	0

Plan 2: A coin tossing experiment is performed until 3 heads *and* 5 tails appear.

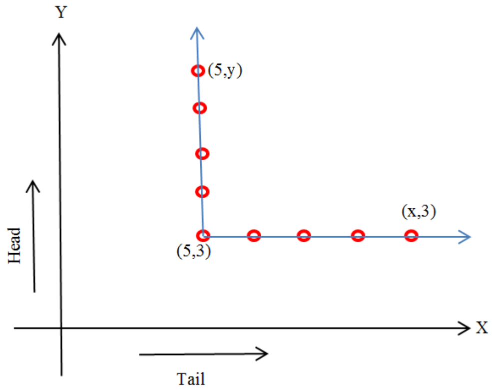

Figure 9: Sampling plan to get 3 heads and 5 tails

From Figure 9 we can find that the boundary points are- (a) (5,3), (b) (x,3); x≥6 and (c) (5,y); y≥4. At first, we compute the values of $K(\alpha)$. For $\alpha = (5,3)$, all the paths are free paths from the point (0,0). Hence the value of $K(\alpha)$ in this case will be $\binom{5+3}{5} = \binom{8}{5} = 56$. Now, for the type (b) boundary points, none of them can be reached from the left side. All of them must be reached only through the moves in the upward direction. Let, a typical boundary point of this type be $\alpha = (x, 3)$. Then it must be reached only through the point (x, 2). Hence in this case, the value of $K(\alpha)$ is $\binom{x+2}{2}$. Similarly, the type (c) boundary points can be reached only from the left side. Hence, for these types of the boundary points the value of $K(\alpha)$ is $\binom{4+y}{4}$. Closure of P is readily verified by the following arguments:

(i) Boundary points defined by the Inverse Bernoulli plan described through "*Y=3*" contain all the boundary points [(x, 3); x ≥ 5] and path counts remain the same.

(ii) Boundary points defined by the Inverse Bernoulli plan described through "*X=5*" contain all the boundary points [(5, y); y ≥ 3] and path counts remain the same.

(iii) Further, the two plans in (i) and (ii) together comprise the given Plan in Figure 9 and the plan in Figure 8.

(iv) Since Figure 8 refers to a closed plan, we necessary have closure of the plan in Figure 9.

Now we discuss about estimation of 'p'. For that, we perform one-step pull down method along Y-axis towards X-axis and for the revised plan, we compute the values of $K(\alpha_{-1})$ using similar logic.Also the closure property of P_{-1} is readily verified.Therefore, Table 3 contains the boundary points α, 1-step pull down boundary points α_{-1}, $K(\alpha)$, $K(\alpha_{-1})$ and $\hat{p}$ for different values of α.

Table 3: Estimation of p

Boundary points, α	$\boldsymbol{\alpha_{-1}}$	$\boldsymbol{k(\alpha)}$	$\boldsymbol{k(\alpha_{-1})}$	$\boldsymbol{\hat{p} = k(\alpha_{-1})/k(\alpha)}$
(5,3)	(5,2)	56	21	0.375
(x,3);x≥6	(x,2);x≥6	$\binom{x+2}{2}$	$\binom{x+1}{1}$	$\frac{2}{x+2}$
(5,y); y≥4	(5,y); y≥3	$\binom{4+y}{y}$	$\binom{3+y}{y-1}$	$\frac{y}{4+y}$

Likewise, we can provide an unbiased estimate of q. We skip the details.

Again if we want to estimate pq, one way would be to estimate p and p^2 first using 1-step pull down and 2-step pull down sampling plans towards X-axis and then use the results to estimate $p(1-p) = p - p^2$. Another way is to estimate it directly following 1-step pull down sampling plan towards X-axis and towards Y-axis simultaneously. For such pull down sampling plan, we denote the new boundary points as $\alpha_{-1}^{**} = (x-1, y-1)$. Therefore, we have to compute $K(\alpha_{-1}^{**})$ for the above three types of boundary points to estimate pq. Here again we skip the details.

2.4. Miscellaneous results

We start by stating an interesting result.

Necessary and sufficient condition for the existence of a closed sampling plan to enable unbiased estimation of $p^a q^b$ is that at least one point on the line $x + y = a + b$ is an accessible/boundary point [a and b being finite positive integers, say $(a, b) = (4,3)$]

It is clear from the above that application of appropriate pull-down plans serves to provide unbiased estimate of$p^a q^b$for a > 0 and b > 0.

Now we consider the other three situations to suggest unbiased estimate of $p^a q^b$, where a, or b, or both are negative finite integers. We confine to (a,b) = (+/-4, +/-3), excluding (4, 3).

Case I: Estimation of $p^4 q^{-3}$

Upon binomial expansion, we write

$$p^4 q^{-3} = q^{-3} - 4q^{-2} + 6q^{-1} - 4 + q$$

In this situation following 1-step pull down sampling plan towards Y-axis q can be estimated. Following 1-step push-up sampling plan along X-axis q^{-1} can be estimated, following 2-step push-up sampling plan q^{-2} can be estimated, and following 3-step push-up sampling plan q^{-3} can be estimated. Therefore, replacing those estimated values in the above expanded form $p^4 q^{-3}$ can be estimated.

All the relevant ‘derived’ plans are closed under the plan in Figure 9.

Case II: Estimation of $p^{-4} q^3$

It follows along analogous arguments. We skip the details.

Case III: Estimation of $p^{-4} q^{-3}$

In this situation, it is not easy to estimate the parametric function since it cannot be expanded as a linear combination of finite number of powers of p or q. To show more flexibility in the choice of a closed sampling plan to ensure this, let us consider a coin tossing experiment which is performed until 5 heads and 4 tails are observed. Then the boundary points are on the lines as in Figure 10. Now we follow 4-step push-up sampling plan along Y-axis and 3-step push-up sampling plan along X-axis. Then the new boundary points will be on the solid line of Figure 11.

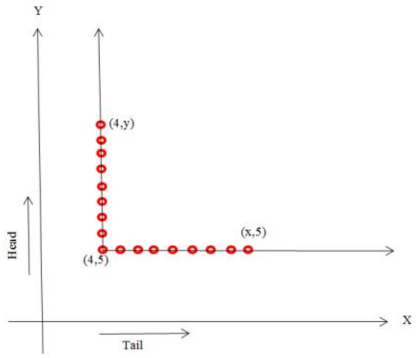

Figure 10: Boundary points for getting 5 heads and 4 tails.

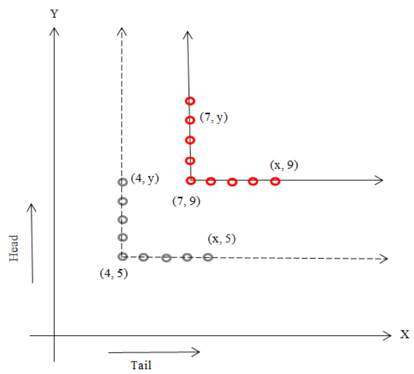

Figure 11: New boundary points after 4-step push-up sampling plan along Y-axis and 3-step push-up sampling plan along X-axis.

From Figure 10 we can find that the boundary points are- (a) (4,5), (b) (x,5); x≥5 and (c) (4,y); y≥6. We have to estimate $\theta = p^{-4}q^{-3}$ at these three points using the formula $\hat{\theta} = k(\alpha^{**})/k(\alpha)$, where α^{**} are the new boundary points (a) (7,9), (b) (x,9); x≥8 and (c) (7,y); y≥10 (Figure 11).

Table 4: Estimation of $\theta = p^{-4}q^{-3}$

$\boldsymbol{\alpha}$	$\boldsymbol{k(\alpha)}$	$\boldsymbol{\alpha^*}$	$\boldsymbol{k(\alpha^*)}$	$\boldsymbol{\hat{\theta} = k(\alpha^*)/k(\alpha)}$
(4,5)	126	(8,7)	6435	51.07
(x,5);x≥5	$\binom{x+4}{4}$	(x,9); x≥7	$\binom{x+8}{8}$	$\binom{x+8}{8}/\binom{x+4}{4}$
(4,y); y≥6	$\binom{3+y}{3}$	(7,y); y≥10	$\binom{6+y}{6}$	$\binom{6+y}{6}/\binom{3+y}{3}$

Closure properties of relevant plans are readily verified.

3. Estimation of Parameters under Trinomial Setup

3.1. Notations and Nomenclature

Let $(W_i, i = 1,2,\cdots)$ be a sequence of trials which have three possible outcomes with corresponding probabilities p, q, r such that $0 < p, q, r < p + q + r = 1$. Then any experiment can be exhibited as a lattice path in the (X, Y, Z)-space, where a particle moves from the origin one step to the right (along X-axis) if the incoming observation is an outcome having probability p, one step above (along Y-axis) if it is an outcome having probability q and one step above (along Z-axis) if it is an outcome having probability r. A stopping rule can be viewed as a sequence of functions φ_k, where φ_k is a function of $(W_1, \cdots, W_k)$; $(W_1, \cdots, W_k)$. being an i.i.d. sequence of Trinomial variates. Each φ_k takes the value 0 or 1; given $(w_1, \cdots, w_k)$, φ_k $(w_1, \cdots, w_k)$ = 1 indicates that we take one more observation and φ_k $(w_1, \cdots, w_k)$ = 0 indicates that we stop at this stage. Similar to Bernoulli trials a point $\alpha = (x, y, z)$ is a continuation point if there exists one sequence of realization $(w_1, \cdots, w_{x+y+z})$ leading to α such that $\varphi_j(w_1, \cdots, w_j) = 1\ \forall j \leq x + y + z$. A point $\alpha = (x, y, z)$ is a boundary point if there exists one sequence of realization $(w_1, \cdots, w_{x+y+z})$ leading to α such that $\varphi_j(w_1, \cdots, w_j) = 1\ \forall j < x + y + z$ and $\varphi_{x+y+z}(w_1, \cdots, w_{x+y+z}) = 0$. A point may be a boundary point or a continuation point depending on the path. A point is an accessible point if it is either a boundary point or a continuation point. Points which are not accessible are inaccessible points.

3.2. Estimation of Parameters for Trinomial Distribution

Consider an experiment with three possible outcomes with finite number of independent identical repetitions. Denote the occurrence of the event having probability p as x times, the occurrence of the event having probability q as y times and the occurrence of the event having probability r as z times. Sampling plan for such experiment is defined as $P(0,0,0)$ which is $x + y + z = n$. i.e. for a fixed $x = t, (0 \leq t \leq n)$, the boundary points are points of the line $y + z = n - t$. It can be represented graphically as Figure 12.

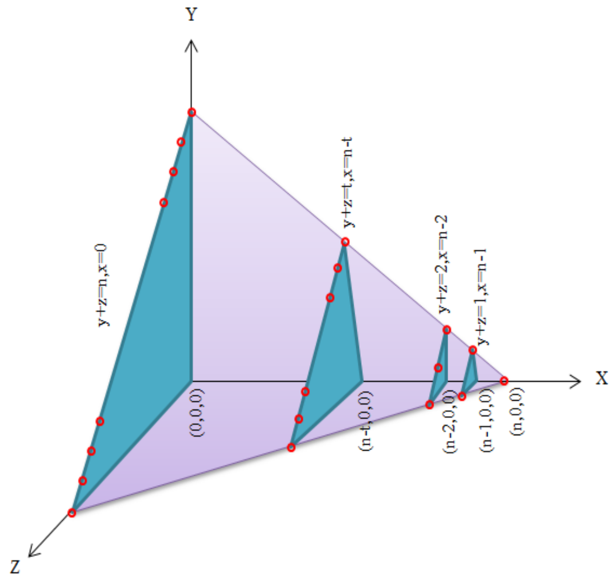

Figure 12: Trinomial Sampling Plan

The above diagram exhibits the trinomial sampling plan, i.e. for a fixed value of $x = t, 0 < t < n$, the boundary points are the integral solutions to the equation $y + z = n - t$, while for $x = n$ only $(n, 0,0)$ is the boundary point.

Clearly the plan is closed as $\forall \alpha = (x, y, z), \sum_{\alpha \in B} p^x q^y r^z = 1$.

From now on we will use the notation for boundary point $\alpha = (x, y, z)$ as $\alpha_{(0,0,0)}$.

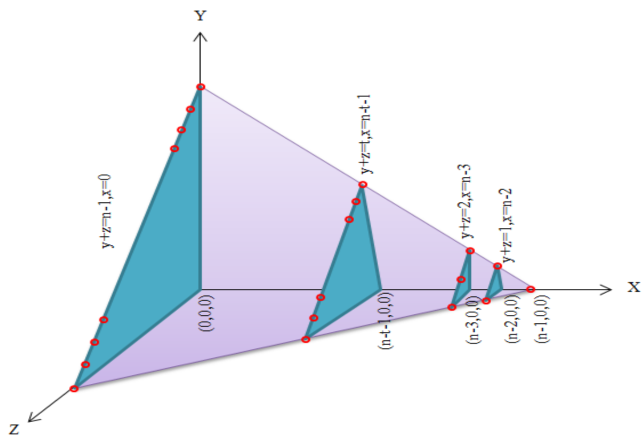

Figure 13: One-step pull down Trinomial Sampling Plan along X-axis

Now, to estimate p one can pull down the plan one-step along X-axis towards (Y-Z) plane, i.e., pull down plan $P(-1,0,0)$ has to be adopted [Figure 13]. Let us denote the new boundary point as $\alpha_{(-1,0,0)} = (x-1, y, z)$. It can be seen then $P(-1,0,0)$ is closed. Also, a typical $k(\alpha)$ is given by $(x, y, z) = \frac{n!}{x!y!z!}$. Hence, the estimator is given by the following steps:

$\sum_{P_{(0,0,0)}} k\big(\alpha_{(0,0,0)}\big) p^x q^y r^z = 1,$ since $P_{(0,0,0)}$ is closed

$\Rightarrow \sum_{P_{(-1,0,0)}} k\big(\alpha_{(-1,0,0)}\big) p^{x-1} q^y r^z = 1,$ since $P_{(-1,0,0)}$ is closed

$$\Rightarrow \sum_{P_{(-1,0,0)}} \left[\frac{k\big(\alpha_{(-1,0,0)}\big)}{k\big(\alpha_{(0,0,0)}\big)}\right] k\big(\alpha_{(0,0,0)}\big) p^x q^y r^z = p$$

$$\therefore \hat{p} = \frac{k\big(\alpha_{(-1,0,0)}\big)}{k\big(\alpha_{(0,0,0)}\big)} = \frac{\frac{(n-1)!}{(x-1)!y!z!}}{\frac{n!}{x!y!z!}} = \frac{x}{n}$$

Similarly, estimator of q and r can be obtained using the one-step pull down sampling plan $P(0,-1,0)$ which is along Y-axis and $P(0,0,-1)$ which is along Z-axis, respectively.

$$\therefore \hat{q} = \frac{k\big(\alpha_{(0,-1,0)}\big)}{k\big(\alpha_{(0,0,0)}\big)} = \frac{\frac{(n-1)!}{x!(y-1)!z!}}{\frac{n!}{x!y!z!}} = \frac{y}{n} \text{ and } \hat{r} = \frac{k\big(\alpha_{(0,0,-1)}\big)}{k\big(\alpha_{(0,0,0)}\big)} = \frac{\frac{(n-1)!}{x!y!(z-1)!}}{\frac{n!}{x!y!z!}} = \frac{z}{n}.$$

Now, to estimate pq, we need one-step pull down sampling plan along X-axis *and* also Y-axis simultaneously towards Z-axis which is $P(-1,-1,0)$. Therefore, we get the estimator as,

$$\widehat{pq} = \frac{k\big(\alpha_{(-1,-1,0)}\big)}{k\big(\alpha_{(0,0,0)}\big)} = \frac{\frac{(n-2)!}{(x-1)!(y-1)!z!}}{\frac{n!}{x!y!z!}} = \frac{xy}{n(n-1)}.$$

Similarly, to estimate pqr, we need one-step pull down sampling plan along all the three directions, which is $P(-1,-1,-1)$. Therefore, we get the estimator as

$$\widehat{pqr} = \frac{k\big(\alpha_{(-1,-1,-1)}\big)}{k\big(\alpha_{(0,0,0)}\big)} = \frac{\frac{(n-3)!}{(x-1)!(y-1)!(z-1)!}}{\frac{n!}{x!y!z!}} = \frac{xyz}{n(n-1)(n-2)}.$$

3.3. Estimation of Parameters for Inverse Trinomial Distribution

Consider, in trinomial setup, the inverse trinomial sampling plan $x = k$, where the occurrence of the event having probability p is considered as $x\ (= k)$ times as mentioned earlier. Let us consider a typical boundary point $\alpha = (x, y, z)$. Under the sampling plan we have,

$$k(\alpha) = \frac{(k + y + z - 1)!}{(k - 1)!\ y!\ z!}.$$

Here, $(Y - Z)\ plane$ serves as an infinite collection of boundary points, represented graphically as in Figure 14.

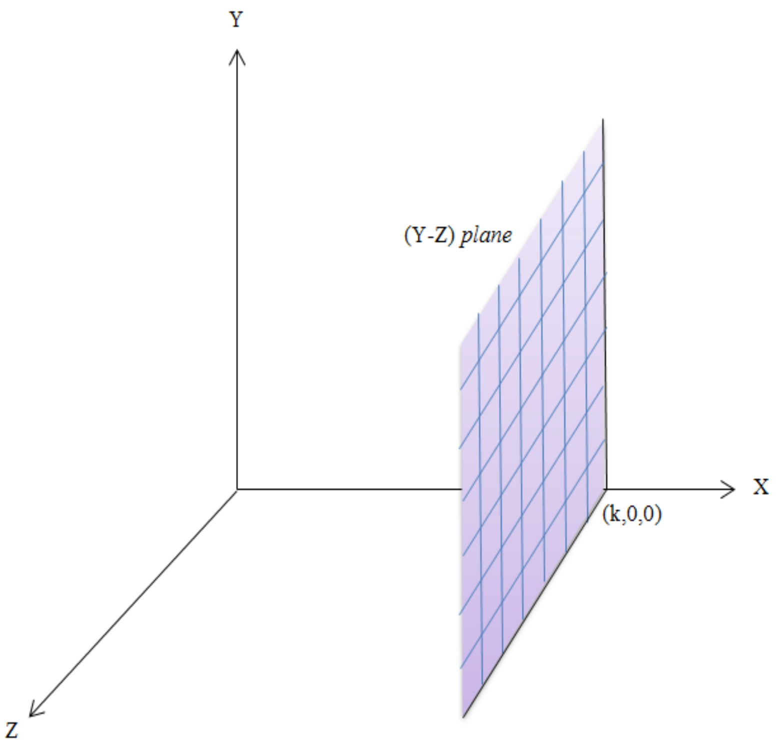

Figure 14: Inverse Trinomial Sampling Plan, $x = k$

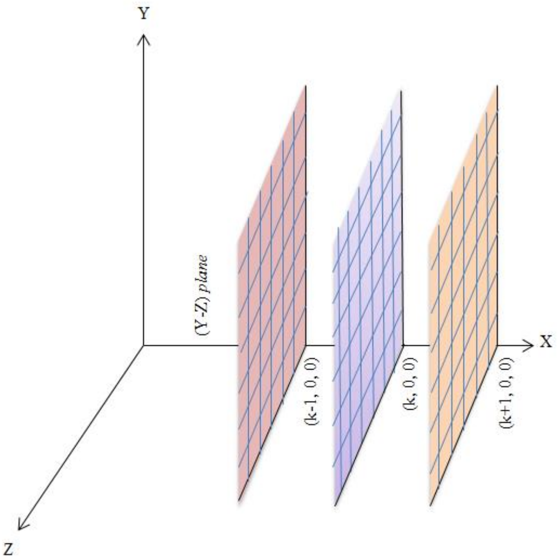

Figure 15: One-step Pull Down and push-up Inverse Trinomial Sampling Plan

3. 3.1 Estimation of p, q, r, pq, pqr

Since, $P(0,0,0)$ which is $x = k$, is a closed plan, we can argue that the pull down plan $P(-1,0,0)$also satisfies the closure property as $P(-1,0,0)$ is nothing but the plan consisting of boundary points on $(Y-Z)\ plane$ for $x = k-1$ [Figure 15]. The new boundary point is $\alpha_{(-1,0,0)}$. Hence, finding unbiased estimator of p is routine by application of path counting formula. The estimator is derived by the following steps,

$\sum_{P_{(0,0,0)}} k(\alpha_{(0,0,0)}) p^x q^y r^z = 1,$ since $P_{(0,0,0)}$ is closed

$\Rightarrow \sum_{P_{(-1,0,0)}} k(\alpha_{(-1,0,0)}) p^{x-1} q^y r^z = 1,$ since $P_{(-1,0,0)}$ is closed

$$\Rightarrow \sum_{P_{(-1,0,0)}} \left[\frac{k(\alpha_{(-1,0,0)})}{k(\alpha_{(0,0,0)})}\right] k(\alpha_{(0,0,0)}) p^x q^y r^z = p$$

$$\therefore \hat{p} = \left[\frac{k(\alpha_{(-1,0,0)})}{k(\alpha_{(0,0,0)})}\right] = \frac{\frac{(k+y+z-2)!}{(k-2)!\,y!\,z!}}{\frac{(k+y+z-1)!}{(k-1)!\,y!\,z!}} = \frac{k-1}{k+y+z-1}$$

Similarly, estimator**s** of q and r can be obtained using the one-step pull down sampling plan $P(0,-1,0)$ which is along Y-axis and $P(0,0,-1)$ which is along Z-axis, respectively.

Now, to estimate pq, we need one-step pull down sampling plan along X-axis *and* along Y-axis simultaneously – both being towards Z-axis which is $P(-1,-1,0)$. The new boundary point is $\alpha_{(-1,-1,0)}$. Therefore, we get the estimator as

$$\widehat{pq} = \frac{k(\alpha_{(-1,-1,0)})}{k(\alpha_{(0,0,0)})} = \frac{\frac{(k+y+z-3)!}{(k-2)!(y-1)!z!}}{\frac{(k+y+z-1)!}{(k-1)!\ y!\ z!}} = \frac{y(k-1)}{(k+y+z-1)(k+y+z-2)}.$$

Similarly, to estimate pqr, we need one-step pull down sampling plan along X-axis, Y-axis and Z-axis simultaneously which is $P(-1,-1,-1)$. The new boundary point is $\alpha_{(-1,-1,-1)}$. Therefore, we get the estimator as

$$\widehat{pqr} = \frac{k(\alpha_{(-1,-1,-1)})}{k(\alpha_{(0,0,0)})} = \frac{\frac{(k+y+z-4)!}{(k-2)!(y-1)!(z-1)!}}{\frac{(k+y+z-1)!}{(k-1)!\ y!\ z!}} = \frac{yz(k-1)}{(k+y+z-1)(k+y+z-2)(k+y+z-3)}.$$

Remark 2: We can estimate p, pq, pqr in trinomial distribution using first principle method, too. This is just verification for the beginners. However, path counting allows us to get unbiased estimators of other complicated forms of parameters.

3.3.2. Estimation of p^{-1}

Now, consider the trinomial sampling plan $P(0,0,0)$ as shown in Figure 12, where the boundary points are $\alpha_{(0,0,0)} = (x, y, z),\quad s.t.\quad x + y + z = n$. We want to estimate p^{-1}. For that we have to push-up (move forward) the sampling plan one-step along X-axis. The new plan is $P(+1,0,0)$ and the new boundary points are $\alpha_{(+1,0,0)} = (x+1, y, z),\quad s.t.\quad x + y + z = n + 1$. Similar to one-step push-up binomial sampling plan, the boundary points on $y + z = n + 1,\ x = 0$ are absent in the plan [Figure 16]. Therefore, the plan $P(+1,0,0)$ is not closed. Hence, estimator of $\frac{1}{p}$ can not be obtained. Find Bose-Sinha (1984) work.

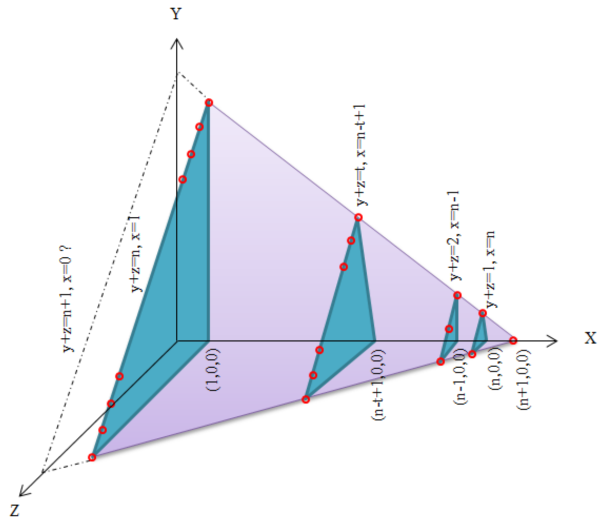

Figure 16: One-step push-up Binomial Sampling Plan along X-axis

However, in case of inverse trinomial sampling plan as shown in Figure 14, the one-step push-up (forward) plan along X-axis is $P(+1,0,0)$ where the boundary points are $\alpha_{(0,0,0)} = (k+1, y, z)$. Therefore, with reference to Figure 14 plan $P(+1,0,0)$ is nothing but the plan consisting of boundary points on $(Y - Z)$ $plane$ for $x = k + 1$ as shown in Figure 15. Hence the plan $P(+1,0,0)$ is closed and $\frac{1}{p}$ is estimable. Therefore, the unbiased estimator of $\frac{1}{p}$ is

$$\widehat{\frac{1}{p}} = \frac{k(\alpha_{(+1,0,0)})}{k(\alpha_{(0,0,0)})} = \frac{\frac{(k+y+z)!}{k!\,y!\,z!}}{\frac{(k+y+z-1)!}{(k-1)!\,y!\,z!}} = \frac{k+y+z}{k}.$$

Remark 3. From the inverse trinomial sampling plan $x = k$ we can not estimate $\frac{1}{q}$directly by push-up sampling method as the plan $P(0,+1,0)$ is no more closed. To estimate $\frac{1}{q}$ we have to switch the inverse trinomial sampling plan from $x = k$ to $y = k$, k being a generic constant. It will not alter the result as the plan is symmetric. Therefore, using one-step push-up sampling plan along Y-axis we can get $\widehat{\frac{1}{q}}$. On the other hand to estimate $\frac{1}{r}$ we have to switch the inverse trinomial sampling plan from $x = k$ to $z = k$. Therefore, using one-step push-up sampling plan along Z-axis we can get $\widehat{\frac{1}{r}}$.

At this stage, one might ask the following question: Is there a single plan which takes care of unbiased estimation of all the three parameters viz., 1/p, 1/q and 1/r ?

The answer is in the affirmative and there are many plans to achieve this. One simple plan is an extension of P[(5, 3)] sampling plan involving Heads and Tails in case of binomials. Recall P[(5, 3)] plan which states : Continue drawing observations on Heads and Tails until 5 Heads *and* 3 Tails are observed. We consider a natural extension like P[(5, 5, 5)] below.

3.3.3. Unbiased estimator of $p^a q^b r^c$

To get an unbiased estimator of $\theta = p^a q^b r^c$ we need to pull down the sampling plan a-steps along X-axis, b-steps along Y-axis and c-steps along Z-axis simultaneously. The new boundary point is $\alpha_{(-a,-b,-c)}$ in the sampling plan $P(-a,-b,-c)$. Therefore, we can get the estimator as

$$\hat{\theta} = \frac{k(\alpha_{(-a,-b,-c)})}{k(\alpha_{(0,0,0)})}.$$

In case of trinomial sampling plan we will get the estimator as

$$\hat{\theta} = \frac{k(\alpha_{(-a,-b,-c)})}{k(\alpha_{(0,0,0)})} = \frac{\frac{(n-a-b-c)!}{(x-a)!(y-b)!(z-c)!}}{\frac{n!}{x!y!z!}}.$$

And in case of inverse trinomial sampling plan [with boundary along x = k] we will get the estimator as

$$\hat{\theta} = \frac{k(\alpha_{(-a,-b,-c)})}{k(\alpha_{(0,0,0)})} = \frac{\frac{(k+y+z-1-a-b-c)!}{(k-1-a)!(y-b)!(z-c)!}}{\frac{(k+y+z-1)!}{(k-1)!\ y!\ z!}}.$$

Example: For simplicity we consider $\theta = p^2 q^3 r^4$ to be estimated and we confine to usual trinomial distribution.

First we consider the trinomial plan $P(0,0,0)$ with $x, y\ and\ z$ such that $x + y + z = n(\geq 2 + 3 + 4 = 9)$. We denote the boundary point as $\alpha_{(0,0,0)} = (x, y, z)$. We use pull down sampling plan 2-steps along X-axis, 3-steps along Y-axis and 4-steps along Z-axis. Then sampling plan becomes $P(-2,-3,-4)$ and the new boundary point is $\alpha_{(-2,-3,-4)} = (x-2, y-3, z-4)$. Using path counting formula we can get $k(\alpha_{(0,0,0)}) = \frac{n!}{x!y!z!}$ and $k(\alpha_{(-2,-3,-4)}) = \frac{(n-9)!}{(x-2)!(y-3)!(z-4)!}$. Therefore, estimator of $\theta = p^2q^3r^4$ is

$$\hat{\theta} = \frac{k(\alpha_{(-2,-3,-4)})}{k(\alpha_{(0,0,0)})} = \frac{\frac{(n-9)!}{(x-2)!(y-3)!\,(z-4)!}}{\frac{n!}{x!\,y!\,z!}} = \frac{x^{(2)}y^{(3)}z^{(4)}}{n^{(9)}}.$$

For example, we consider a trinomial plan with $x+y+z=15$. Therefore, a typical boundary point is $\alpha_{(0,0,0)} = (3,4,8)$, for example. We want to find an estimator at $\alpha_{(0,0,0)} = (3,4,8)$. We use pull down sampling plan 2-step along X-axis, 3-step along Y-axis and 4-step along Z-axis. Then the new boundary point is $\alpha_{(-2,-3,-4)} = (1,1,4)$ which can be obtained using path counting. Therefore, the estimator at $\alpha_{(0,0,0)}$= $(3,4,8)$ is

$$\hat{\theta} = \frac{k(\alpha_{(-2,-3,-4)})}{k(\alpha_{(0,0,0)})} = \frac{\frac{(n-9)!}{(x-3)!(y-3)!(z-4)!}}{\frac{n!}{x!y!z!}} = \frac{\frac{6!}{1!1!4!}}{\frac{15!}{3!4!8!}} = 0.00053.$$

Remark4. It is to be noted that for the existence of such estimator it is not necessary for every boundary point (x,y,z) to satisfy : x $\geq a$, $y \geq b$ and $z \geq c$. Since $n \geq a+b+c$, the point (a,b,c) is an accessible/boundary point and the estimator is positive for some boundary points. Recall similar results for binomial distribution. Example: $\hat{\theta} = 0$ for $\alpha_{(0,0,0)} = (2, 10, 3)$.

3.3.4.Estimation for inverse trinomial distribution:

We **c**onsider the inverse trinomial sampling plan $P(0,0,0)$ with fixed number of success $x = k$; $y, z = 0,1,2,\cdots$. The boundary point is denoted as $\alpha_{(0,0,0)} = (k, y, z)$ for such sampling plan. We use pull down sampling plan 2-step along X-axis, 3-step along Y-axis and 4-step along Z-axis. Then the new boundary point is $\alpha_{(-2,-3,-4)} = (k-2, y-3, z-4)$. Using path counting formula we can get $k(\alpha_{(0,0,0)}) = \frac{(k+y+z-1)!}{(k-1)!\ y!\ z!}$ and $k(\alpha_{(-2,-3,-4)}) = \frac{(k+y+z-10)!}{(k-3)!(y-3)!(z-4)!}$. Therefore, estimator of $\theta = p^2q^3r^4$ is

$$\hat{\theta} = \frac{k(\alpha_{(-2,-3,-4)})}{k(\alpha_{(0,0,0)})} = \frac{\frac{(k+y+z-10)!}{(k-3)!(y-3)!(z-4)!}}{\frac{(k+y+z-1)!}{(k-1)!\ y!\ z!}} = \frac{(k-1)^{(2)}y^{(3)}z^{(4)}}{(k+y+z-1)^{(9)}}.$$

For example, let $x = k = 5$ (and $y = 5, z = 5$). Now we pull down the sampling plan 2-step along X-axis, 3-step along Y-axis and 4-step along Z-axis. Then the new boundary point is $\alpha_{(-2,-3,-4)} = (2,1,0)$. Therefore, the estimator at $\alpha_{(0,0,0)}$=(5,5,5) is

$$\hat{\theta} = \frac{k\left(\alpha_{(-2,-3,-4)}\right)}{k\left(\alpha_{(0,0,0)}\right)} = \frac{\frac{(k+y+z-10)!}{(k-3)!(y-3)!(z-4)!}}{\frac{(k+y+z-1)!}{(k-1)!\; y!\; z!}} = \frac{\frac{5!}{2!2!1!}}{\frac{14!}{4!5!5!}} = 0.00012$$

Remark 5: It is to be noted that for the existence of such an estimator it is not necessary for the boundary point to satisfy: $y \geq b$ and $z \geq c$. But, the plan parameter $'k'$ should be so chosen that $k \geq a + b + c$ should be satisfied.

3.3.5 Estimation of $\frac{1}{pq}$ and $\frac{1}{pqr}$ when an experiment is performed until (5,5,5) is achieved

Let us consider, an experiment is performed until $x = 5$ and $y = 5$ and $z = 5$ are observed. Boundary points of such sampling plan are given in Figure 17 (red colour).

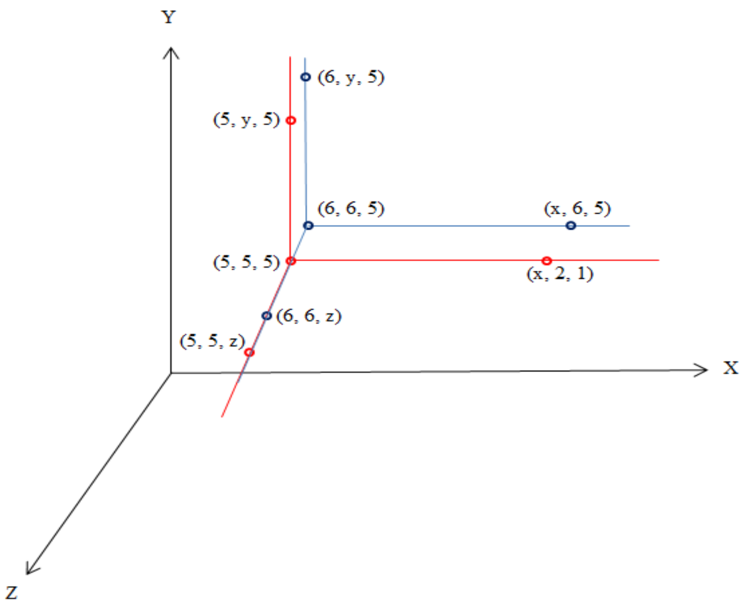

Figure 17: Sampling plan to get (5,5,5)

From Figure 17 we can find that the boundary points (red colour) are : (a) (5,5,5), (b) (x,5,5); x≥6, (c) (5,y,5); y≥6 and (d) (5,5,z); z≥6. At first, we compute the values of $k(\alpha_{(0,0,0)})$. For $\alpha_{(0,0,0)} = (5,5,5)$, all the paths are free paths from the point (0,0,0) through the point (4,5,5) to (5,5,5). To reach (5,5,5), we can also approach via (5,4,5) or (5,5,4). Hence following the path counting formula, the value of $k(\alpha_{(0,0,0)})$ in this case will be $3 \times \frac{14!}{4!5!5!} = 756756$.

Now, for the type (b), let, a typical boundary point be $\alpha_{(0,0,0)} = (x, 5,5)$. Then it must be reached only through the points (x,4,5) or (x,5,4). Hence in this case, the value of $k\left(\alpha_{(0,0,0)}\right)$ is $2 \times \frac{(x+9)!}{x!4!5!}$.

Similarly, for the type (c), a typical boundary point is $\alpha_{(0,0,0)} = (5, y, 5)$ which must be reached only through the point (4,y,5) or (5,y,4). Hence in this case, the value of $k(\alpha_{(0,0,0)})$ is $2 \times \frac{(y+9)!}{4!y!5!}$

Lastly, for type (d), let a typical boundary point be $\alpha_{(0,0,0)} = (5,5, z)$. Then it must be reached only through the point (5,4,z) or (4,5,z). Hence in this case, the value of $k\left(\alpha_{(0,0,0)}\right)$ is $2 \times \frac{(z+9)!}{4!5!z!}$.

Now to estimate $\frac{1}{pq}$, we need to push-up one-step along X-axis and push-up one-step along Y-axis. Therefore, the new boundary points $\alpha_{(+1,+1,0)}$ are [blue colour in Figure 17] (a) (6,6,5), (b) (x,6,5); x≥7, (c) (6,y,5); y≥7 and (d) (6,6,z); z≥6. On the other hand, to estimate $\frac{1}{pqr}$ we can push-up one-step towards X-axis, one-step towards Y-axis and one-step towards Z-axis. Therefore, the new boundary points $\alpha_{(+1,+1,+1)}$ are (a) (6,6,6), (b) (x,6,6); x≥7, (c) (6,y,6); y≥7 and (d) (6,6,6); z≥7. Boundary points and the estimators are summarized in Table 5.

We may argue, as in the case on P[(5, 3)] plan, about the closure of the sampling plan P[(5, 5, 5)]. We skip the details.

Table 5: Estimation of $\frac{1}{pq}$ and $\frac{1}{pqr}$

$\alpha_{(0,0,0)}$	$\alpha_{(+1,+1,0)}$	$\alpha_{(+1,+1,+1)}$	$k(\alpha_{(0,0,0)})$	$k(\alpha_{(+1,+1,0)})$	$k(\alpha_{(+1,+1,+1)})$	$\widehat{\frac{1}{pq}} = \frac{k(\alpha_{(+1,+1,0)})}{k(\alpha_{(0,0,0)})}$	$\widehat{\frac{1}{pqr}} = \frac{k(\alpha_{(+1,+1,+1)})}{k(\alpha_{(0,0,0)})}$
(5,5,5)	(6,6,5)	(6,6,6)	$3 \times \frac{14!}{4!\,5!\,5!}$	$\frac{16!}{6!\,6!\,4!} + 2 \times \frac{16!}{6!\,5!\,5!}$	$3 \times \frac{17!}{6!\,6!\,5!}$	7.55	22.67
(x,5,5); x≥6	(x,6,5); x≥7	(x,6,6); x≥7	$2 \times \frac{(x+9)!}{x!\,4!\,5!}$	$\frac{(x+10)!}{x!\,6!\,4!} + \frac{(x+10)!}{x!\,5!\,5!}$	$2 \times \frac{(x+11)!}{x!\,5!\,6!}$	$\frac{11(x+10)}{60}$	$\frac{(x+10)(x+11)}{30}$
(5,y,5); y≥6	(6,y,5); y≥7	(6,y,6); y≥7	$2 \times \frac{(y+9)!}{4!\,y!\,5!}$	$\frac{(y+10)!}{6!\,y!\,4!} + \frac{(y+10)!}{5!\,y!\,5!}$	$2 \times \frac{(y+11)!}{5!\,y!\,6!}$	$\frac{11(y+10)}{60}$	$\frac{(y+10)(y+11)}{30}$
(5,5,z); z≥6	(6,6,z); z≥6	(6,6,z); z≥7	$2 \times \frac{(z+9)!}{4!\,5!\,z!}$	$2 \times \frac{(z+11)!}{5!\,6!\,z!}$	$2 \times \frac{(z+11)!}{5!\,6!\,z!}$	$\frac{(z+10)(z+11)}{30}$	$\frac{(z+10)(z+11)}{30}$

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